A SELECTION PRINCIPLE FOR THE SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITY

MARCO CICALESE AND GIAN PAOLO LEONARDI

Abstract. We introduce a new variational method for the study of isoperimetric inequalities with quantitative terms. The method is general as it relies on a penalization technique combined with the regularity theory for quasiminimizers of the perimeter. Two notable applications are presented. First we give a new proof of the sharp quantitative isoperimetric inequality in $\mathbb{R}^n$. Second we positively answer to a conjecture by Hall concerning the best constant for the quantitative isoperimetric inequality in $\mathbb{R}^2$ in the small asymmetry regime.

1. Introduction

Let E be a Borel set in $\mathbb{R}^n$, $n \geq 2$, with positive and finite Lebesgue measure $|E|$. Denoting by B_E the open ball centered at 0 such that $|B_E| = |E|$, and by $P(E)$ the distributional perimeter of E (in the sense of Caccioppoli-De Giorgi), we define the *isoperimetric deficit* of E as

$$\delta P(E) = \frac{P(E) - P(B_E)}{P(B_E)}.$$

By the classical isoperimetric inequality in $\mathbb{R}^n$, $\delta P(E)$ is non-negative and zero if and only if E coincides with B_E up to null sets and to a translation. A natural issue arising from the optimality of the ball in the isoperimetric inequality, is that of stability estimates of the type

$$\delta P(E) \geq \varphi(E),$$

where $\varphi(E)$ is a measure of how far E is from a ball. Such inequalities, called *Bonnesen-type inequalities* by Osserman ([31]), have been widely studied after the results by Bernstein ([5]) and Bonnesen ([6, 7]) in the convex, 2-dimensional case (see also [22] and [11] for extensions to convex sets in higher dimensions). Among inequalities of this kind, the well-known *quantitative isoperimetric inequality* states that there exists a constant $C = C(n) > 0$, such that

$$\delta P(E) \geq C\alpha(E)^2, \tag{1}$$

where

$$\alpha(E) = \inf\left\{\frac{|E \triangle (x + B_E)|}{|B_E|},\ x \in \mathbb{R}^n\right\}$$

and $V \triangle W = (V \setminus W) \cup (W \setminus V)$. We recall that $\alpha(E)$ is known as the *Fraenkel asymmetry* of E (see [25]). Observe that both $\delta P(E)$ and $\alpha(E)$ are invariant under isometries and dilations. For this reason, denoting by B the unit open ball in $\mathbb{R}^n$, in studying (1) we are allowed to restrict ourselves to sets E with $|E| = |B|$.

Before the complete proof of the inequality (1) by Fusco, Maggi and Pratelli [19], a number of partial results came one after the other. A first stability result outside the convex setting was proved by Fuglede in [16] (see also [17]), who gave a proof of (1) in the class of *nearly-spherical* sets in $\mathbb{R}^n$. A set E is *nearly-spherical* in the sense of Fuglede if ∂E can be represented as the normal graph of a Lipschitz function u defined on ∂B and such that $\|u\|_{W^{1,\infty}(\partial B)}$ is suitably small. More specifically, the following

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inequality between the isoperimetric deficit $\delta P(E)$ and the Sobolev norm of u is proved in [16] under the assumption that E is nearly-spherical and has the same barycenter as B:

$$\delta P(E) \geq C\|u\|^2_{W^{1,2}(\partial B)}, \tag{2}$$

where $C = C(n) > 0$. By (2) one easily obtains (1) (see Section 4).

A few years later, Hall proved in [23] the inequality (1) for sets with an axis of rotational symmetry (axisymmetric sets). Combining this result with a previous estimate obtained in [25], he was able to prove the quantitative isoperimetric inequality for all sets in $\mathbb{R}^n$, but with a sub-optimal exponent (4 instead of 2) for the asymmetry.

The full proof of the quantitative isoperimetric inequality (with the sharp exponent 2, as conjectured by Hall in [23]) has been recently accomplished by Fusco, Maggi and Pratelli in [19], via a ingenious geometric construction by which the proof of (1) is reduced to sets having more and more symmetries and eventually to axisymmetric sets, for which Hall's result leads to the conclusion.

Since the publication of [19] the study of quantitative forms of various geometric and functional inequalities has received a new impulse (see for instance [18], [20], [13], [14], [8] and the review paper [26]). Among the recent results on this subject, the one by Figalli, Maggi and Pratelli [15] is of particular interest since the authors develop a new technique to study the stability in isoperimetric inequalities. More precisely, they show a more general version of (1), namely a quantitative version of the Wulff theorem, and their analysis relies on Gromov's proof of the isoperimetric inequality [28] and on the theory of optimal mass transportation.

In this paper we present the *Selection Principle*, a variational technique designed for studying isoperimetric inequalities with quantitative terms by reducing their verification to a narrower and "optimal" class of competitors. The Selection Principle basically combines a suitable penalization technique with the regularity theory for quasiminimizers of the perimeter. A couple of comments are in order. First, it is worth noting that the penalization step could be accomplished by the more abstract Ekeland's variational principle, but this has at least two major drawbacks: indeed, the sets that are selected via the Ekeland's principle are not "optimal" (in the sense of statement (ii) below); moreover, Ekeland's functional depends upon its unique minimizer (hence, it is an "implicit" functional, thus not particularly useful for direct applications). Second, the main ideas of this method can work in more general frameworks, however we present them here in a form which is tailored to study the stability for the isoperimetric inequality. Indeed, as a first application of our technique we show a new proof of the sharp quantitative isoperimetric inequality in $\mathbb{R}^n$.

We start from the simple observation that (1) is equivalent to

$$\frac{\delta P(E)}{\alpha(E)^2} \geq C \tag{3}$$

when $\alpha(E) > 0$ (i.e., when E is not a ball up to null sets). On the other hand, since $\alpha(E) < 2$, it is enough to show (3) under a smallness assumption on $\delta P(E)$. This, in turn, translates into a smallness assumption on $\alpha(E)$ (see Section 3). Therefore, we only need to estimate from below the left-hand side of (3) in the *small asymmetry regime*, that is, as $\alpha(E)$ gets smaller and smaller. To study the quotient on the left hand side of (3) in this regime we introduce the functional Q defined as

$$Q(E) = \inf\Big\{ \liminf_k \frac{\delta P(F_k)}{\alpha(F_k)^2} : \ |F_k| = |E|,\ \alpha(F_k) > 0,\ |F_k \triangle E| \to 0 \Big\}.$$

By the definition of Q, the inequality (3) in the small asymmetry regime turns out to be equivalent to the inequality

$$Q(B) > 0. \tag{4}$$

The Selection Principle, that we state below, allows us to compute $Q(B)$ as the limit of $Q(E_j)$, as $j \to \infty$, and where $(E_j)_j$ is an "optimal" sequence of sets with asymmetry going to zero. More precisely, we prove in Section 3 the following result:

Selection Principle. *There exists a sequence of sets $(E_j)_j$ with the following properties:*

(i) $|E_j| = |B|$, $0 < \alpha(E_j) \to 0$ *and* $Q(E_j) \to Q(B)$ *as* $j \to \infty$;
(ii) E_j *is "optimal", i.e. it minimizes the isoperimetric deficit among all sets* F *with* $\alpha(F) = \alpha(E_j)$;
(iii) *there exists a function* $u_j \in C^1(\partial B)$ *such that* $\partial E_j = \{(1+u_j(x))x : x \in \partial B\}$ *and* $u_j \to 0$ *in the* C^1*-norm, as* $j \to \infty$;
(iv) ∂E_j *has (scalar) mean curvature* $H_j \in L^\infty(\partial E_j)$ *and* $\|H_j - 1\|_{L^\infty(\partial E_j)} \to 0$ *as* $j \to \infty$.

As a consequence of the Selection Principle, in Theorem 4.3 we obtain a new and very short proof of the quantitative isoperimetric inequality in $\mathbb{R}^n$. Indeed, thanks to (iii) and for j large enough, E_j is a nearly spherical set, and thus, by Fuglede's estimate (2), we have that $Q(E_j) \geq C$ for some $C = C(n) > 0$. Eventually passing to the limit in j, we get (4). Incidentally we also prove the lower bound

$$Q(B) \geq \frac{n+1}{2n^2}$$

in any dimension $n \geq 2$. This lower bound is contained in Lemma 4.2 and its proof makes use of a strategy developed by Fuglede in [16]. It is interesting to observe that our estimate cannot be recovered form those known for the best constant of (1), that is $\inf Q$. Infact, in [19] and [15] it is proved that, for some $c > 0$, $\inf Q \geq c/4^n$ and $\inf Q \geq c/n^6$, respectively.

In Theorem 4.6 we give a positive answer to another conjecture posed by Hall in [23] which asserts that for any measurable set in $\mathbb{R}^2$ with positive and finite Lebesgue measure the following *Taylor-type lower bound* holds:

$$\delta P(E) \geq C_0 \alpha(E)^2 + o(\alpha(E))^2, \tag{5}$$

with *optimal asymptotic constant* $C_0 = \frac{\pi}{8(4-\pi)}$. Clearly this means that $Q(B) = C_0$ can be explicitly computed in dimension $n = 2$. The inequality (5) was already established in [25, 24] for convex sets in the plane. By property (iv) of the Selection Principle and for j large enough, it turns out that E_j is a convex set, hence by the validity of (5) for convex sets,

$$Q(E_j) \geq C_0 + o(1).$$

Passing to the limit as $j \to \infty$ we get $Q(B) \geq C_0$ which immediately implies (5) for all Borel sets in $\mathbb{R}^2$ with positive and finite Lebesgue measure. Actually, an even more precise estimate than (5) can be proved. Indeed, in the forthcoming paper [9], relying on a more refined version of the Selection Principle, we show in a rather direct way how to compute any order of the optimal Taylor-type lower bound of the isoperimetric deficit in terms of powers of the asymmetry (this result extends to all Borel sets an earlier one obtained in [2] for convex sets in the plane).

We conclude this introduction by first briefly describing the main ideas behind the proof of the Selection Principle. The first step of the proof is the construction of a suitable sequence of penalized functionals $(Q_j)_j$ defined as

$$Q_j(E) = Q(E) + \left(\frac{\alpha(E)}{\alpha(W_j)} - 1\right)^2,$$

where $(W_j)_j$ is a recovery sequence for $Q(B)$. Then, in Lemma 3.3 we check that Q_j admits a minimizer E_j enjoying a number of useful properties. First of all, the sequence $(E_j)_j$ is a recovery sequence for $Q(B)$, that is (ii) in the statement of the Selection Principle. Moreover, we can show in Lemma 3.5 that each E_j is a quasiminimizer of the perimeter (more specifically, a strong Λ-minimizer, see Section 3 and [3]). Therefore, in Lemma 3.6, we can appeal to the regularity theory for quasiminimizers of the perimeter (see [10], [27], [32], [33], [1]) to get the property (iii) stated in the Selection Principle. In addition, by a first variation argument, in Lemma 3.7, we obtain (iv). We finally note that the theory of quasiminimizers has already been successfully applied to derive local strong minimality for strictly stable minimal surfaces (see in particular [34] and [30]).

2. Notation and preliminaries

Given a Borel set $E \subset \mathbb{R}^n$, we denote by $|E|$ its Lebesgue measure. Let $x \in \mathbb{R}^n$ and $r > 0$ be given, then we denote $B(x,r)$ as the open ball in $\mathbb{R}^n$ centered at x and of radius r. We also set $B = B(0,1)$ and $\omega_n = |B|$. Given $E \in \mathbb{R}^n$ we also denote by χ_E its characteristic function and we say that a sequence of sets E_j converges to E with respect to the L^1 or the $L^1_{\rm loc}$-convergence of sets if $\chi_{E_j} \to \chi_E$ in L^1 or in $L^1_{\rm loc}$, respectively. We recall that the *perimeter* of a Borel set E inside an open set $\Omega \subset \mathbb{R}^n$ is

$$P(E,\Omega) := \sup\left\{ \int_E \operatorname{div} g(x)\,dx : \ g \in C^1_c(\Omega;\mathbb{R}^n),\ |g| \le 1 \right\}.$$

This definition extends the natural notion of $(n-1)$-dimensional area of a smooth (or Lipschitz) boundary ∂E. We will say that E has *finite perimeter* in Ω if $P(E,\Omega) < \infty$. Equivalently, E is a set of finite perimeter in Ω if the distributional derivative $D\chi_E$ of its characteristic function χ_E is a vector-valued Radon measure in Ω with finite total variation $|D\chi_E|(\Omega)$. We will simply write $P(E)$ instead of $P(E,\mathbb{R}^n)$, and we will say that E is a set of finite perimeter if $P(E) < \infty$. From the well-known De Giorgi's Rectifiability Theorem (see [4], [12]), $D\chi_E = \nu_E\,\mathcal{H}^{n-1}\lfloor\partial^* E$ where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\partial^* E$ is the *reduced boundary* of E, i.e., the set of points $x \in \partial E$ such that the *generalized inner normal* $\nu_E(x)$ is defined, that is,

$$\nu_E(x) = \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))} \quad \text{and} \quad |\nu_E(x)| = 1.$$

We recall (see for instance [21]) that, given E of finite perimeter in $\mathbb{R}^n$, for all $x \in \partial^* E$

$$\lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{|B(x,r)|} = \frac{1}{2}, \tag{6}$$

and for a.e. $r \in \mathbb{R}$ it holds that

$$P(E,B(0,r)) = P(E\cap B(0,r)) - \mathcal{H}^{n-1}(E\cap \partial B(0,r)). \tag{7}$$

We now recall some classical definitions and properties of quasiminimizers of the perimeter (see [32], [33], [3]). Given $E \subset \mathbb{R}^n$ of finite perimeter and $A \subset \mathbb{R}^n$ an open bounded set, we define the *deviation from minimality* of E in A as

$$\Psi(E,A) = P(E,A) - \inf\{P(F,A) : \ F \triangle E \subset\subset A\},$$

where $F \triangle E = (F\setminus E)\cup(E\setminus F)$ and $S \subset\subset A$ iff S is a relatively compact subset of A. Note that $\Psi(E,A) \ge 0$, with equality if and only if E minimizes the perimeter in A (w.r.t. all of its compact variations F). We set $\Psi(E,x,r) = \Psi(E,B(x,r))$ and, given $\gamma \in (0,1)$, $R > 0$ and $\Lambda > 0$, we call *quasiminimizer* of the perimeter (in $\mathbb{R}^n$) any set E of finite perimeter for which

$$\Psi(E,x,r) \le \Lambda\omega_{n-1} r^{n-1+2\gamma} \tag{8}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$ (see [32, 3]). We will also equivalently write $E \in \mathcal{QM}(\gamma,R,\Lambda)$ to highlight the key parameters occurring in the above definition of quasiminimality. If $E \in \mathcal{QM}(\frac{1}{2},R,\Lambda)$, i.e. when $\gamma = \frac{1}{2}$ in (8), then we call E a Λ-*minimizer* (see [3]). Finally, if E satisfies

$$P(E,B(x,r)) \le P(F,B(x,r)) + \Lambda\omega_{n-1}\frac{|E\triangle F|}{\omega_n}$$

for all $x \in \mathbb{R}^n$, $0 < r < R$ and all Borel sets F such that $E \triangle F \subset\subset B(x,r)$, then E is said to be a *strong* Λ-minimizer. It is easy to check that any strong Λ-minimizer is also a Λ-minimizer, hence a quasiminimizer of the perimeter (we refer to [33] for a clear treatment of the subject).

We now extend the definition of quasiminimality to sequences of sets of finite perimeter. We say that a sequence $(E_h)_h$ of sets of finite perimeter is a *uniform sequence of quasiminimizers* if $E_h \in \mathcal{QM}(\gamma,R,\Lambda)$ for some fixed parameters $\gamma \in (0,1)$, $R > 0$ and $\Lambda > 0$, and for all $h \in \mathbb{N}$.

Before going on, we recall the notion of *convergence in the Kuratowski sense.* Let $(S_h)_h$ be a sequence of sets in $\mathbb{R}^n$, then we say that S_h converges in the Kuratowski sense to a set $S \subset \mathbb{R}^n$ as $h \to \infty$, if the following two properties hold:

- if a sequence of points $x_h \in S_h$ converges to a point x as $h \to \infty$, then $x \in S$;
- for any $x \in S$ there exists a sequence $x_h \in S_h$ such that x_h converges to x as $h \to \infty$.

In addition, given $(S_h)_h$ an equibounded sequence of compact sets, the convergence of S_h to S in the Kuratowski sense is equivalent to the convergence in the Hausdorff metric.

In the following proposition we recall some crucial properties of uniform sequences of quasiminimizers (see for instance Theorem 1.9 in [33]).

Proposition 2.1 (Properties of quasiminimizers)**.** *Let $(E_h)_h$ be a uniform sequence of quasiminimizers, i.e. assume there exist $\gamma \in (0,1)$, $R > 0$ and $\Lambda > 0$ such that $E_h \in \mathcal{QM}(\gamma, R, \Lambda)$ for all $h \in \mathbb{N}$. Then, if E_h converges to E in L^1, the following facts hold.*

(i) *∂E_h converges to ∂E in the Kuratowski sense, as $h \to \infty$. If in addition ∂E is compact, then ∂E_h converges to ∂E in the Hausdorff metric.*
(ii) *If $x \in \partial^* E$ and $x_h \in \partial E_h$ is such that $x_h \to x$, then there exists $\bar{h}$, such that $x_h \in \partial^* E_h$ for all $h \geq \bar{h}$. Moreover, $\nu_{E_h}(x_h) \to \nu_E(x)$ as $h \to \infty$.*

The deviation from minimality (and, thus, the concept of quasiminimality described above) turns out to be closely related to another key quantity in De Giorgi's regularity theory: the *excess.* Given $x \in \mathbb{R}^n$, $r > 0$ and E of locally finite perimeter, the excess of E in $B(x,r)$ is defined as

$$\begin{aligned} \operatorname{Exc}(E,x,r) &= r^{1-n}\left(P(E,B(x,r)) - |D\chi_E(B(x,r))|\right) \\ &= \frac{r^{1-n}}{2} \min_{\substack{\xi \in \mathbb{R}^n \\ |\xi|=1}} \left\{ \int_{\partial^* E \cap B(x,r)} |\nu_E(y) - \xi|^2 \, d\mathcal{H}^{n-1}(y) \right\}. \end{aligned}$$

In the following proposition we state a useful continuity property of the excess and the fundamental regularity result for quasiminimizers (see for instance Proposition 4.3.1 in [3] and Theorem 1.9 in [33]). Before stating the proposition, we introduce some extra notation. Given a point $x \in \mathbb{R}^n$ and a unit vector $\nu \in \mathbb{R}^n$, we write with a little abuse of notation $x = x_\nu^\perp + x_\nu \nu = (x_\nu^\perp, x_\nu)$, where $x_\nu^\perp$ is the projection of x onto the orthogonal complement of ν and $x_\nu = \langle x, \nu \rangle$. Given $r > 0$ and a unit vector $\nu \in \mathbb{R}^n$, we define the *cylinder* $\mathcal{C}_{\nu,r} = \{x = (x_\nu^\perp, x_\nu) : \ \max(|x_\nu^\perp|, |x_\nu|) < r\}$. Following our notation, $\mathcal{C}_{\nu,r}$ can be defined as the Cartesian product $B_{\nu,r} \times (-r,r) \cdot \nu$, where $B_{\nu,r}$ is the open ball of radius r in the orthogonal complement of ν. Given a function $f : B_{\nu,r} \to \mathbb{R}$, we define its *graph* as

$$\operatorname{gr}(f) = \{(x_\nu^\perp, f(x_\nu^\perp)\nu) : \ x_\nu^\perp \in B_{\nu,r}\}.$$

Proposition 2.2 (Excess and regularity for quasiminimizers)**.** *Given $\gamma \in (0,1)$, $R > 0$ and $\Lambda > 0$, the following facts hold.*

(i) *Let $(E_h)_h$ be a sequence in $\mathcal{QM}(\gamma, R, \Lambda)$ and assume $E_h \to E$ in L^1_{loc}, as $j \to \infty$. Then*
$$\lim_j \operatorname{Exc}(E_j, x, r) = \operatorname{Exc}(E, x, r),$$
for all $x \in \mathbb{R}^n$ and $0 < r < R$ for which $P(E, \partial B(x,r)) = 0$.
(ii) *There exists $\varepsilon_0 = \varepsilon_0(n, \gamma, R, \Lambda) > 0$ with the following property: if $E \in \mathcal{QM}(\gamma, R, \Lambda)$, $x_0 \in \partial E$, and if $\operatorname{Exc}(E, x_0, 2r) < \varepsilon_0$ for some $0 < r < R/2$, then $x_0 \in \partial^* E$ and, setting $\nu = \nu_E(x_0)$, one has that*
$$(\partial E - x_0) \cap \mathcal{C}_{\nu,r} = \operatorname{gr}(f),$$
where $f \in C^{1,\gamma}(B_{\nu,r}) \to \mathbb{R}$, with $f(0) = |\nabla f(0)| = 0$. Moreover, one has the Hölder estimate
$$|\nabla f(v) - \nabla f(w)| \leq C|v - w|^\gamma \tag{9}$$
for all $v, w \in B_{\nu,r}$ and for a suitable constant $C = C(n, \gamma, R, \Lambda) > 0$.

Remark 2.3. Given a quasiminimizer $E \in \mathcal{QM}(\gamma, R, \Lambda)$, and owing to Proposition 2.2 and the fact that for any $x_0 \in \partial^* E$ one has $\mathrm{Exc}(E, x_0, r) \to 0$ as $r \to 0$, we conclude that $\partial^* E$ is a smooth hypersurface of class $C^{1,\gamma}$. Moreover, by Federer's blow-up argument (see [21]), the Hausdorff dimension of the *singular set* $\partial E \setminus \partial^* E$ cannot exceed $n-8$. Finally, one can show via standard elliptic estimates for weak solutions to the mean curvature equation with bounded prescribed curvature (see Section 7.7 in [4]) that, if E is a strong Λ-minimizer, then $\partial^* E$ is of class $C^{1,\eta}$ for all $0 < \eta < 1$ (and of class $C^{1,1}$ in dimension $n = 2$).

In what follows we will denote by $\mathcal{S}^n$ the class of Borel subsets of $\mathbb{R}^n$ with positive and finite Lebesgue measure. Given $E \in \mathcal{S}^n$, we define its *isoperimetric deficit* $\delta P(E)$ and its *Fraenkel asymmetry* $\alpha(E)$ as follows:

$$\delta P(E) := \frac{P(E) - P(B_E)}{P(B_E)} \tag{10}$$

and

$$\alpha(E) := \inf \left\{ \frac{|E \triangle (x + B_E)|}{|B_E|},\ x \in \mathbb{R}^n \right\}, \tag{11}$$

where B_E denotes the ball centered at the origin such that $|B_E| = |E|$ and $E \triangle F$ denotes the symmetric difference of the two sets E and F. Since both $\delta P(E)$ and $\alpha(E)$ are invariant under isometries and dilations, from now on we will set $|E| = |B|$ so that $B_E = B$. By definition, the Fraenkel asymmetry $\alpha(E)$ satisfies $\alpha(E) \in [0, 2)$ and it is zero if and only if E coincides with B in measure-theoretic sense and up to a translation. Notice that the infimum in (11) is actually a minimum.

3. The Selection Principle

Given a Borel set E in $\mathbb{R}^n$ with $|E| = |B|$, the classical isoperimetric inequality states that

$$P(E) \geq P(B), \tag{12}$$

with equality if and only if $\alpha(E) = 0$ (i.e., if E coincides with the ball B up to translations and to negligible sets), that is to say, the isoperimetric deficit $\delta P(E)$ is always non-negative and zero if and only if $\alpha(E) = 0$.

In the next section we will provide a new proof of the sharp quantitative isoperimetric inequality in $\mathbb{R}^n$ which is a quantitative refinement of (12) and asserts the existence of a positive constant C such that, for any $E \in \mathcal{S}^n$ it holds

$$\delta P(E) \geq C\alpha^2(E). \tag{13}$$

With the aim of presenting the main ideas of the method that will lead to the proof of (13), we start with some relatively elementary comments. As we have recalled before, the equality case in the isoperimetric inequality (12) is attained precisely when E coincides with a ball in measure-theoretic sense. This uniqueness property can be equivalently stated as the implication

$$\delta P(E) = 0 \ \Rightarrow\ \alpha(E) = 0. \tag{14}$$

By Lemma 2.3 and Lemma 5.1 in [19] (or via a standard concentration-compactness type argument in [1] Lemma VI.15) it is possible to strengthen (14) and state the following

Lemma 3.1. *For all $\alpha_0 > 0$ there exists $\delta_0 > 0$ such that, for any $E \in \mathcal{S}^n$, if $\delta P(E) < \delta_0$ then $\alpha(E) < \alpha_0$.*

It is worth noticing that, as a consequence of Lemma 3.1, to prove (13) it is enough to work in the *small asymmetry regime*, i.e. to show that there exist $\alpha_0 > 0$ and $C_0 > 0$ such that

$$\frac{\delta P(E)}{\alpha^2(E)} \geq C_0 \tag{15}$$

for all $E \in \mathcal{S}^n$ with $0 < \alpha(E) < \alpha_0$. In fact, assume otherwise that $\alpha(E) \geq \alpha_0$ and let δ_0 be as in Lemma 3.1. Then, since $\alpha(E) < 2$, it holds that $\frac{\delta P(E)}{\alpha^2(E)} \geq \frac{\delta_0}{4}$, and thus (13) follows by taking $C = \min\{C_0, \frac{\delta_0}{4}\}$.

In order to study the small asymmetry regime, it is convenient to introduce the functional $Q : \mathcal{S}^n \to [0, +\infty]$ defined as

$$Q(E) = \inf \Big\{ \liminf_k \frac{\delta P(F_k)}{\alpha(F_k)^2} : \ (F_k)_k \subset \mathcal{S}^n, \ |F_k| = |E|, \ \alpha(F_k) > 0, \ |F_k \triangle E| \to 0 \Big\}. \tag{16}$$

The functional Q is the lower semicontinuous envelope of the quotient $\frac{\delta P(E)}{\alpha(E)^2}$ with respect to the L^1-convergence of sets and, by the lower semicontinuity of the perimeter and the continuity of the asymmetry with respect to this convergence, $Q(E) = \frac{\delta P(E)}{\alpha(E)^2}$ whenever $\alpha(E) > 0$. Let us now observe that, by the definition of Q, the inequality (15) in the small asymmetry regime (and, in turn, (13)) turns out to be equivalent to

$$Q(B) > 0. \tag{17}$$

In order to prove (17) one may study a recovery sequence for $Q(B)$, that is a sequence of sets $(W_j)_j$ such that $|W_j| = |B|$, $\alpha(W_j) > 0$ and $|W_j \triangle B| \to 0$, for which $Q(B) = \lim_j Q(W_j)$. However, such a sequence may not be "good enough" to handle in order to get the desired estimate (15). To overcome this problem, we take advantage of the following theorem, which is the main result of this section, and asserts the existence of a recovery sequence $(E_j)_j$ for $Q(B)$ satisfying some useful additional properties which simplify the computation of $Q(B)$.

Theorem 3.2 (Selection Principle)**.** *There exists a sequence of sets $(E_j)_j \subset \mathcal{S}^n$, such that*

(i) $|E_j| = |B|$, $0 < \alpha(E_j) \to 0$ *and* $Q(E_j) \to Q(B)$ *as* $j \to \infty$*;*
(ii) E_j *minimizes the isoperimetric deficit among all sets* F *with* $\alpha(F) = \alpha(E_j)$*;*
(iii) *for each* j *there exists a function* $u_j \in C^1(\partial B)$ *such that* $\partial E_j = \{(1 + u_j(x))x : \ x \in \partial B\}$ *and* $u_j \to 0$ *in the* C^1*-norm, as* $j \to \infty$*;*
(iv) ∂E_j *has (scalar) mean curvature* $H_j \in L^\infty(\partial E_j)$ *and* $\|H_j - 1\|_{L^\infty(\partial E_j)} \to 0$ *as* $j \to \infty$*.*

The rest of the section will be devoted to the proof of Theorem 3.2. The latter will be a consequence of several intermediate results, most of them having their own independent interest and being suitable for applications to more general frameworks. The main ingredients of the proof of Theorem 3.2 involve a penalization argument combined with some properties of quasiminimizers of the perimeter.

Let $(W_j)_j$ be a recovery sequence for $Q(B)$ having

$$\alpha(W_j) \le \frac{1}{4(Q(B) + 2)} \tag{18}$$

and satisfying

$$|Q(W_j) - Q(B)| < \frac{1}{j} \qquad \text{for all } j \ge 1. \tag{19}$$

Note that, as pointed out in [23], by selecting a suitable sequence of ellipsoids converging to B, one can show that $Q(B) < +\infty$ (see also [26]).
We now define the sequence of functionals $(Q_j)_j : \mathcal{S}^n \to [0, +\infty)$ as

$$Q_j(E) = Q(E) + \left(\frac{\alpha(E)}{\alpha(W_j)} - 1 \right)^2. \tag{20}$$

The following lemma holds

Lemma 3.3 (Penalization)**.** *For any integer* $j \ge 1$*,*

(i) Q_j *is lower semicontinuous with respect to the* L^1*-convergence of sets;*
(ii) *there exists a bounded minimizer of the functional* Q_j*, i.e. a bounded set* E_j *such that* $|E_j| = |B|$ *and* $Q_j(E_j) \le Q_j(F)$ *for all* $F \in \mathcal{S}^n$*;*
(iii) $E_j \to B$ *in* L^1*,* $Q(E_j) \to Q(B)$ *and* $\frac{\alpha(E_j)}{\alpha(W_j)} \to 1$*, as* $j \to \infty$*;*

Proof. (i) follows from the lower semicontinuity of the perimeter and the continuity of $\frac{\alpha(\cdot)}{\alpha(W_j)}$ with respect to L^1-convergence of sets.

The proof of (ii) borrows some ideas from Lemma VI.15 in [1] (see also [29]). Let j be fixed and let $(V_{j,k})_k \subset \mathcal{S}^n$ be a minimizing sequence for Q_j satisfying $|V_{j,k}| = |B|$, $Q_j(V_{j,k}) \le \inf Q_j + 1/k$, and such that $\alpha(V_{j,k}) = \frac{|V_{j,k} \triangle B|}{|B|}$ for all $k \ge 1$. Since $\inf Q_j \le Q_j(W_j) = Q(W_j)$ and $Q(W_j) \to Q(B)$ as $j \to \infty$, we may assume without loss of generality that, for all $k \ge 1$,

$$Q_j(V_{j,k}) \le Q(B) + 1. \tag{21}$$

In particular, this implies that there exists $M > 0$ such that $\sup_k P(V_{j,k}) \le M$. By the well-known compactness properties of sequences of sets with equibounded perimeter, we can assume that there exists $V_j \in \mathcal{S}^n$ such that (up to subsequences) $V_{j,k} \to V_j$ in the L^1_{loc} convergence of sets, which in particular implies that $|V_j| \le \liminf_k |V_{j,k}| = |B|$. Moreover, by the lower semicontinuity of the perimeter, we have also that $P(V_j) \le M$. By the definition of Q_j, thanks to (18) and (21), we have that

$$\frac{|V_{j,k} \triangle B|}{|B|} = \alpha(V_{j,k}) \le ((Q(B)+1)^{\frac{1}{2}} + 1)\alpha(W_j) \le \frac{(Q(B)+1)^{\frac{1}{2}} + 1}{4(Q(B)+2)} < \frac{1}{4}.$$

Therefore

$$|V_{j,k} \cap B| > \frac{3}{4}|B|, \tag{22}$$

for all $k \in \mathbb{N}$. We now show that

$$P(V_j) \le P(F), \tag{23}$$

for all sets $F \in \mathcal{S}^n$ such that $F \triangle V_j \subset\subset \mathbb{R}^n \setminus B(0,3)$ and $|F| = |V_j|$. Let us assume by contradiction that (23) does not hold, i.e., there exist $\delta > 0$ and F as above, such that

$$P(F) \le P(V_j) - \delta. \tag{24}$$

Given $0 < r < R$, we set $C(r,R) = B(0,R) \setminus \overline{B(0,r)}$ and define $(\hat{V}_{j,k})_k \subset \mathcal{S}^n$ as

$$\hat{V}_{j,k} = (V_{j,k} \setminus C(r,R)) \cup (F \cap C(r,R)).$$

Note that, by the definition of F, by the L^1_{loc} convergence of $V_{j,k}$ to V_j and thanks to (7), we can choose r and R such that

(a) $3 < r < R$,
(b) $F \triangle V_j \subset\subset C(r,R)$,
(c) $\mathcal{H}^{n-1}((V_{j,k} \triangle V_j) \cap \partial C(r,R)) \to 0$ as $k \to \infty$,
(d) $P(\hat{V}_{j,k}) = P(V_{j,k}, \mathbb{R}^n \setminus \overline{C(r,R)}) + P(F, C(r,R)) + \mathcal{H}^{n-1}((V_{j,k} \triangle V_j) \cap \partial C(r,R))$.

Let us observe that, since $P(V_j, C(r,R)) \le \liminf_k P(V_{j,k}, C(r,R))$, on combining (c) and (d), and thanks to (24), there exists $k_j \in \mathbb{N}$ such that, for all $k \ge k_j$ we get

$$P(\hat{V}_{j,k}) \le P(V_{j,k}) - \frac{2\delta}{3}. \tag{25}$$

Moreover, by the definition of $\hat{V}_{j,k}$ we also have that

$$\begin{aligned} |\hat{V}_{j,k}| &= |F \cap C(r,R)| + |V_{j,k} \setminus C(r,R)| \\ &= |V_{j,k}| + |V_j \cap C(r,R)| - |V_{j,k} \cap C(r,R)| \\ &= |B| + |V_j \cap C(r,R)| - |V_{j,k} \cap C(r,R)|, \end{aligned}$$

therefore, passing to the limit as $k \to \infty$, one obtains

$$\lim_k |\hat{V}_{j,k}| = |B|. \tag{26}$$

Let us now fix $x_j \in \partial^* F \cap C(r,R)$. Thanks to (26) and (6), for k large enough there exists $0 \le \rho_{j,k} < \left(\frac{\delta}{3n\omega_n}\right)^{\frac{1}{n-1}}$, such that, defining $(\tilde V_{j,k})_k$ as

$$\tilde V_{j,k} = \begin{cases} \hat V_{j,k} \cup B(x_j,\rho_{j,k}) & \text{if } |\hat V_{j,k}| \le |B| \\ \hat V_{j,k} \setminus B(x_j,\rho_{j,k}) & \text{if } |\hat V_{j,k}| > |B|, \end{cases} \tag{27}$$

we get $|\tilde V_{j,k}| = |B|$, $B(x_j,\rho_{j,k}) \subset\subset C(r,R)$, and

$$|P(\hat V_{j,k}) - P(\tilde V_{j,k})| \le P(B(x_j,\rho_{j,k})) = n\omega_n(\rho_{j,k})^{n-1} < \frac{\delta}{3}. \tag{28}$$

By (25) and (28), we eventually get

$$P(\tilde V_{j,k}) \le P(V_{j,k}) - \frac{\delta}{3}. \tag{29}$$

This, in turn, would contradict the fact that $V_{j,k}$ is a minimizing sequence for Q_j, once we prove that, for k sufficiently large,

$$\alpha(\tilde V_{j,k}) = \alpha(V_{j,k}). \tag{30}$$

Indeed, by (22) and (27) we have

$$\begin{aligned} |\tilde V_{j,k} \triangle B| &= |V_{j,k} \triangle B| \\ &= 2(|B| - |V_{j,k} \cap B|) \\ &\le |B|/2. \end{aligned} \tag{31}$$

On the other hand, if $x \in \mathbb{R}^n \setminus B(0,2)$ then $V_{j,k} \cap B \subset \tilde V_{j,k} \triangle (x+B)$, and therefore by (22) we get

$$|\tilde V_{j,k} \triangle (x+B)| \ge |V_{j,k} \cap B| > \frac{3}{4}|B|. \tag{32}$$

On combining (31) and (32), one shows that the asymmetry of $\tilde V_{j,k}$ is attained on a ball centered in $x \in B(0,2)$, that is (30) holds, as wanted.

Thanks to (23), and by well-known results about minimizers of the perimeter subject to a volume constraint, there exists $R > 1$ such that $V_j \subset B(0,R)$.

We now distinguish two cases.

Case 1. $|V_j| = |B|$. In this case the local convergence is equivalent to convergence in $L^1(\mathbb{R}^n)$, hence by the lower semicontinuity of Q_j we have that V_j is a minimizer of Q_j, thus we conclude taking $E_j = V_j$.

Case 2. $|V_j| < |B|$. In this case the sequence $(V_{j,k})_k$ “looses volume at infinity”. We now claim that, setting $x_0 = (R+2,0,\dots,0) \in \mathbb{R}^n$ and $0 < t < 1$ such that $\omega_n t^n + |V_j| = |B|$, the set $E_j := V_j \cup B(x_0,t)$ is a minimizer for Q_j. To this end, note that, since $V_j \subset B(0,R)$, there exists a null set $\mathcal{N} \subset (R,R+1)$ such that, for all $j \ge 1$ and $\rho \in (R,R+1) \setminus \mathcal{N}$, we have that

$$P(V_{j,k}, B(0,\rho)) = P(V_{j,k} \cap B(0,\rho)) - \mathcal{H}^{n-1}(V_{j,k} \cap \partial B(0,\rho)), \ \forall k \ge 1, \tag{33}$$

thanks to (7), and

$$\lim_k \mathcal{H}^{n-1}(V_{j,k} \cap \partial B(0,\rho)) = 0 \tag{34}$$

since $|V_{j,k} \setminus B(0,\rho)| \to 0$ as $k \to \infty$.

By (33) and (34), and owing to the isoperimetric inequality in $\mathbb{R}^n$, we get

$$\begin{aligned} P(E_j) &= P(V_j, B(0,\rho)) + n\omega_n t^{n-1} \\ &\le \liminf_k P(V_{j,k}, B(0,\rho)) + n\omega_n t^{n-1} \\ &= \liminf_k (P(V_{j,k} \cap B(0,\rho)) - \mathcal{H}^{n-1}(V_{j,k} \cap \partial B(0,\rho))) + n\omega_n t^{n-1} \\ &= \liminf_k (P(V_{j,k}) - P(V_{j,k} \setminus B(0,\rho))) + n\omega_n t^{n-1} \\ &\le \liminf_k (P(V_{j,k}) - n\omega_n t_k^{n-1}) + n\omega_n t^{n-1} \\ &= \liminf_k P(V_{j,k}), \end{aligned} \tag{35}$$

where we have denoted by t_k the radius of a ball equivalent to $V_{j,k} \setminus B(0,\rho)$ and used the fact that $t_k \to t$ as $k \to \infty$. Taking into account (22), one can check that the asymmetry of E_j is attained on balls that are disjoint from $B(x_0,t)$, hence

$$0 < \alpha(E_j) = \lim_k \alpha(V_{j,k}). \tag{36}$$

Then by (35) and (36) we conclude that $Q_j(E_j) \le \liminf_k Q_j(V_{j,k}) = \inf Q_j$, as claimed.

Finally, to prove (iii) we take E_j a minimizer for Q_j and observe that

$$Q(E_j) \le Q_j(E_j) \le Q_j(W_j) = Q(W_j).$$

This implies that $\alpha(E_j) = \alpha(W_j) + o(\alpha(W_j))$ and that $\lim Q(E_j) = \lim Q_j(E_j) = Q(B)$. Eventually, by the invariance of Q_j under translation we may assume that E_j converges to B, thus completing the proof. $\square$

We omit the elementary proof of the next lemma. It follows quite directly from the definition of asymmetry and from the triangular inequality

$$|A \triangle B| \le |A \triangle C| + |C \triangle B| \tag{37}$$

which holds in particular for any $A,\ B,\ C \in \mathcal{S}^n$.

Lemma 3.4. *Let $E \in \mathcal{S}^n$ with $|E| = |B| = \omega_n$. For all $x \in \mathbb{R}^n$ and for any $F \in \mathcal{S}^n$ with $E \triangle F \subset\subset B(x,\frac{1}{2})$, it holds that $|\alpha(E) - \alpha(F)| \le \frac{2^{n+2}}{(2^n-1)\omega_n}|E \triangle F|$.*

We now establish a fundamental property of the sequence $(E_j)_j$ of Lemma 3.3, i.e., the fact that it is a uniform sequence of Λ-minimizers (see Section 2).

Lemma 3.5 (Uniform Λ-minimality)**.** *There exist $\Lambda = \Lambda(n) > 0$ and $j_0 \in \mathbb{N}$ with the following property: for all $j \ge j_0$ and for any minimizer E_j of the functional Q_j satisfying $|E_j| = |B|$, $Q(E_j) \le Q(B)+1$, and such that $|\alpha(E_j) - \alpha(W_j)| \le \alpha(W_j)/2$, we obtain that E_j is a strong Λ-minimizer of the perimeter.*

Proof. Let $x \in \mathbb{R}^n$ be fixed and let $F \subset \mathcal{S}^n$ be such that $F \triangle E_j \subset\subset B(x,1/2)$. We want to prove that

$$P(E_j) \le P(F) + \Lambda\omega_{n-1}\frac{|E_j \triangle F|}{\omega_n}$$

for some $\Lambda = \Lambda(n) > 0$. Without loss of generality let us assume that $P(F) \le P(E_j)$ and that $\alpha(E_j) = \frac{|E_j \triangle B|}{|B|}$. We divide the proof in two cases.

Case 1. $\alpha(E_j)^2 \le |E_j \triangle F|$. In this case, by the assumption $Q(E_j) \le Q(B)+1$, we get

$$\begin{aligned} P(E_j) &\le P(B) + (Q(B)+1)P(B)\alpha(E_j)^2 \\ &\le P(B) + (Q(B)+1)P(B)|E_j \triangle F| \end{aligned} \tag{38}$$

By the previous inequality, denoting by B_F the ball equivalent to F centered at the origin, using the isoperimetric inequality in $\mathbb{R}^n$ and the triangular inequality (37) we have

$$\begin{aligned} P(E_j) &\le P(F) + P(B) - P(B_F) + (Q(B)+1)P(B)|E_j \triangle F| \\ &\le P(F) + n\omega_n^{\frac{1}{n}}(|E_j|^{\frac{n-1}{n}} - |F|^{\frac{n-1}{n}}) + (Q(B)+1)P(B)|E_j \triangle F| \\ &\le P(F) + n\omega_n^{\frac{1}{n}}((|F| + |E_j \triangle F|)^{\frac{n-1}{n}} - |F|^{\frac{n-1}{n}}) + (Q(B)+1)P(B)|E_j \triangle F| \\ &= P(F) + n\omega_n^{\frac{1}{n}}|F|^{\frac{n-1}{n}}((1 + \frac{|E_j \triangle F|}{|F|})^{\frac{n-1}{n}} - 1) + (Q(B)+1)P(B)|E_j \triangle F|. \end{aligned} \tag{39}$$

Using Bernoulli's inequality and the fact that, by construction, $|F| \ge \frac{3}{4}\omega_n$, by (39) we get

$$\begin{aligned} P(E_j) &\le P(F) + (n-1)\omega_n^{\frac{1}{n}}|F|^{\frac{-1}{n}}|E_j \triangle F| + (Q(B)+1)P(B)|E_j \triangle F| \\ &\le P(F) + (n-1)(4/3)^{\frac{1}{n}}|E_j \triangle F| + (Q(B)+1)P(B)|E_j \triangle F| \\ &= P(F) + \Lambda_1\omega_{n-1}\frac{|E_j \triangle F|}{\omega_n}, \end{aligned} \tag{40}$$

where we have set $\Lambda_1 = \frac{\omega_n((n-1)(4/3)^{\frac{1}{n}}+(Q(B)+1)P(B))}{\omega_{n-1}}$.

Case 2. $|E_j \triangle F| < \alpha(E_j)^2$. By the inequality $Q_j(E_j) \le Q_j(F)$ we obtain

$$\delta P(E_j) \le \delta P(F) + \left(\frac{\alpha(E_j)^2}{\alpha(F)^2} - 1\right)\delta P(F) + \eta, \tag{41}$$

where

$$\eta := \alpha(E_j)^2 \frac{(\alpha(F) - \alpha(E_j))(\alpha(F) + \alpha(E_j) - 2\alpha(W_j))}{\alpha(W_j)^2}.$$

By noting that the assumption $|\alpha(E_j) - \alpha(W_j)| \le \alpha(W_j)/2$ implies $\alpha(E_j) \le 3\alpha(W_j)/2$, and by exploiting Lemma 3.4, we have that

$$\begin{aligned} \eta &\le \tfrac{9}{4}(\alpha(F) - \alpha(E_j))(\alpha(F) + \alpha(E_j) - 2\alpha(W_j)) \\ &\le C_1|E_j \triangle F|, \end{aligned} \tag{42}$$

for some $C_1 = C_1(n) > 0$. By Lemma 3.4 we have that

$$\left(\frac{\alpha(E_j)^2}{\alpha(F)^2} - 1\right)\delta P(F) \le \frac{2^{n+4}}{(2^n-1)\omega_n} Q(F)|E_j \triangle F|. \tag{43}$$

Observe now that, combining the hypothesis $|E_j \triangle F| < \alpha^2(E_j)$ with Lemma 3.4 and recalling that $\alpha(E_j) \to 0$, we have that there exists $C > 0$ and $j_0 \in \mathbb{N}$ such that, for all $j \ge j_0$ it holds that

$$\left|\frac{P(B)}{P(B_F)} - 1\right| \le C\alpha(E_j)^2, \quad \left|\frac{\alpha(E_j)}{\alpha(F)} - 1\right| \le C\alpha(E_j). \tag{44}$$

By the previous estimates, using that, by assumption on F, $P(F) \le P(E_j)$ we also get that

$$Q(F) \le \frac{P(B)\alpha(E_j)^2}{P(B_F)\alpha(F)^2} Q(E_j) + \left(\frac{P(B)}{P(B_F)} - 1\right)\frac{1}{\alpha(F)^2}.$$

By the previous inequality, using (44), we have for j large enough

$$Q(F) \le 2Q(E_j) + 2 \le 2(Q(B)+1) + 2,$$

Therefore, (43) becomes

$$\left(\frac{\alpha(E_j)^2}{\alpha(F)^2} - 1\right)\delta P(F) \le C_2|E_j \triangle F|, \tag{45}$$

with $C_2 = C_2(n) > 0$. In conclusion, starting from (41) we have proved that

$$\delta P(E_j) \le \delta P(F) + (C_1 + C_2)|E_j \triangle F|,$$

that is

$$\begin{aligned} P(E_j) &\le P(F) + \left(\frac{P(B)}{P(B_F)} - 1\right)P(F) + (C_1+C_2)P(B)|E_j \triangle F| \\ &\le P(F) + \Lambda_2 P(B)|E_j \triangle F|, \end{aligned}$$

with $\Lambda_2 = (C_1 + C_2)P(B) + 1$.

The conclusion follows by setting $\Lambda = \max(\Lambda_1, \Lambda_2)$. □

In the next lemma, we prove the $C^{1,\gamma}$ regularity of ∂E_j for j large enough, as well as the fact that ∂E_j converges to ∂B in the C^1-topology, as $j \to \infty$. Here, by convergence of ∂E_j to ∂B in the C^1-topology, we mean the following: there exist $r > 0$ and an open covering of ∂B by a finite family of cylinders $\{\nu_k + \mathcal{C}_{\nu_k,r}\}_{k=1}^N$, with $\nu_k \in \partial B$ such that it holds

- $\partial E_j \subset \bigcup_{k=1}^N (\nu_k + \mathcal{C}_{\nu_k,r})$ for j large;
- $\partial E_j \cap \mathcal{C}_{\nu_k,r} = \mathrm{gr}(g_{j,k})$ for some function $g_{j,k} \in C^1(B_{\nu_k,r})$, $k = 1,\dots,N$, and for j large;
- $g_{j,k} \to g_k$ in C^1 as $j \to \infty$, where $g_k \in C^1(B_{\nu_k,r})$ is such that $\partial B \cap \mathcal{C}_{\nu_k,r} = \mathrm{gr}(g_k)$, for $k = 1,\dots,N$.

Lemma 3.6 (Regularity)**.** *There exists $j_1 \in \mathbb{N}$ such that, for all $j \geq j_1$ and for any minimizer E_j of Q_j, ∂E_j is of class $C^{1,\eta}$ for any $\eta \in (0,1)$. Moreover, ∂E_j converges to ∂B in the C^1-topology, as $j \to \infty$.*

Proof. First, we set $e_n = (0,\dots,0,1) \in \mathbb{R}^n$ and for a given $x \in \mathbb{R}^n$ we write $x = (x', x_n) = (x_{e_n}^{\perp}, x_{e_n})$ following the notation introduced in Section 2. For a given $r > 0$ we set

$$A_r = \{x' \in \mathbb{R}^{n-1} : \; |x'| < r\}.$$

We recall that, owing to Lemma 3.5 and for $j \geq j_0$, $E_j \in \mathcal{QM}(\frac{1}{2}, \frac{1}{2}, \Lambda)$. Then, recalling the above definition of C^1 convergence of smooth boundaries, it is enough to prove that there exists $j_1 \geq j_0$ and a small $r_1 > 0$, such that one can find a sequence of functions $(g_j)_j$, with $g_j \in C^{1,\frac{1}{2}}(A_{r_1})$ for all $j \geq j_1$, and satisfying the following two properties:

$$(\partial E_j - e_n) \cap \mathcal{C}_{e_n, r_1} = \mathrm{gr}(g_j) \quad \forall\, j \geq j_1, \tag{46}$$

where $\mathcal{C}_{e_n,r_1} = A_{r_1} \times (-r_1, r_1)$;

$$\|g_j - g\|_{C^1(A_{r_1})} \to 0 \quad \text{as } j \to \infty, \tag{47}$$

where we have set $g(x') = \sqrt{1 - |x'|^2} - 1$. Then, the proof of the lemma will be completed on taking into account Remark 2.3.

To prove (46) and (47) above, we choose $0 < r < 1$ such that $\mathrm{Exc}(B, e_n, 4r) < \frac{\varepsilon_0}{2^{n-1}}$, where ε_0 is as in Proposition 2.2 (ii) relative to $\mathcal{QM}(\frac{1}{2}, \frac{1}{2}, \Lambda)$. Thanks to Propositions 2.2 (i) and 2.1 (i)–(ii), we can find $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$

(a) $\partial E_j \cap B(e_n, r) \neq \emptyset$,
(b) $\mathrm{Exc}(E_j, e_n, 4r) < \frac{\varepsilon_0}{2^{n-1}}$,
(c) there exists $x_j \in \partial^* E_j \cap B(e_n, r)$ such that $x_j \to e_n$ and $\nu_j := \nu_{E_j}(x_j) \to e_n$.

By the definition of the excess, by the inclusion $B(x_j, 2r) \subset B(e_n, 4r)$, and by (b) above, we have

$$\begin{aligned}
\mathrm{Exc}(E_j, x_j, 2r) &= \frac{(2r)^{1-n}}{2} \inf_{|\xi|=1} \int_{\partial^* E_j \cap B(x_j, 2r)} |\nu_{E_j}(z) - \xi|^2 \, d\mathcal{H}^{n-1}(z) \\
&\leq \frac{(2r)^{1-n}}{2} \int_{\partial^* E_j \cap B(x_j, 2r)} |\nu_{E_j}(z) - \xi_j|^2 \, d\mathcal{H}^{n-1}(z) \\
&\leq \frac{(2r)^{1-n}}{2} \int_{\partial^* E_j \cap B(e_n, 4r)} |\nu_{E_j}(z) - \xi_j|^2 \, d\mathcal{H}^{n-1}(z) \\
&= 2^{n-1} \, \mathrm{Exc}(E_j, e_n, 4r) \\
&< \varepsilon_0
\end{aligned}$$

for $\xi_j = \frac{D\chi_{E_j}(B(e_n,4r))}{|D\chi_{E_j}|(B(e_n,4r))}$ and for all $j \geq j_0$. Thanks to Lemma 3.5 and Proposition 2.2 (ii), there exists a sequence of functions $f_j \in C^{1,\frac{1}{2}}(B_{\nu_j, r})$, such that $f_j(0) = |\nabla f_j(0)| = 0$ and $(\partial E_j - x_j) \cap \mathcal{C}_{\nu_j, r} = \mathrm{gr}(f_j)$. At this point, one can check that, setting $r_1 = r/2$ and taking a larger j_1 if needed, the following facts hold:

(d) $\mathcal{C}_{e_n, r_1} \subset \mathcal{C}_{\nu_j, r}$ for $j \geq j_1$,
(e) we can find $g_j \in C^{1,\frac{1}{2}}(A_{r_1})$ for $j \geq j_1$ such that $\mathrm{gr}(g_j) = (x_j - e_n + \mathrm{gr}(f_j)) \cap \mathcal{C}_{e_n, r_1}$,
(f) $\|g_j - g\|_{L^\infty(A_{r_1})} \to 0$, where $g(x') = \sqrt{1 - |x'|^2} - 1$.

Indeed, (d) is a direct consequence of Proposition 2.1 (ii). Then, (e) follows on recalling that $x_j \to e_n$ by (c) and that ∇f_j is $\frac{1}{2}$-Hölder continuous (uniformly in j), thanks to (9). Finally, (f) can be proved on using (c) and Proposition 2.1 (i).

Owing to (e) above and to the properties of f_j, we obtain (46). Then, thanks to (d), (e) and (f) combined with (9), we get

$$|\nabla g_j(v) - \nabla g_j(w)| \leq C |v - w|^{\frac{1}{2}} \tag{48}$$

for all $v, w \in A_{r_1}$ and for a constant $C > 0$ independent of j. By a contradiction argument using (iii), (48), and Ascoli-Arzelà's Theorem, we finally conclude that

$$\|g_j - g\|_{C^1(A_{r_1})} \to 0$$

as $j \to \infty$, thus proving (47). This completes the proof of the lemma. $\square$

In the following lemma, we show that the (scalar) mean curvature H_j of ∂E_j is in $L^\infty(\partial E_j)$. Then, we compute a first variation inequality of Q_j at E_j that translates into a quantitative estimate of the oscillation of the mean curvature.

Lemma 3.7. *Let $j \geq j_1$, with j_1 as in Lemma 3.6, and let E_j be a minimizer of Q_j. Then*

(i) *∂E_j has scalar mean curvature $H_j \in L^\infty(\partial E_j)$ (with orientation induced by the inner normal to E_j). Moreover, for $\mathcal{H}^{n-1}$-a.e. $x, y \in \partial E_j$, one has*

$$|H_j(x) - H_j(y)| \leq \frac{2n}{n-1}\left(Q(E_j)\alpha(E_j) + \frac{\alpha(E_j)^2}{\alpha(W_j)^2}|\alpha(E_j) - \alpha(W_j)|\right); \tag{49}$$

(ii) $\lim_j \|H_j - 1\|_{L^\infty(\partial E_j)} = 0$.

Proof. To prove the theorem we consider a “parametric inflation-deflation”, that will lead to the first variation inequality (49) and, in turn, to (ii).

Let us fix $x_1,\ x_2 \in \partial E_j$ such that $x_1 \neq x_2$. By Lemma 3.6, for $j \geq j_1$ there exist $r > 0$, two unit vectors $\nu_1, \nu_2 \in \mathbb{R}^n$, and two functions $f_1 \in C^1(B_{\nu_1,r})$ and $f_2 \in C^1(B_{\nu_2,r})$, such that $(x_1 + \mathcal{C}_{\nu_1,r}) \cap (x_2 + \mathcal{C}_{\nu_2,r}) = \emptyset$ and

$$(\partial E_j - x_m) \cap \mathcal{C}_{\nu_m,r} = \mathrm{gr}(f_m), \quad m = 1, 2.$$

For $m = 1, 2$ we take $\varphi_m \in C^1_c(B_{\nu_m,r})$ such that $\varphi_m \geq 0$ and

$$\int_{B_{\nu_m,r}} \varphi_m = 1. \tag{50}$$

Let $\varepsilon > 0$ be such that, setting $f_{m,t}(w) = f_m(w) + t\varphi_m(w)$ for $w \in B_{\nu_m,r}$, one has $\mathrm{gr}(f_{m,t}) \subset \mathcal{C}_{\nu_m,r}$ for all $t \in (-\varepsilon, \varepsilon)$. We use the functions $f_{m,t}$, $m = 1, 2$, to modify the set E_j, i.e. we define

$$E_{j,t} = \Big(E_j \setminus \bigcup_{m=1,2}(x_m + \mathcal{C}_{\nu_m,r})\Big) \cup \big(x_1 + \mathrm{sgr}(f_{1,t})\big) \cup \big(x_2 + \mathrm{sgr}(f_{2,-t})\big),$$

where

$$\mathrm{sgr}(f_{m,s}) = \{(w, l) \in \mathcal{C}_{\nu_m,r} :\ l < f_{m,s}(w)\}.$$

By (50) one immediately deduces that $|E_{j,t}| = |E_j|$. Moreover, by a standard computation one obtains

$$\frac{1}{n-1}\frac{d}{dt}P(E_{j,t})_{|t=0} = \int_{B_{\nu_1,r}} h_1\varphi_1 - \int_{B_{\nu_2,r}} h_2\varphi_2, \tag{51}$$

where for $m = 1, 2$

$$h_m(v) := H_j(v, f_m(v)) = -\frac{1}{n-1}\,\mathrm{div}\left(\frac{\nabla f_m(v)}{\sqrt{1 + |\nabla f_m(v)|^2}}\right).$$

Then, by Theorem 4.7.4 in [3], the L^∞-norm of H_j over ∂E_j turns out to be bounded by the constant $4\Lambda/(n-1)$.

By the definition of $E_{j,t}$ one can verify that, for $t > 0$

$$|\alpha(E_{j,t}) - \alpha(E_j)| \leq \frac{t}{\omega_n}. \tag{52}$$

By (51) and (52), and for $t > 0$, we also have that

$$\begin{aligned} Q(E_{j,t}) &= \frac{P(E_{j,t}) - P(B)}{P(B)\alpha(E_{j,t})^2} \le \frac{P(E_{j,t}) - P(B)}{P(B)} \cdot \frac{1}{\alpha(E_j)^2 \left(1 - \frac{t}{\alpha(E_j)\omega_n}\right)^2} \\ &\le Q(E_j) \cdot \frac{1}{1 - \frac{2t}{\alpha(E_j)\omega_n}} + \frac{t}{P(B)\alpha(E_j)^2} \frac{d}{dt} P(E_{j,t})_{|t=0} + o(t) \\ &\le Q(E_j) + \frac{t}{\omega_n \alpha(E_j)} \left(2Q(E_j) + \frac{1}{n\alpha(E_j)} \frac{d}{dt} P(E_{j,t})_{|t=0} \right) + o(t). \end{aligned}$$

On using again (52), we get

$$\begin{aligned} Q_j(E_{j,t}) \le\ & Q_j(E_j) + \frac{t}{\omega_n \alpha(E_j)} \left(2Q(E_j) + \frac{1}{n\alpha(E_j)} \frac{d}{dt} P(E_{j,t})_{|t=0} \right) \\ & + \frac{2t}{\omega_n \alpha(W_j)^2} |\alpha(E_j) - \alpha(W_j)| + o(t). \end{aligned}$$

Exploiting now the minimality hypothesis $Q_j(E_j) \le Q_j(E_{j,t})$ in the previous inequality, dividing by $t > 0$, multiplying by $n\omega_n \alpha(E_j)^2$, and finally taking the limit as t tends to 0, we obtain

$$0 \le 2nQ(E_j)\alpha(E_j) + \frac{d}{dt} P(E_{j,t})_{|t=0} + 2n \frac{\alpha(E_j)^2}{\alpha(W_j)^2} |\alpha(E_j) - \alpha(W_j)|. \tag{53}$$

Let now $w_m \in B_{\nu_m, r}$ be a Lebesgue point for h_{f_m}, $m = 1, 2$. On choosing a sequence $(\varphi_m^k)_k \subset C_c^1(B_{\nu_m,r})$ of non-negative mollifiers, such that

$$\lim_k \int_{B_{\nu_m,r}} h_{f_m} \varphi_m^k = h_{f_m}(w_m)$$

for $m = 1, 2$, we obtain that for $E_{j,t}^k$ defined as before, but with φ_m^k replacing φ_m, it holds

$$\begin{aligned} \frac{1}{n-1} \lim_k \frac{d}{dt} P(E_{j,t}^k)_{|t=0} &= \lim_k \int_{B_{\nu_1,r}} h_{f_1} \varphi_1^k - \int_{B_{\nu_1,r}} h_{f_2} \varphi_2^k \\ &= h_{f_1}(w_1) - h_{f_2}(w_2). \end{aligned} \tag{54}$$

Moreover, from (53) with $E_{j,t}^k$ in place of $E_{j,t}$ and thanks to (54), we get

$$h_{f_2}(w_2) - h_{f_1}(w_1) = -\frac{1}{n-1} \lim_k \frac{d}{dt} P(E_{j,t}^k)_{|t=0} \le \frac{2n}{n-1} \left(Q(E_j)\alpha(E_j) + \frac{\alpha(E_j)^2}{\alpha(W_j)^2} |\alpha(E_j) - \alpha(W_j)| \right). \tag{55}$$

The proof of (49), and therefore of claim (i), is achieved by exchanging the roles of x_1 and x_2.

Finally, to prove (ii) we recall that $\sup_j \|H_j\|_{L^\infty(\partial E_j)} \le 4\Lambda/(n-1)$. Moreover, by (49) we have that

$$\lim_j \operatorname*{ess\,sup}_{x,y \in \partial E_j} |H_j(x) - H_j(y)| = 0. \tag{56}$$

Thanks to (56) we conclude that, up to subsequences, there exists a constant H such that $\|H_j - H\|_{L^\infty(\partial E_j)} \to 0$ as $j \to \infty$. By Lemma 3.6, ∂E_j converges to ∂B in C^1 and thus we can consider $U = B_{e_n, \frac{1}{2}} \subset \mathbb{R}^{n-1}$ such that, for j large enough, the portion of the boundary of E_j inside the open set $U \times (0, +\infty)e_n \subset \mathbb{R}^n$ is the graph of a function $f_j \in C^1(U)$ converging to the function $f(w) = \sqrt{1 - |w|^2}$ in the C^1-norm, as $j \to \infty$. As a consequence, adopting the Cartesian notation for the mean curvature as in (i),

$$\begin{aligned}\lim_j \int_U h_{f_j}\varphi &= \lim_j \int_U \frac{\langle \nabla f_j, \nabla\varphi\rangle}{\sqrt{1+|\nabla f_j|^2}}\\ &= \int_U \frac{\langle \nabla f, \nabla\varphi\rangle}{\sqrt{1+|\nabla f|^2}}\\ &= \int_U h_f\varphi,\end{aligned}$$

for any $\varphi \in C^1_c(U)$. This proves that H coincides with the mean curvature of the ball B, i.e. $H = h_f = 1$. It is then easy to conclude that the whole sequence H_j must converge to $H = 1$, and this completes the proof of (ii). □

Proof of Theorem 3.2. Statement (i) of the thesis follows by Lemma 3.3. Statement (ii) is immediate, as it stems from the peculiar choice of penalization we have adopted. The proof of statement (iii) is an elementary consequence of Lemma 3.6, while (iv) follows by Lemma 3.7. □

4. Two applications of the Selection Principle

In this section we describe two applications of Theorem 3.2. The first one is a new proof of the sharp quantitative isoperimetric inequality in $\mathbb{R}^n$. The second one is a positive answer to a conjecture by Hall [23], concerning the optimal asymptotic constant for (1) in $\mathbb{R}^2$, when the asymmetry vanishes.

4.1. **The Sharp Quantitative Isoperimetric Inequality.** We start by recalling the definition of *nearly spherical set* introduced by Fuglede in [17] (see also [16]). A Borel set E in $\mathbb{R}^n$ is nearly spherical if $|E| = |B|$, the barycenter of E is 0, and ∂E is the normal graph of a Lipschitz function $u : \partial B \to (-1, +\infty)$ (i.e., $\partial E = \{(1 + u(x))x : \ x \in \partial B\}$) with $\|u\|_{L^\infty(\partial B)} \le \frac{1}{20n}$ and $\|\nabla u\|_{L^\infty(\partial B)} \le \frac{1}{2}$. In [17] (see also [16] for a proof in dimension 2 and 3) Fuglede proved the following crucial estimate, whence the sharp quantitative isoperimetric inequality easily follows:

Theorem 4.1 (Fuglede's estimate)**.** *Let $E \subset \mathbb{R}^n$ be a nearly spherical set with $\partial E = \{(1 + u(x))x : \ x \in \partial B\}$ and $u \in W^{1,\infty}(\partial B)$ as above, then there exists $C = C(n) > 0$ such that*

$$\delta P(E) \ge C\|u\|^2_{W^{1,2}(\partial B)}.$$

By appealing to the Selection Principle and to the estimate above, we could directly provide the complete proof of the sharp quantitative isoperimetric inequality (see the proof of Theorem 4.3). Instead, in Lemma 4.2 we follow the argument exploited by Fuglede in the proof of Theorem 4.1, thus proving an asymptotic estimate of the isoperimetric deficit in terms of the asymmetry, valid for nearly spherical sets that get closer and closer to the ball B.

Let us first recall the following facts. Let $E \subset \mathbb{R}^n$ be such that $\partial E = \{(1 + u(x)x), \ x \in \partial B)\}$ for some $u : \partial B \to (-1, +\infty)$ of class C^1, then the perimeter $P(E)$, the Lebesgue measure $|E|$, the symmetric difference $|E \triangle B|$ and the barycenter $\mathrm{bar}(E)$ of E can be computed by exploiting the following formulas:

$$\frac{P(E)}{P(B)} = \int_{\partial B} (1+u)^{n-1}\sqrt{1+(1+u)^{-2}|\nabla u|^2}\, d\sigma, \tag{57}$$

$$\frac{|E|}{|B|} = \int_{\partial B} (1+u)^n\, d\sigma, \tag{58}$$

$$|E \triangle B| = n|B| \int_{\partial B} (|u| + O(u^2))\, d\sigma, \tag{59}$$

$$\mathrm{bar}(E) = \int_{\partial B} (1+u(x))^{n+1}\, x\ d\sigma(x), \tag{60}$$

where we have set $\sigma = \frac{1}{P(B)}\mathcal{H}^{n-1}$.

In the following lemma we prove a weak (for C^1 nearly-spherical sets) version of Fuglede's asymptotic estimate established in [17]. Such an estimate shows a dependence of the asymptotic best constant $Q(B)$ of the quantitative isoperimetric inequality on the space dimension n of order n^{-1}. It is interesting to observe that such a dependence cannot be obtained by the others estimates proved in [19] and [15] where, as a result of the techniques used in the proof of the quantitative isoperimetric inequality, its best constant, namely $\inf Q$, scales with n as n^{-6} and 4^{-n}, respectively.

Lemma 4.2 (Asymptotic estimate)**.** *Let $E \subset \mathbb{R}^n$ and $u : \partial B \to (-1, +\infty)$ of class C^1 be such that $\partial E = \{(1 + u(x))x,\ x \in \partial B)\}$, $|E| = |B|$ and $\mathrm{bar}(E) = 0$. Then for all $\eta > 0$ there exists $\varepsilon > 0$ such that, if $\|u\|_{L^\infty(\partial B)} + \|\nabla u\|_{L^\infty(\partial B)} < \varepsilon$, one has*

$$\delta P(E) \geq \frac{(n + 1 - \eta)}{2n^2} \alpha(E)^2. \tag{61}$$

Proof. By applying Taylor's formula in (57), and thanks to the bound on the sum of the L^∞-norms of u and ∇u, we have that

$$\begin{aligned} \frac{P(E)}{P(B)} = \int_{\partial B} \left(1 + \frac{|\nabla u|^2}{2} + (n-1)u + \frac{(n-1)(n-2)}{2} u^2\right) \, d\sigma \\ + O(\varepsilon)(\|u\|^2_{L^2(\partial B)} + \|\nabla u\|^2_{L^2(\partial B)}). \end{aligned} \tag{62}$$

By the hypothesis $|E| = |B|$, which is equivalent to $\int_{\partial B} ((1 + u)^n - 1) \, d\sigma = 0$, it turns out, again by Taylor's formula, that

$$\int_{\partial B} u \, d\sigma = -\left(\frac{n-1}{2} + O(\varepsilon)\right) \|u\|^2_{L^2(\partial B)}. \tag{63}$$

Combining (62) and (63) we get

$$\begin{aligned} \delta P(E) &= \int_{\partial B} \left(\frac{|\nabla u|^2}{2} + (n-1)u + \frac{(n-1)(n-2)}{2} u^2\right) \, d\sigma + O(\varepsilon)(\|u\|^2_{L^2(\partial B)} + \|\nabla u\|^2_{L^2(\partial B)}) \\ &= \frac{1}{2} \int_{\partial B} (|\nabla u|^2 - (n-1)u^2) \, d\sigma + O(\varepsilon)(\|u\|^2_{L^2(\partial B)} + \|\nabla u\|^2_{L^2(\partial B)}). \end{aligned} \tag{64}$$

Thanks to the previous estimate, in order to prove the thesis it is only left to prove that, for all $\eta > 0$

$$\|\nabla u\|^2_{L^2(\partial B)} - (n-1)\|u\|^2_{L^2(\partial B)} \geq \frac{(n + 1 - \eta)}{n^2} \alpha(E)^2 \tag{65}$$

if $\varepsilon > 0$ is chosen small enough depending on η. To this end, it will be sufficient to consider the Fourier series of u over the orthonormal basis of spherical harmonics $\{Y_k :\ k = 0, 1, \dots\}$, namely

$$u = \sum_{k=0}^{\infty} a_k Y_k,$$

and estimate the first two coefficients a_0 and a_1. We start by recalling that

$$Y_0 = 1, \qquad Y_1(x) = x \cdot \nu \tag{66}$$

for a suitably chosen $\nu \in \mathbb{R}^n$. Thus the first two coefficients a_0, a_1 of the Fourier expansion of u are given by

$$a_0 = \int_{\partial B} u Y_0 \, d\sigma = \int_{\partial B} u \, d\sigma \quad \text{and} \quad a_1 = \int_{\partial B} u Y_1 \, d\sigma = \int_{\partial B} u(x) x \cdot \nu \, d\sigma.$$

We first estimate a_0. Taking into account that $\|u\|_{L^\infty(\partial B)} < \varepsilon$, we have that

$$a_0^2 = O(\varepsilon^2)\|u\|^2_{L^2(\partial B)}. \tag{67}$$

We now estimate a_1. Observing that, by (66) and by the hypothesis $\mathrm{bar}(E) = 0$

$$\int_{\partial B} Y_1 \, d\sigma = 0,$$

and that

$$\int_{\partial B}(1+u)^{n+1}Y_1\,d\sigma = \mathrm{bar}(E)\cdot\nu = 0$$

we first obtain that

$$\int_{\partial B}((1+u)^{n+1}-1)Y_1\,d\sigma = \int_{\partial B}\left((n+1)u+\sum_{k=2}^{n+1}\binom{n+1}{k}u^k\right)Y_1\,d\sigma = 0. \tag{68}$$

Then, from (68) we derive

$$a_1 = \int_{\partial B} uY_1\,d\sigma = -\sum_{k=2}^{n+1}\binom{n}{k}\int_{\partial B} u^k\,d\sigma = O(\|u\|^2_{L^2(\partial B)})$$

and

$$a_1^2 = O(\varepsilon^2)\|u\|^2_{L^2(\partial B)}. \tag{69}$$

Since

$$\|u\|^2_{L^2(\partial B)} = \sum_{k=0}^{\infty} a_k^2 \quad\text{and}\quad \|\nabla u\|^2_{L^2(\partial B)} = \sum_{k=1}^{\infty}\lambda_k a_k^2, \tag{70}$$

where

$$\lambda_k = k(k+n-2) \tag{71}$$

denotes the k-th eigenvalue of the Laplace-Beltrami operator on ∂B (relative to the k-th eigenfunction Y_k), on gathering together (67) and (69) we obtain

$$\|u\|^2_{L^2(\partial B)} \le (1+O(\varepsilon^2))\sum_{k=2}^{\infty} a_k^2, \qquad \|\nabla u\|^2_{L^2(\partial B)} \le (1+O(\varepsilon^2))\sum_{k=2}^{\infty}\lambda_k a_k^2. \tag{72}$$

On observing that by (71) one has $\lambda_k \ge 2n$ for all $k\ge 2$, and on using (72) the left-hand side of (65) can be estimated as follows:

$$\begin{aligned}\int_{\partial B}(|\nabla u|^2-(n-1)u^2)\,d\sigma &= \sum_{k=2}^{\infty}(\lambda_k-n+1)a_k^2 + O(\varepsilon^2)\|u\|^2_{L^2(\partial B)}\\ &\ge \sum_{k=2}^{\infty}(n+1)a_k^2 + O(\varepsilon^2)\|u\|^2_{L^2(\partial B)}\\ &\ge (n+1+O(\varepsilon^2))\|u\|^2_{L^2(\partial B)}.\end{aligned} \tag{73}$$

On the other hand, by (59) and by Hölder inequality one has

$$|E\,\triangle\,B|^2 = n^2|B|^2\left(\int_{\partial B}(|u|+O(u^2))\,d\sigma\right)^2 \le (n^2|B|^2+O(\varepsilon))\|u\|^2_{L^2(\partial B)},$$

which gives in particular

$$\alpha(E)^2 \le \frac{|E\,\triangle\,B|^2}{|B|^2} \le (n^2+O(\varepsilon))\|u\|^2_{L^2(\partial B)}. \tag{74}$$

By combining (73) with (74) and by choosing ε small enough, we get the desired estimate (65), and hence the thesis of the lemma. □

We are now ready to prove the main result of the section:

Theorem 4.3 (The Sharp Quantitative Isoperimetric Inequality in $\mathbb{R}^n$)**.** *There exists a positive constant C such that, for any $E\in\mathcal{S}^n$ it holds*

$$\delta P(E) \ge C\alpha(E)^2. \tag{75}$$

Proof. We claim that

$$Q(B) \geq \frac{n+1}{2n^2}. \tag{76}$$

Suppose the claim proved. Then, by definition of $Q(B)$, there exists $\alpha_0 > 0$ such that, for all $E \in \mathcal{S}^n$ with $\alpha(E) < \alpha_0$ it holds that

$$Q(E) \geq \frac{Q(B)}{2}. \tag{77}$$

If otherwise E is such that $\alpha_0 \leq \alpha(E) < 2$, then by Lemma 3.1 there exists $\delta_0 > 0$ such that $\delta P(E) \geq \delta_0$, which implies

$$Q(E) = \frac{\delta P(E)}{\alpha(E)^2} \geq \frac{\delta_0}{4}. \tag{78}$$

On combining (77) and (78), we obtain (75) by choosing $C = \min\{\frac{Q(B)}{2}, \frac{\delta_0}{4}\}$.

We are thus left with the proof of (76). To compute $Q(B)$, we will exploit the sequence $(E_j)_j \subset \mathcal{S}^n$ provided by the Selection Principle (Theorem 3.2). Since $\mathrm{bar}(E_j) \to 0$ as $j \to \infty$, without loss of generality (that is, up to replacing E_j by the sequence $E_j - \mathrm{bar}(E_j)$) we may suppose that E_j fulfills the hypotheses of Lemma 4.2, which gives

$$Q(E_j) \geq \frac{n+1-\eta_j}{2n^2} \tag{79}$$

for some η_j infinitesimal as $j \to \infty$. By taking the limit in (79) we eventually get (76) and thus conclude the proof of the theorem. □

Remark 4.4. It is worth noticing that, by the definition of $Q(B)$, for any $E \in \mathcal{S}^n$ the following estimate holds true:

$$\delta P(E) \geq Q(B)\alpha(E)^2 + o(\alpha(E)^2).$$

In other words, $Q(B)$ is the best (asymptotic) constant in the sharp isoperimetric inequality in $\mathbb{R}^n$, as the asymmetry goes to zero. We have seen that the lower bound (76) holds in any dimension $n \geq 2$. In the next subsection it will be shown that in dimension $n = 2$ one has precisely $Q(B) = \frac{\pi}{8(4-\pi)}$.

4.2. **Optimal asymptotic constant in dimension** 2. In Theorem 4.6 we prove a conjecture posed by Hall in [23] and asserting that, for any measurable set in $\mathbb{R}^2$ with positive and finite Lebesgue measure, the following estimate holds:

$$\delta P(E) \geq C_0\alpha(E)^2 + o(\alpha(E))^2, \tag{80}$$

with $C_0 = \frac{\pi}{8(4-\pi)}$ optimal. Note that C_0 is precisely the value of $Q(B)$ in dimension $n = 2$.
We start by recalling a result conjectured in [25] Section V, and proved in [24] Theorem 1:

Theorem 4.5 (Hall-Hayman-Weitsmann)**.** *Let $E \in \mathcal{S}^2$ be a convex set, then* (80) *holds true.*

As an immediate consequence of the Selection Principle and of the above theorem, we now prove (80).

Theorem 4.6 (Hall's conjecture)**.** *Let $E \in \mathcal{S}^2$. Then* (80) *holds true.*

Proof. By (iv) in Theorem 3.2, there exists a sequence of sets $(E_j)_j \subset \mathcal{S}^2$ such that

$$Q(E_j) \to Q(B) \quad \text{and} \quad \|H_j - 1\|_{L^\infty(\partial E_j)} \to 0, \tag{81}$$

where H_j stands for the curvature of ∂E_j. This in particular implies the existence of $j_0 > 0$, such that E_j is a convex set for all $j \geq j_0$. By Theorem 4.5 we have

$$Q(E_j) \geq C_0 + o(1).$$

Passing to the limit as j tends to ∞, and thanks to (81), we eventually get $Q(B) \geq C_0$ which in turn implies (80) by the definition of $Q(B)$. □

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Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, Monte S. Angelo, I-80126 Napoli, Italy

E-mail address: cicalese@unina.it

Dipartimento di Matematica Pura e Applicata "G. Vitali", Università degli Studi di Modena e Reggio Emilia, Via Campi 213/b, I-41100 Modena, Italy

E-mail address: gianpaolo.leonardi@unimore.it