Abelian *one*–factorizations in infinite graphs

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January 29, 2010

Abstract

For each finitely generated abelian infinite group G, we construct a *one*–factorization of the countable complete graph admitting G as an automorphism group acting sharply transitively on vertices.

Keywords: infinite graphs, *one*–factorizations, sharply transitive permutation groups.

MSC 2000: 05C70, 20B25

1 Introduction

A *one*-factorization of the complete graph is a partition of the edge–set into *one*–factors, that is into *one*–regular spanning subgraphs. For a survey on *one*–factorizations of the finite complete graph, see [9].

One–factorizations of the complete graph K_v, with v even, are easy to construct. Despite the purely combinatorial nature of the problem, the constructions of many *one*–factorizations are often based on considerations of symmetry. In [6] it has been observed that many of the known constructions have a cyclic symmetry. An extension of that observation yields a general point of view of symmetry for the problem. In practice, one asks for the existence of a *one*–factorization admitting an automorphism group acting sharply transitively on vertices. Recall that an *automorphism group* of a *one*–factorization of K_v is a subgroup of the symmetric group $Sym(v)$ leaving the *one*–factorization invariant.

In the cited paper [6], Hartman and Rosa investigate the existence of cyclic *one*–factorizations of K_v, that is *one*–factorizations with a cyclic automorphism group acting sharply transitively on vertices. They prove that

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when $v = 2^t$, with $t \geq 3$, there are no cyclic *one*–factorizations of K_v and provide a cyclic *one*–factorization for all other cases.

In [4] Buratti generalizes the result to all finite abelian group of even order. More specifically, given an abelian group G of even order v, which is not isomorphic to a cyclic 2–group of order greater than 4, the author constructs a *one*–factorization of K_v admitting G as an automorphism group acting sharply transitively on vertices.

There are many other results about sharply transitive *one*–factorizations of K_v, see for instance [7, 8]. Also finite non–abelian groups have been considered, see [1, 2, 3, 10].

Bearing in mind the results in [6] and in [4], in this paper we investigate the existence of *one*–factorizations of an infinite complete graph possessing an abelian automorphism group acting sharply transitively on vertices. Obviously, the notion of automorphism group given for K_v can be easily extended to the infinite case, hence we can speak of sharply transitive *one*–factorizations of the infinite complete graph.

We extend the result in [4] to all infinite abelian groups which are finitely generated. By the "Fundamental theorem of finitely generated abelian groups", every infinite abelian group which is finitely generated is the direct sum of a finite abelian group and $s \geq 1$ copies of the cyclic group $\mathbb{Z}$, see [5]. These groups are countable and have a sharply transitive action on the vertex–set of the graph, hence we can identify the element–set of these groups with the vertex–set of the graph. Whence the infinite complete graph we consider is countable: it is the graph whose vertex–set is countable and the edge–set is given by every possible pair of distinct vertices. We denote it by $K_{\aleph_0}$.

In Section 2 we construct a *one*–factorization $\mathcal{F}_{\mathbb{Z}}$ of $K_{\aleph_0}$ admitting the cyclic group $\mathbb{Z}$ as an automorphism group acting sharply transitively on vertices. The *one*–factorization $\mathcal{F}_{\mathbb{Z}}$ is given by the orbit of a *one*–factor F_0 of $K_{\aleph_0}$ with respect to the cyclic group $\mathbb{Z}$.

From F_0 we get a near *one*–factor N_0 and then a near *one*–factorization of $K_{\aleph_0}$. We recall that a *near one–factor* of the complete graph is a set of non–adjacent edges which cover all vertices but one. A *near one–factorization* $\mathcal{N}$ of the complete graph is a partition of the edge set into near *one*–factors with the property that every vertex of the graph is the missing vertex of exactly one near *one*–factor of $\mathcal{N}$.

We note that if the complete graph is not finite, it is possible to find examples of sets of near *one*–factors which partition the edge–set of the graph, but do not satisfy the property required in the previous definition of near *one*–factorization.

Given an arbitrary group H, we will use the *one*–factor F_0 and the near *one*–factor N_0 to construct a *one*–factorization of $K_{\aleph_0}$ admitting $H \oplus \mathbb{Z}$ as an automorphism group acting sharply transitively on vertices, see Lemma 2 and 3.

The constructions we give in the cited lemmas can be extended to every infinite abelian group which is finitely generated. We prove (see Theorem 1) that every infinite abelian group which is finitely generated can be represented as an automorphism group of a *one*–factorization of $K_{\aleph_0}$ acting sharply transitively on vertices.

For abelian groups which are not finitely generated no complete characterization results are known so far. Some very familiar examples are the groups $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$.

In Section 3 we will show a *one*–factorization of the infinite complete graph admitting $(\mathbb{R}, +)$ as an automorphism group acting sharply transitively on vertices. Note that in this case the graph considered has an uncountable number of vertices.

2 Abelian *one*–factorizations of $K_{\aleph_0}$

In this section we will denote by G an arbitrary group. Since we are interested in abelian groups, we shall use the additive notation and we will maintain the same notation also for non-abelian groups. We shall denote by 0_G the identity element of G. Obviously, when G is the cyclic group $\mathbb{Z}$, the additive notation means the classical addition between integer numbers and the identity element is the integer 0.

We assume that G acts sharply transitively on the vertices of the complete graph. Thus we can identify the vertices of the graph with the elements of G and consider the complete graph $K_G = (G, \binom{G}{2})$.

The action of G on the vertices of K_G is given by the right regular permutation representation of G, that is $g(x) = x + g$ for every $x, g \in G$.

If $G = \mathbb{Z}$ and $\{x, y\}$ is and edge of $K_{\mathbb{Z}}$, it is understood that $x < y$. Furthermore, if $S \subseteq E(K_{\mathbb{Z}})$ we will denote by ΔS the multiset $\{y - x | \{x, y\} \in S\}$.

If $\{x, y\}$ is an edge of $E(K_G)$ and J is a subgroup of G, we shall denote by $\{x, y\}^J = \{\{x + g, y + g\} : g \in J\}$ the edge–orbit of $\{x, y\}$ under the action of J. Similarly, if F is a (near) *one*–factor of K_G, we shall denote by F^J the J–orbit of F, that is the orbit of F under the action of J.

Throughout the paper, we say that there exists a sharply transitive (near) *one*–factorization of K_G meaning that there exists a (near) *one*–factorization of K_G admitting G as an automorphism group acting sharply transitively on vertices.

Proposition 1. *If F_0 is a (near) one–factor of $K_{\mathbb{Z}}$ such that $\Delta F_0 = \mathbb{Z}^+$, then the $\mathbb{Z}$–orbit of F_0 is a sharply transitive (near) one–factorization of $K_{\mathbb{Z}}$.*

Proof. Given $\{x, y\} \in E(K_{\mathbb{Z}})$, let $\{a, b\}$ be the unique edge of F_0 such that $y - x = b - a$. Then the unique (near) *one*–factor of $F_0^{\mathbb{Z}}$ containing $\{x, y\}$ is $F_0 + (x - a)$. $\square$

Lemma 1. *If $\{b_i | i \in \mathbb{N}\}$ is a sequence of positive integers, then there exists a sequence $\{t_i | i \in \mathbb{N}\}$ such that the closed intervals $[t_i, t_i + b_i]$ partition $\mathbb{Z}$.*

Proof. It suffices to take $t_0 = 0$ and

$$t_{2i} = i + \sum_{k=0}^{i-1} b_{2k} \qquad t_{2i-1} = -i - \sum_{k=0}^{i-1} b_{2k+1}$$

for every positive integer i. $\square$

Proposition 2. *There exists a sharply transitive one–factorization of $K_{\mathbb{Z}}$.*

Proof. We construct a *one*–factorization of $K_{\mathbb{Z}}$ admitting the cyclic group $\mathbb{Z}$ as an automorphism group acting sharply transitively on vertices.

Let $\{B_i\}_{i=0}^{\infty}$ be a sequence of subsets of the edge–set of $K_{\mathbb{Z}}$ defined as follows:

$$B_i = \left(\bigcup_{j=0}^{\frac{3^i-3}{2}} \{\{j, \tfrac{3^{i+1}-3}{2} - j\}\} \right) \cup \left(\bigcup_{j=0}^{\frac{3^i-1}{2}} \{\{\tfrac{3^i-1}{2} + j, 2 \cdot 3^i - 1 - j\}\} \right)$$

Observe that ΔB_i is the closed interval $[\frac{3^i+1}{2}, \frac{3^{i+1}-1}{2}]$. It easily follows that

$$\bigcup_{i \in \mathbb{N}} \Delta B_i = \mathbb{Z}^+. \tag{1}$$

Now note that the edges of B_i partition $[0, b_i]$ where $b_i = 2 \cdot 3^i - 1$. We consider the sequence $\{b_i | i \in \mathbb{N}\}$. By Lemma 1, there exists a sequence $\{t_i | i \in \mathbb{N}\}$ such that the closed intervals $[t_i, b_i + t_i]$ partition $\mathbb{Z}$. We set $A_i = B_i + t_i$ and $F_0 = \bigcup_{i \in \mathbb{N}} A_i$. The edges of A_i partition $[t_i, b_i + t_i]$ (see for instance A_2 in Figure 1). Hence, in view of the choice of the sequence $\{t_i | i \in \mathbb{N}\}$, the edges of F_0 partition $\mathbb{Z}$, that is F_0 is a *one*–factor of $K_{\mathbb{Z}}$. Of course we have $\Delta A_i = \Delta B_i$ for every i and hence, by (1), $\Delta F_0 = \mathbb{Z}^+$. The assertion follows from Proposition 1. $\square$

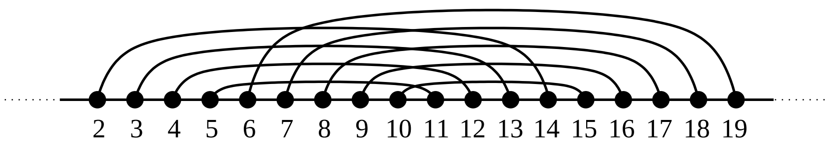

Figure 1: The subset of edges A_2.

Proposition 3. *There exists a sharply transitive near one–factorization of $K_{\mathbb{Z}}$.*

Proof. We construct a near *one*–factorization of $K_{\mathbb{Z}}$ admitting the cyclic group $\mathbb{Z}$ as an automorphism group acting sharply transitively on vertices.

Consider the *one*–factor F_0 of $K_{\mathbb{Z}}$ constructed in Proposition 2. Observe that the edges of F_0, other than $\{0, 1\}$, have both vertices of the same sign. More specifically, the vertices of an edge of A_i, with $i > 0$, are both positive or both negative according to whether i is even or odd, respectively.

We set

$$N_0 = \{\{a, b\} \in F_0 : a, b < 0\} \cup \{\{a + 1, b + 1\} : \{a, b\} \in F_0, a, b \geq 0\}.$$

By the previous remark, in N_0 all vertices of $K_{\mathbb{Z}}$ other than 0 are covered, that is N_0 is a near *one*–factor of $K_{\mathbb{Z}}$. Furthermore, since N_0 is obtained from F_0 by selecting the edges $\{a, b\} \in F_0$ with $a, b < 0$, and by replacing the edges $\{a, b\} \in F_0$, with $a, b \geq 0$, with edges in the same $\mathbb{Z}$–orbit, we have $\Delta N_0 = \Delta F_0 = \mathbb{Z}^+$. Hence $N_0^{\mathbb{Z}}$ is a sharply transitive near *one*–factorization of $K_{\mathbb{Z}}$, since Proposition 1 holds. □

Lemma 2. *Let H be an arbitrary group. Let $G = H \oplus \mathbb{Z}$. If K_H admits a sharply transitive near one–factorization, then there exists a sharply transitive one–factorization of K_G.*

Proof. Assume K_H admits a sharply transitive near *one*–factorization $\mathcal{F}_H$. Since H acts sharply transitively on the vertices of K_H and $\mathcal{F}_H$ is a near *one*–factorization which is invariant with respect to H, there exists a unique representative, say N, for the H–orbits of near *one*–factor of $\mathcal{F}_H$. Without loss of generality we can assume that in N all vertices of K_H other than 0_H are covered.

We construct a *one*–factor M of K_G by the near *one*–factor N of K_H, the *one*–factor F_0 and the near *one*–factor N_0 of $K_{\mathbb{Z}}$. We set

$$\begin{aligned} M = \ & \{\{(0_H, a), (0_H, b)\} : \{a, b\} \in F_0\} \cup \{\{(x, 0), (y, 0)\} : \{x, y\} \in N\} \cup \\ & \cup \{\{(x, c), (y, d)\}, \{(x, d), (y, c)\} : \{c, d\} \in N_0, \{x, y\} \in N\}. \end{aligned}$$

Observe that every edge $\{a, b\} \in F_0$ gives rise to the edge $\{(0_H, a), (0_H, b)\}$ of K_G, every edge $\{x, y\} \in N$ gives rise to the edge $\{(x, 0), (y, 0)\}$, while every edge $\{c, d\} \in N_0$ gives rise to two non–adjacent edges of K_G, namely $\{(x, c), (y, d)\}$, $\{(x, d), (y, c)\}$, for every $\{x, y\} \in N$.

We prove that $\mathcal{F}_G = M^G$ is a sharply transitive *one*–factorization of K_G. We show that $\mathcal{F}_G$ is a partition of the edge–set of K_G.

Let $\{u, w\}$ be an edge of K_G. We set $u = (u_1, u_2)$, $w = (w_1, w_2)$, with $u_1, w_1 \in H$ and $u_2, w_2 \in \mathbb{Z}$.

Assume u_1, w_1 be both different from 0_H. Since $\mathcal{F}_H$ is a partition of the edge–set of K_H and $\mathcal{F}_H = N^H$, there exist $\{x, y\} \in N$ and $h \in H$ such that $\{u_1, w_1\} = \{x, y\} + h$.

Since $\mathcal{F}_{\mathbb{Z}}$ is a partition of the edge–set of $K_{\mathbb{Z}}$ and $\mathcal{F}_{\mathbb{Z}} = F_0^{\mathbb{Z}}$, there exist $\{a, b\} \in F_0$ and $m \in \mathbb{Z}$ such that $\{u_2, w_2\} = \{a, b\} + m$.

Then $\{u, w\} = \{(x+h, a+m), (y+h, b+m)\} = \{(x, a), (y, b)\} + (h, m) \in M + (h, m)$. Observe that if $u_1 = w_1 = 0_H$, then $n \neq 0$ and $\{u, w\} \in M + (0_H, m)$.

The *one*–factor $M + (h, m) \in \mathcal{F}_G$, hence $\mathcal{F}_G$ is a partition of the edge–set of K_G. By construction $\mathcal{F}_G$ is invariant with respect to G. □

Lemma 3. *Let H be an arbitrary group. Let $G = H \oplus \mathbb{Z}$. If K_H admits a sharply transitive one–factorization, then there exists a sharply transitive one–factorization of K_G.*

Proof. Assume that K_H admits a sharply transitive *one*–factorization $\mathcal{F}_H$. Let $F_1, \ldots, F_r$, with $r \geq 1$, denote a complete system of distinct representatives for the H–orbits of *one*–factors of $\mathcal{F}_H$.

For every $i = 1, \ldots, r$ we construct a *one*–factor M_i of K_G by "mixing" the *one*–factor F_i of K_H with the near *one*–factor N_0 of $K_{\mathbb{Z}}$. More specifically, we set

$$M_i = \{\{(x, 0), (y, 0)\}, \{(x, c), (y, d)\}, \{(x, d), (y, c)\} : \{x, y\} \in F_i, \{c, d\} \in N_0\}$$

and

$$M_0 = \{\{(x, a), (x, b)\} : \{a, b\} \in F_0, x \in H\}.$$

The set $\mathcal{F}_G = (\cup_{i=1}^r M_i^G) \cup M_0^G$ is a sharply transitive *one*–factorization of K_G. We omit the proof of the fact that $\mathcal{F}_G$ is a G–invariant partition of the edge–set of K_G because it is similar to the proof given in the previous lemma. □

In the previous lemmas one can replace the cyclic group $\mathbb{Z}$ by an arbitrary infinite group admitting a sharply-transitive (near) *one*-factorization. Since our aim is to prove Theorem 1, we have stated the two lemmas only for the case of our interest.

Lemma 4. *Let H be a cyclic 2–group of order $v = 2^t$, with $t \geq 3$. Let $G = H \oplus \mathbb{Z}$. There exists a sharply transitive one–factorization of K_G.*

Proof. We identify H with the group $\mathbb{Z}_v$ of the residues modulo v. We denote by K the subgroup of H of index 2. For every $i = 0, \ldots, v/4 - 1$ we set

$$F_{2i+1} = \{0, 2i + 1\}^K$$

and

$$F_2 = \cup_{i=1}^{v/4-1} \{\{i, -i\}, \{i + v/2, -i + v/2\}\} \cup \{\{0, v/2\}\}.$$

Observe that each F_{2i+1} is a *one*–factor of K_H, while F_2 is not a *one*–factor of K_H since the vertices $v/4$ and $-v/4$ are not covered.

For every $i = 0, \ldots, v/4-1$, we construct a *one*–factor T_{2i+1} of K_G by the *one*–factor F_{2i+1} of K_H and the near *one*–factor N_0 of $K_{\mathbb{Z}}$ as in Proposition 3. More specifically, for every $i = 0, \ldots, v/4-1$ we set

$$T_{2i+1} = \{\{(x,0),(y,0)\},\{(x,c),(y,d)\},\{(x,d),(y,c)\} : \{x,y\} \in F_{2i+1}, \{c,d\} \in N_0\}$$

We construct a *one*–factor T_2 of K_G by the set F_2, the *one*–factor F_0 and the near *one*–factor N_0 of $K_{\mathbb{Z}}$. In particular, we set

$$\begin{aligned} T_2 = \quad & \{\{(v/4,a),(v/4,b)\},\{(-v/4,a),(-v/4,b)\} : \{a,b\} \in F_0\} \cup \\ & \cup \{\{(x,0),(y,0)\},\{(x,c),(y,d)\},\{(x,d),(y,c)\} : \{x,y\} \in F_2, \{c,d\} \in N_0\} \end{aligned}$$

Now we prove that $\mathcal{F}_G = (\cup_{i=0}^{v/4-1} T_{2i+1}^G) \cup T_2^G$ is a sharply transitive *one*–factorization of K_G.

Firstly, we show that $\mathcal{F}_G$ is a partition of the edge–set of K_G.

Let $\{u,w\}$ be an edge of K_G. We set $u = (u_1,u_2)$, $w = (w_1,w_2)$, with $u_1, w_1 \in H$ and $u_2, w_2 \in \mathbb{Z}$.

There exists $g \in G - \{0_G\}$ such that $\{u,v\} \in \{0_G, g\}^G$. We distinguish the cases $g = (2i+1,n)$ or $g = (2j,n)$.

Assume $\{u,w\} \in \{0_G,(2i+1,n)\}^G$. We have that the edge $\{u_1,w_1\}$ of K_H belongs to the edge–orbit $\{0_H, 2i+1\}^H$, with $i \in \{0,\ldots,v/4-1\}$. The edge $\{u_2,w_2\}$ of $K_{\mathbb{Z}}$ belongs to the edge–orbit $\{0,n\}^{\mathbb{Z}}$.

Since F_{2i+1}^H is a partition of $\{0_H, 2i+1\}^H$, there exist $\{x,y\} \in F_{2i+1}$ and $h \in H$ such that $\{u_1,w_1\} = \{x,y\} + h$.

Since $\mathcal{F}_{\mathbb{Z}} = F_0^{\mathbb{Z}}$ is a partition of the edge–set of $K_{\mathbb{Z}}$, there exist $\{a,b\} \in F_0$ and $m \in \mathbb{Z}$ such that $\{u_2,w_2\} = \{a,b\} + m$.

Then $\{u,v\} = \{(x+h,a+m),(y+h,b+m)\} = \{(x,a),(y,b)\} + (h,m) \in T_{2i+1} + (h,m)$.

Assume $\{u,w\} \in \{0_G,(2j,n)\}^G$, with $j \in \{0,\ldots,v/4\}$. We have that the edge $\{u_1,w_1\}$ of K_H belongs to the edge–orbit $\{0_H,2j\}^H$. The set F_2 shares exactly one edge with $\{0_H,2j\}^H$ if $j = v/4$, otherwise it shares two edges. Then there exist $\{x,y\} \in F_2$ and $h \in H$ such that $\{u_1,w_1\} \in F_2 + h$.

As in the previous case, the edge $\{u_2,w_2\} = \{a,b\} + m$ for some integer $m \in \mathbb{Z}$ and $\{a,b\} \in F_0$. Then $\{u,v\} \in T_2 + (h,m)$.

We have proved that for every edge $\{u,v\}$ of K_G there exists a *one*–factor of $\mathcal{F}_G$ containing it. In other words, $\mathcal{F}_G$ is a partition of the edge–set of K_G. By construction $\mathcal{F}_G$ is invariant with respect to G. □

Theorem 1. *For every infinite abelian group G which is finitely generated there exists a sharply transitive one–factorization of K_G.*

Proof. Let G be an infinite abelian group which is finitely generated. As mentioned in Section 1, by the "Fundamental theorem of finitely generated abelian grupos", [5], we can write G as the direct sum of a finite abelian group H and $s \geq 1$ copies of the cyclic group $\mathbb{Z}$, that is $G = H \oplus \mathbb{Z}^s$.

We prove that there exists a sharply transitive *one*–factorization of K_G. We proceed by induction on the number s of copies of $\mathbb{Z}$.

For $s = 1$ the assertion follows from [11, pp. 76–77] and Lemma 2 if H has odd order. If H has even order, it follows from Lemma 4 or from [4, Theorem 3.3] and Lemma 3 according to whether H is a cyclic 2–group of order greater than 4 or not.

Consider $s > 1$ and set $L = H \oplus \mathbb{Z}^{s-1}$. By the previous remarks we can assume that there exists a sharply transitive *one*–factorization of K_L. Since $G = L \oplus \mathbb{Z}$, by Lemma 3 there exists a sharply transitive *one*–factorization of K_G. □

3 Abelian *one*–factorizations of non–finitely generated abelian groups

Theorem 1 leaves completely open the problem of the existence of a sharply transitive *one*–factorization of the complete graph K_G, when G is a non–finitely generated abelian group.

In this section, we give an example of a sharply transitive *one*–factorization of the complete graph K_G when G is the group $(\mathbb{R}, +)$, that is G is a non–finitely generated abelian group with an uncountable number of elements. We will denote by $K_{\mathbb{R}}$ the complete graph whose vertex–set is $\mathbb{R}$ and edge–set is the set of all possible pairs of real numbers.

Let $S = \{n/2 : n \in \mathbb{Z}\}$. The map $\phi(n) = n/2$ is a bijection from $\mathbb{Z}$ to S. Hence the set $\phi(F_0) = \cup_{\{a,b\} \in F_0} \{\phi(a), \phi(b)\}$ is a *one*–factor of the complete graph on S, where F_0 is stated in Proposition 2.

Since F_0 shares exactly one edge with the edge–orbit $\{0, n\}^{\mathbb{Z}}$ (see the proof of Proposition 3), for every $n \in \mathbb{Z}$, $n \neq 0$, we have that $\phi(F_0)$ shares exactly one edge with the edge–orbit $\{0, n/2\}^{\mathbb{R}}$, for every $n \in \mathbb{Z}$, $n \neq 0$.

Let Y denote the set of all positive real numbers belonging to $\mathbb{R} \setminus S$. We set

$$M = \cup_{y \in Y} \{y, -y\}$$

We have that M shares exactly one edge with the edge–orbit $\{0, 2y\}^{\mathbb{R}}$, for every $y \in Y$.

By the previous remarks, the set $\phi(F_0) \cup M$ is a *one*–factor of $K_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}} = (\phi(F_0) \cup M)^{\mathbb{R}}$ is a *one*–factorization of $K_{\mathbb{R}}$ admitting $(\mathbb{R}, +)$ as an automorphism group acting sharply transitively on vertices.

Finally, note that $\mathcal{F}_{\mathbb{R}}$ contains a *one*–factorization of $K_{\aleph_0}$ admitting $(\mathbb{Q}, +)$ as an automorphism group acting sharply transitively on vertices. In fact, it suffices to consider the subset M' of M consisting of the edges $\{y, -y\}$, with $y \in Y \cap \mathbb{Q}$, and then to take the set $\mathcal{F}_{\mathbb{Q}} = (\phi(F_0) \cup M')^{\mathbb{Q}}$.

Acknoledgment

The authors would like to thank one anonymous referee for his/her useful comments, which led to a substantial improvement on the original paper.

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