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L^P CONTINUITY OF WAVE OPERATORS IN \mathbb{Z}

SCIPIO CUCCAGNA

ABSTRACT. We recover for discrete Schrödinger operators on the lattice \mathbb{Z} , stronger analogues of the results by Weder [W1] and by D'Ancona & Fanelli [DF] on \mathbb{R} .

§1 INTRODUCTION

We consider the discrete Schrödinger operator

$$(1.1) \quad (Hu)(n) = -(\Delta u)(n) + q(n)u(n)$$

with the discrete Laplacian Δ in \mathbb{Z} , $(\Delta u)(n) = u(n+1) + u(n-1) - 2u(n)$ and a potential $q = \{q(n), n \in \mathbb{Z}\}$ with $q(n) \in \mathbb{R}$ for all n . In $\ell^2(\mathbb{Z})$ the spectrum is $\sigma(-\Delta) = [0, 4]$. Let for $\langle n \rangle = \sqrt{1+n^2}$

$$\begin{aligned} \ell^{p,\sigma} &= \ell^{p,\sigma}(\mathbb{Z}) = \{u = \{u_n\} : \|u\|_{\ell^{p,\sigma}}^p = \sum_{n \in \mathbb{Z}} \langle n \rangle^{p\sigma} |u(n)|^p < \infty\} \text{ for } p \in [1, \infty) \\ \ell^{\infty,\sigma} &= \ell^{\infty,\sigma}(\mathbb{Z}) = \{u = \{u(n)\} : \|u\|_{\ell^{\infty,\sigma}} = \sup_{n \in \mathbb{Z}} \langle n \rangle^\sigma |u(n)| < \infty\}. \end{aligned}$$

We set $\ell^p = \ell^{p,0}$. If $q \in \ell^{1,1}$ then H has at most finitely many eigenvalues, see the Appendix. The eigenvalues are simple and are not contained in $[0, 4]$, see for instance Lemma 5.3 [CT]. We denote by $P_c(H)$ the orthogonal projection in ℓ^2 on the space orthogonal to the space generated by the eigenvectors of H . $P_c(H)$ defines a projection in ℓ^p for any $p \in [1, \infty]$, see Lemma 2.6 below. We set $\ell_c^p(H) := P_c(H)\ell^p$. By $q \in \ell^1$, q is a trace class operator. Then, by Pearson's Theorem, see Theorem XI.7[RS], the following two limits exist in ℓ^2 , for $w \in \ell_c^2(H)$ and $u \in \ell^2$:

$$(1.2) \quad Wu = \lim_{t \rightarrow +\infty} e^{itH} e^{it\Delta} u, \quad Zw = \lim_{t \rightarrow +\infty} e^{-it\Delta} e^{-itH} w.$$

The operators W and Z intertwine $-\Delta$ acting in ℓ_2 with H acting in $\ell_c^2(H)$. Our main result is the following:

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Theorem 1.1. *Consider the operators W initially defined in $\ell^2 \cap \ell^p$ and Z initially defined in $\ell^2(H) \cap \ell^p$.*

- (1) *Assume H does not have resonances in 0 and 4. Then for $q \in \ell^{1,1}$ the operators extend into isomorphisms $W : \ell^p \rightarrow \ell_c^p(H)$ and $Z : \ell_c^p(H) \rightarrow \ell^p$ for all $1 < p < \infty$.*
- (2) *Assume H has resonances in 0 and/or 4. Then the above conclusion is true for $q \in \ell^{1,2}$.*
- (3) *Assume that $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$. Then W and Z extend into isomorphisms also for $p = 1, \infty$ exactly when both 0 and 4 are resonances and the transmission coefficient $T(\theta)$, defined for $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, satisfies $T(0) = T(\pi) = 1$.*

Remark 1. W extends into a bounded operator for $p = 1, \infty$ when the sum of the operators (3.1)–(3.4) is bounded and this can happen only for $T(0) = T(\pi) = 1$.

Remark 2. We do not know if Claim 3 holds with $\sigma = 0$.

Remark 3. $\lambda = 0$ or $\lambda = 4$ is a resonance exactly if $Hu = \lambda u$ admits a nonzero solution in ℓ^∞ . We say that H is generic if both 0 and 4 are not resonances.

Remark 4. Since $Z = W^*$, by duality it will be enough to consider W .

Theorem 1.1 provides dispersive estimates for solutions of the Klein Gordon equation $u_{tt} + Hu + m^2u = 0$. In particular in the case of Claim 3, we obtain the optimal $\ell^1 \rightarrow \ell^\infty$ estimate, thanks also to [SK] which deals with the $H = -\Delta$ case. The result for $T(0) = 1$ by [W1] proved crucial to us for a nonlinear problem in [C]. There is a close analogy between the theories in \mathbb{Z} and in \mathbb{R} . Claims 1 and 2 in Theorem 1.1 are analogous to the result in [DF] for \mathbb{R} while claim 3 is related to analysis in [W1]. Our proof mixes the approach in [W1] with estimates [CT], which in turn is inspired by [GS,DT]. Some effort is spent proving formulas for which we do not know references in the discrete case. The main theme here and in [CT], is that cases \mathbb{Z} and \mathbb{R} are very similar. In particular one can see in [CT] a theory of Jost functions in \mathbb{Z} very similar to the one for \mathbb{R} , following the treatment in [DT]. The present paper is inspired by various recent papers on dispersion theory for the group e^{itH} , see [SK,KKK,PS,CT]. In particular the bound $|e^{it\Delta}(n, m)| \leq C\langle t \rangle^{-1/3}$ was proved in [SK]. The bound $|P_c(H)e^{itH}(n, m)| \leq C\langle t \rangle^{-1/3}$ was proved in [PS] for $q \in \ell^{1,\sigma}(\mathbb{Z})$ with $\sigma > 4$ and for H without resonances. This result was extended by [CT] to $q \in \ell^{1,1}$ for H without resonances and to $q \in \ell^{1,2}$ if 0 or 4 is a resonance. [CT] is able produce for \mathbb{Z} essentially the same argument introduced in [GS] for \mathbb{R} , thanks to a theory of Jost functions in \mathbb{Z} which is basically the same of that for \mathbb{R} . Here we recall that [GS] for Schrödinger operators on \mathbb{R} improves an earlier result in [W2]. Theorem 1.1 is the natural transposition to \mathbb{Z} , with some improvements, of the theory of wave operators for \mathbb{R} in [W1,GY,DF]. We simplify the argument in [DF] for claims (1) and (2) of Theorem 1.1 and, for claim (3), we use weaker decay hypotheses on the potential than [W1].

We end with some notation. Given an operator A we set $R_A(z) = (A - z)^{-1}$. $\mathcal{S}(\mathbb{Z})$ is the set of functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ with $f(n)$ rapidly decreasing as $|n| \nearrow \infty$. For $u \in \ell^2$ we set $F_0[u](\theta) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{-in\theta} u(n)$. We set $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. $2\mathbb{Z}$ is the set of even integers; $2\mathbb{Z} + 1$ is the set of odd integers. We set

$$\eta(\mu) = \sum_{\nu=\mu}^{\infty} |q(\nu)| \text{ and } \gamma(\mu) = \sum_{\nu=\mu}^{\infty} (\nu - \mu) |q(\nu)|.$$

Given $f \in L^1(\mathbb{T})$ we set $\hat{f}(\nu) = \int_{-\pi}^{\pi} e^{-i\nu\theta} f(\theta) d\sigma$, with $d\sigma = d\theta/\sqrt{2\pi}$.

§2 FOURIER TRANSFORM ASSOCIATED TO H

We recall that the resolvent $R_{-\Delta}(z)$ for $z \in \mathbb{C} \setminus [0, 4]$ has kernel

$$R_{-\Delta}(m, n, z) = \frac{-i}{2 \sin \theta} e^{-i\theta|n-m|}, \quad m, n \in \mathbb{Z},$$

with θ a solution to $2(1 - \cos \theta) = z$ in $D = \{\theta : -\pi \leq \Re \theta \leq \pi, \Im \theta < 0\}$. In [CT] it is detailed the existence of functions $f_{\pm}(n, \theta)$ with

$$(2.1) \quad H f_{\pm}(\mu, \theta) = z f_{\pm}(\mu, \theta) \text{ with } \lim_{\mu \rightarrow \pm \infty} [f_{\pm}(\mu, \theta) - e^{\mp i\mu\theta}] = 0.$$

We have

$$(2.2) \quad f_{\pm}(\mu, \theta) = e^{\mp i\mu\theta} - \sum_{\nu=\mu}^{\pm \infty} \frac{\sin(\theta(\mu - \nu))}{\sin \theta} q(\nu) f_{\pm}(\nu, \theta).$$

Define m_{\pm} by $f_{\pm}(n, \theta) = e^{\mp in\theta} m_{\pm}(n, \theta)$. Lemma 5.1 [CT] implies that for fixed n

$$(2.3) \quad m_{\pm}(n, \theta) = 1 + \sum_{\nu=1}^{\infty} B_{\pm}(n, \nu) e^{-i\nu\theta}.$$

In Lemma 5.2 [CT] it is proved:

Lemma 2.1. *For $q \in \ell^{1,1}$ and setting $B_{+}(n, 0) = 0$ for all n , we have*

$$\begin{aligned} B_{+}(n, 2\nu) &= \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) B_{+}(j, 2l+1) \\ B_{+}(n, 2\nu-1) &= \sum_{l=n+\nu}^{\infty} q(l) + \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) B_{+}(j, 2l). \end{aligned}$$

We have for $n \geq 0$ the estimate $|B_{+}(n, \nu)| \leq \chi_{[1, \infty)}(\nu) e^{\gamma(0)} \eta(\nu)$. Similarly for $n \leq 0$ we have $|B_{-}(n, \nu)| \leq \chi_{[1, \infty)}(\nu) e^{\tilde{\gamma}(0)} \tilde{\eta}(\nu)$ with $\tilde{\gamma}(\mu)$ and $\tilde{\eta}(\mu)$ defined like $\gamma(\mu)$ and $\eta(\mu)$ but with $q(\nu)$ replaced by $q(-\nu)$.

Lemma 2.1 implies what follows, see the proof of Lemma 5.10 [CT]:

Lemma 2.2. *If $q \in \ell^{1,1+\sigma}$ for $\sigma \geq 0$, then $\|B_{\pm}(n, \cdot)\|_{\ell^{1,\sigma}} \leq C_{\sigma}\|q\|_{\ell^{1,1+\sigma}}$ for $\pm n \geq 0$.*

We recall that for two given functions $u(n)$ and $v(n)$ their Wronskian is $[u, v](n) = u(n+1)v(n) - u(n)v(n+1)$. If u and v are solutions of $Hw = zw$ then $[u, v]$ is constant. In particular we set $W(\theta) := [f_+(\theta), f_-(\theta)]$ and $W_1(\theta) := [f_+(\theta), \overline{f_-(\theta)}]$. By an argument in Lemma 5.10 [CT] we have:

Lemma 2.3. *If for $\sigma \geq 0$ we have $q \in \ell^{1,1+\sigma}$, then $W(\theta), W_1(\theta) \in \ell^{1,\sigma}$.*

Lemma 5.4 [CT] states:

Lemma 2.4. *Let $q \in \ell^{1,1}$. For $\theta \in [-\pi, \pi]$ we have $\overline{f_{\pm}(n, \theta)} = f_{\pm}(n, -\theta)$ and for $\theta \neq 0, \pm\pi$ we have*

$$(1) \quad f_{\mp}(n, \theta) = \frac{1}{T(\theta)} \overline{f_{\pm}(n, \theta)} + \frac{R_{\pm}(\theta)}{T(\theta)} f_{\pm}(n, \theta)$$

where $T(\theta)$ and $R_{\pm}(\theta)$ are defined by (1) and satisfy:

$$(2) \quad [\overline{f_{\pm}(\theta)}, f_{\pm}(\theta)] = \pm 2i \sin \theta,$$

$$(3) \quad T(\theta) = \frac{-2i \sin \theta}{W(\theta)}, \quad R_+(\theta) = -\frac{\overline{W_1(\theta)}}{W(\theta)}, \quad R_-(\theta) = -\frac{W_1(\theta)}{W(\theta)}$$

$$(4) \quad \overline{T(\theta)} = T(-\theta), \quad \overline{R_{\pm}(\theta)} = R_{\pm}(-\theta),$$

$$(5) \quad |T(\theta)|^2 + |R_{\pm}(\theta)|^2 = 1, \quad T(\theta)\overline{R_{\pm}(\theta)} + R_{\mp}(\theta)\overline{T(\theta)} = 0.$$

Lemma 5.5 [CT] states:

Lemma 2.5.

- (1) *For $\theta \in [-\pi, \pi] \setminus \{0, \pm\pi\}$ we have $W(\theta) \neq 0$. We have $|W(\theta)| \geq 2|\sin \theta|$ for all $\theta \in [-\pi, \pi]$ and in the generic case $|W(\theta)| > 0$.*
- (2) *For $j = 0, 1$ and $q \in \ell^{1,1+j}$ then $W(\theta)$ and $W_1(\theta)$ are in $C^j[-\pi, \pi]$.*
- (3) *If $q \in \ell^{1,2}$ and $W(\theta_0) = 0$ for a $\theta_0 \in \{0, \pm\pi\}$, then $\dot{W}(\theta_0) \neq 0$. In particular if $q \in \ell^{1,2}$, then $T(\theta) = -2i \sin \theta / W(\theta)$ can be extended continuously in \mathbb{T} .*

We have the following result:

Lemma 2.6. *Assume that $q \in \ell^{1,1}$ if H is generic and $q \in \ell^{1,2}$ if H has a resonance at 0 or at 4. Then the following statements hold.*

- (1) *H has finitely many eigenvalues.*
- (2) *If λ is an eigenvalue, then $\dim \ker(H - \lambda) = 1$.*
- (3) *If there are eigenvalues they are in $\mathbb{R} \setminus [0, 4]$.*
- (4) *Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues and $\varphi_1, \dots, \varphi_n$ corresponding eigenvectors with $\|\varphi_j\|_{\ell^2} = 1$. Then for fixed $C > 0$ and $a > 0$ we have $|\varphi_j(\nu)| \leq Ce^{-a|\nu|}$ for all $j = 1, \dots, n$ and for all $\nu \in \mathbb{Z}$.*

(5) Let $P_d(H) := \sum_j \varphi_j \langle \cdot, \varphi_j \rangle$. Then $P_d(H)$ and $P_c(H) := 1 - P_d(H)$ are bounded operators in ℓ^p for all $p \in [1, \infty]$.

Proof. (1) is proved in the Appendix. (2) and (3) are in Lemma 5.3 [CT]. (5) follows from (4). (4) follows from the fact that by the proof in Lemma 5.3 [CT] there are constants $A(\pm, j)$ such that $\varphi_j(\nu) = A(\pm, j)f_{\pm}(\nu, \theta_j)$, with $\theta_j \in D$ such that $\lambda_j = 2(1 - \cos(\theta_j))$. The fact that $\lambda_j \notin [0, 4]$ implies $\Im(\theta_j) < 0$ for all j .

By Lemmas 5.6-9 [CT] we have

$$(2.4) \quad \begin{aligned} P_c(H)u &= \frac{1}{2\pi i} \int_0^4 [R_H^+(\lambda) - R_H^-(\lambda)] u d\lambda = \\ &= \frac{1}{2\pi i} \sum_{\nu \in \mathbb{Z}} \int_{-\pi}^{\pi} K(n, \nu, \theta) d\theta u(\nu) \text{ with} \end{aligned}$$

$$(2.5) \quad \begin{aligned} K(n, \nu, \theta) &= f_-(n, \theta) f_+(\nu, \theta) \frac{\sin(\theta)}{W(\theta)} \text{ for } \nu > n \\ K(n, \nu, \theta) &= f_+(n, \theta) f_-(\nu, \theta) \frac{\sin(\theta)}{W(\theta)} \text{ for } \nu \leq n. \end{aligned}$$

Consider now plane waves defined as follows:

Definition 2.7. We consider the following functions:

$$\begin{aligned} \psi(\nu, \theta) &= \frac{1}{\sqrt{2\pi}} T(\theta) e^{-i\nu\theta} m_+(\nu, \theta) \text{ for } \theta \geq 0 \\ \psi(\nu, \theta) &= \frac{1}{\sqrt{2\pi}} T(-\theta) e^{-i\nu\theta} m_-(\nu, -\theta) \text{ for } \theta < 0. \end{aligned}$$

Lemma 2.8. The kernel $P_c(H)(\mu, \nu)$ of $P_c(H)$ can be expressed as

$$(1) \quad P_c(H)(\mu, \nu) = \int_{-\pi}^{\pi} \overline{\psi(\mu, \theta)} \psi(\nu, \theta) d\theta.$$

Proof. We assume $\mu \geq \nu$. By (2.4-5)

$$P_c(H)(\mu, \nu) = \frac{1}{2\pi i} \int_0^{\pi} \left[\frac{f_-(\nu, \theta) f_+(\mu, \theta)}{W(\theta)} - \frac{f_-(\nu, -\theta) f_+(\mu, -\theta)}{W(-\theta)} \right] \sin(\theta) d\theta.$$

We have by Lemma 2.4

$$\begin{aligned}
\overline{f_{\pm}(n, \theta)} &= f_{\pm}(n, -\theta), \quad \overline{T(\theta)} = T(-\theta), \quad \overline{R_{\pm}(\theta)} = R_{\pm}(-\theta), \\
f_{-}(\nu, -\theta) &= T(\theta)f_{+}(\nu, \theta) - R_{-}(\theta)f_{-}(\nu, \theta), \\
f_{+}(\mu, \theta) &= \overline{T(\theta)f_{-}(\mu, \theta) - R_{+}(\theta)f_{+}(\mu, \theta)}.
\end{aligned}$$

Substituting the last two lines in the square bracket in the integral,

$$\begin{aligned}
(2) \quad [\cdots] &= \frac{\overline{T(\theta)f_{-}(\mu, \theta)}f_{-}(\nu, \theta)}{W(\theta)} - \frac{T(\theta)f_{+}(\nu, \theta)f_{+}(\mu, -\theta)}{W(-\theta)} \\
&\quad - \overline{f_{+}(\mu, \theta)}f_{-}(\nu, \theta) \left[\frac{\overline{R_{+}(\theta)}}{W(\theta)} - \frac{R_{-}(\theta)}{W(-\theta)} \right].
\end{aligned}$$

The last line is zero by (5) Lemma 2.4 and by

$$-i \sin(\theta) \left[\frac{\overline{R_{+}(\theta)}}{W(\theta)} - \frac{R_{-}(\theta)}{W(-\theta)} \right] = (T\overline{R_{+}} + \overline{T}R_{-})(\theta) = 0.$$

We have by $T(\theta) = -i \sin(\theta)/W(\theta)$

$$\text{rhs}(2) = \frac{1}{2\pi} |T(\theta)|^2 \overline{f_{+}(\mu, \theta)}f_{+}(\nu, \theta) + \frac{1}{2\pi} |T(\theta)|^2 \overline{f_{-}(\mu, \theta)}f_{-}(\nu, \theta).$$

This yields formula (1) for $\mu \geq \nu$. For $\mu < \nu$ the argument is similar.

Lemma 2.9. *Let $F[u](\theta) := \sum_n \psi(n, \theta)u(n)$. Then:*

- (1) $F : \ell_c^2(H) \rightarrow L^2(\mathbb{T})$ is an isometric isomorphism.
- (2) $F^*[f](n) := \int_{-\pi}^{\pi} \overline{\psi(n, \theta)}f(\theta)d\theta$ is the inverse of F .
- (3) $F[Hu](\theta) = 2(1 - \cos \theta)F[u](\theta)$.

$F[u](\theta)$ is a generalization of Fourier series expansions $F[u_0](\theta)$. Lemma 2.9 is a consequence of Lemma 2.8 except for the fact that we could have $F(\ell_c^2(H)) \subsetneq L^2(\mathbb{T})$. The fact $F(\ell_c^2(H)) = L^2(\mathbb{T})$ follows from $F_0(\ell^2) = L^2(\mathbb{T})$, from the fact that W and Z in (1.2) are isomorphisms between ℓ^2 and $\ell_c^2(H)$ and from Lemma 2.10 below. In the next section the following formula will be important:

Lemma 2.10. *For the operator in (1.2) we have $W = F^*F_0$.*

We have, for $u, v \in \mathcal{S}(\mathbb{Z})$ and $v \in L_c^2(H)$

$$\langle Wu, v \rangle_{\ell^2} - \langle u, v \rangle_{\ell^2} = i \lim_{\epsilon \searrow 0} \int_0^{\infty} \langle e^{itH} q e^{it\Delta} u, v \rangle_{\ell^2} e^{-\epsilon t} dt.$$

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We have for $L^2 = L^2(\mathbb{T})$

$$\langle e^{itH} q e^{it\Delta} u, v \rangle_{\ell^2} = \langle e^{i2t(1-\cos\theta)} F[q e^{it\Delta} u], F[v] \rangle_{L^2} = \langle F[q e^{it(\Delta+2(1-\cos\theta))} u], F[v] \rangle_{L^2}.$$

Then

$$i \int_0^\infty \langle e^{itH} q e^{it\Delta} u, v \rangle_{\ell^2} e^{-\epsilon t} dt = \langle F[q R_{-\Delta}(2 - 2\cos\theta + i\epsilon)u], F[v] \rangle_{L^2}$$

and

$$\begin{aligned} \langle W u, v \rangle_{\ell^2} - \langle u, v \rangle_{\ell^2} &= \\ &= \int_{-\pi}^\pi d\theta \overline{F[v]}(\theta) \sum_{\nu \in \mathbb{Z}} \psi(\nu, \theta) q(\nu) (R_{-\Delta}^+(2 - 2\cos\theta)u)(\nu) = \\ (1) \quad &\int_{-\pi}^\pi d\theta \overline{F[v]}(\theta) \sum_{\nu' \in \mathbb{Z}} u(\nu') \frac{-i}{2\sin|\theta|} \sum_{\nu \in \mathbb{Z}} e^{-i|\theta| |\nu-\nu'|} q(\nu) \psi(\nu, \theta). \end{aligned}$$

We claim we have

$$(2) \quad \psi(\mu, \theta) = e^{-i\mu\theta} / \sqrt{2\pi} + \frac{i}{2\sin\theta} \sum_{\nu \in \mathbb{Z}} e^{-i\theta |\nu-\mu|} q(\nu) \psi(\nu, \theta) \text{ for } \theta > 0$$

$$(3) \quad \psi(\mu, \theta) = e^{-i\mu\theta} / \sqrt{2\pi} - \frac{i}{2\sin\theta} \sum_{\nu \in \mathbb{Z}} e^{i\theta |\nu-\mu|} q(\nu) \psi(\nu, \theta) \text{ for } \theta < 0.$$

Assuming (2)–(3)

$$\begin{aligned} \langle W u, v \rangle_{\ell^2} - \langle u, v \rangle_{\ell^2} &= \int_{-\pi}^\pi \sum_{\nu' \in \mathbb{Z}} d\theta \overline{F[v]}(\theta) u(\nu') \left[e^{-i\nu'\theta} / \sqrt{2\pi} - \psi(\nu', \theta) \right] \\ &= \int_{-\pi}^\pi d\theta \overline{F[v]}(\theta) [F_0[u](\theta) - F[u](\theta)] = \langle F^* F_0 u, v \rangle_{\ell^2} - \langle u, v \rangle_{\ell^2}. \end{aligned}$$

This yields $W = F^* F_0$. Now we focus on (2) and (3). For $\theta > 0$ it is possible to rewrite (2.2) as follows, for some constant $A(\theta)$,

$$(4) \quad f_+(\mu, \theta) = e^{-i\mu\theta} A(\theta) - R_{-\Delta}^+(2 - 2\cos\theta) q f_+(\cdot, \theta)(\mu).$$

Using (2.2) for f_- we obtain $-2i\sin(\theta)A(\theta) = [f_+(\theta), f_-(\mu, \theta)]$. Hence $A(\theta) = 1/T(\theta)$. So multiplying (4) by $T(\theta)/\sqrt{2\pi}$ we obtain (2). We have for $\theta < 0$

$$(5) \quad f_-(\mu, \theta) = e^{i\mu\theta} B(\theta) - R_{-\Delta}^-(2 - 2\cos\theta) q f_-(\cdot, \theta)(\mu)$$

for some constant $B(\theta)$. One checks that $-2i\sin(\theta)B(\theta) = [f_+(\theta), f_-(\mu, \theta)]$. Hence $B(\theta) = 1/T(\theta)$. So multiplying (5) by $T(\theta)/\sqrt{2\pi}$ we obtain

$$\frac{T(\theta)}{\sqrt{2\pi}} f_-(\mu, \theta) = \frac{e^{i\mu\theta}}{\sqrt{2\pi}} - R_{-\Delta}^-(2 - 2\cos\theta) q \frac{T(\theta)}{\sqrt{2\pi}} f_-(\cdot, \theta)(\mu).$$

Taking complex conjugate we obtain (3).

§3 BOUNDS ON W

It is not restrictive to consider $\chi_{[0,\infty]}(n)Wu(n)$ instead of $Wu(n)$. Indeed the proof for $\chi_{(-\infty,0)}(n)Wu(n)$ is similar. Claims 1 and 2 in Theorem 1.1 are a consequences of Lemma 3.1 below. We follow [W1], exploiting at some crucial points results proved in [CT] and inspired by [GS]. We set $n_{\pm}(\mu, \theta) := m_{\pm}(\mu, \theta) - 1$.

Lemma 3.1. *Let $q \in \ell^{1,1}$ in the generic case and $q \in \ell^{1,2}$ in the non generic case. Then $\|\chi_{[0,\infty]}Wu\|_{\ell^p} \leq C_p \|u\|_{\ell^p} \quad \forall p \in (1, \infty)$.*

Proof. Recall $F_0^*[n_{\pm}(\mu, \cdot)](\nu) = B_{\pm}(\mu, \nu)$. Furthermore in Lemma 5.10 [CT] it is proved that $F_0^*[T] \in \ell^1$. One can prove similarly that also $F_0^*[R_{\pm}] \in \ell^1$. For $d\sigma = d\theta/\sqrt{2\pi}$ and by $\overline{m_{\pm}}(\mu, \theta) = m_{\pm}(\mu, -\theta)$, $\overline{T}(\theta) = T(-\theta)$, we consider

$$\begin{aligned} Wf(\mu) &= \int_{-\pi}^{\pi} \overline{\psi(\mu, \theta)} F_0[f](\theta) d\theta = \int_0^{\pi} T(-\theta) e^{i\mu\theta} m_+(\mu, -\theta) F_0[f](\theta) d\sigma \\ &\quad + \int_{-\pi}^0 T(\theta) e^{i\mu\theta} m_-(\mu, \theta) F_0[f](\theta) d\sigma. \end{aligned}$$

We consider only $\mu \geq 0$. We substitute $n_{\pm}(\mu, \theta) := m_{\pm}(\mu, \theta) - 1$ and $T(\theta)m_-(\mu, \theta) = m_+(\mu, -\theta) + e^{-2i\mu\theta} R_+(\theta)m_+(\mu, \theta)$ obtaining

$$\begin{aligned} \chi_{[0,\infty]}(\mu)Wf(\mu) &= \int_{-\pi}^{\pi} e^{i\mu\theta} T(-\theta) \frac{1 + \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma \\ &\quad + \int_{-\pi}^{\pi} e^{i\mu\theta} \frac{1 - \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma + \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) \frac{1 - \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma \\ &\quad + \int_{-\pi}^{\pi} e^{i\mu\theta} T(-\theta) n_+(\mu, -\theta) \frac{1 + \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma \\ &\quad + \int_{-\pi}^{\pi} e^{i\mu\theta} n_+(\mu, -\theta) \frac{1 - \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma \\ &\quad + \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) n_+(\mu, \theta) \frac{1 - \text{sign}(\theta)}{2} F_0[f](\theta) d\sigma. \end{aligned}$$

We have $\chi_{[0,\infty]}(\mu)Wf(\mu) = \widetilde{W}_1 f(\mu) + \widetilde{W}_2 f(\mu)$ where, for $W_j = 2\sqrt{2\pi}\widetilde{W}_j$ for $j = 1, 2$:

$$\begin{aligned} W_1 f(\mu) &= \int_{-\pi}^{\pi} e^{i\mu\theta} T(-\theta) F_0[f](\theta) d\theta + \sqrt{2\pi} f + \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) F_0[f](\theta) d\theta \\ &\quad + \int_{-\pi}^{\pi} e^{i\mu\theta} (T(-\theta) + 1) n_+(\mu, -\theta) F_0[f](\theta) d\theta + \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) n_+(\mu, \theta) F_0[f](\theta) d\theta; \end{aligned}$$

$$W_2 f(\mu) = \int_{-\pi}^{\pi} e^{i\mu\theta} (T(-\theta) - 1) m_+(\mu, -\theta) \text{sign}(\theta) F_0[f](\theta) d\theta - \\ - \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) m_+(\mu, \theta) \text{sign}(\theta) F_0[f](\theta) d\theta.$$

W_1 is bounded for $p \in [1, \infty]$. Indeed for example,

$$\begin{aligned} & \left\| \chi_{[0, \infty)}(\cdot) F_0^* [R_+(\theta) n_+(\mu, \theta) F_0[f](\theta)] (-\cdot) \right\|_{\ell^p} \leq \\ & \left\| \chi_{[0, \infty)}(\cdot) \left(|F_0^* [R_+]| * \chi_{[1, \infty)} e^{\gamma(0)} \eta * |f| \right) (-\cdot) \right\|_{\ell^p} \\ & \leq e^{\gamma(0)} \gamma(0) \|F_0^* [R_+]\|_{\ell^1} \|f\|_{\ell^p}, \end{aligned}$$

where we have used $|B_+(\mu, \nu)| \leq \chi_{[1, \infty)}(\nu) e^{\gamma(0)} \eta(\nu)$ for $\mu \geq 0$. Other terms of W_1 can be treated similarly. By the same argument W_2 is bounded for $p \in (1, \infty)$. For W_2 we cannot include $p = 1, \infty$ because $\text{sign}(\theta)$ is the symbol of the Calderon-Zygmund operator

$$\mathcal{H}v(\nu) = \int_{-\pi}^{\pi} e^{i\nu\theta} F_0[v](\theta) d\sigma = \frac{2i}{\pi} \sum_{\nu' \in \nu + 2\mathbb{Z} + 1} \frac{v(\nu')}{\nu - \nu'}$$

which is unbounded in ℓ^1 and in ℓ^∞ . So the proof of Lemma 3.1 is completed.

Consider now $W_2 f(\mu) = \chi_{[0, \infty)}(\mu) W_2 f(\mu)$

Lemma 3.2. *Let $q \in \ell^{1, 2+\sigma}$ with $\sigma > 0$. Then W_2 extends into a bounded operator also for $p = 1, \infty$ exactly when both 0 and 4 are resonances and the transmission coefficient $T(\theta)$ defined in \mathbb{T} satisfies $T(0) = T(\pi) = 1$.*

Proof. We consider a partition of unity $1 = \chi + (1 - \chi)$ on \mathbb{T} with χ even, $\chi = 1$ near 0 and $\chi = 0$ near π . Correspondingly we have $W_2 = U_1 + U_2$ with U_1 written below and U_2 given by the same formula with χ replaced by $1 - \chi$. We focus on U_1 . We have $U_1 = U_{11} + U_{12}$ with for $\mu \geq 0$

$$\begin{aligned} U_{11} f(\mu) &= U_{111} f(\mu) + U_{112} f(\mu) \\ U_{111} f(\mu) &= m_+(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} (T(-\theta) - T(0)) \text{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \\ &\quad - m_+(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} (R_+(\theta) - R_+(0)) \text{sign}(\theta) F_0[f](\theta) d\theta \\ U_{112} f(\mu) &= \int_{-\pi}^{\pi} e^{i\mu\theta} (T(-\theta) - 1) (n_+(\mu, -\theta) - n_+(\mu, 0)) \text{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \\ &\quad - \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) (n_+(\mu, \theta) - n_+(\mu, 0)) \text{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \end{aligned}$$

and

(3.1)

$$\begin{aligned}
U_{12}f(\mu) &= \chi_{[0,\infty)}(\mu) (T(0) - 1) m_+(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} \text{sign}(\theta) \chi(\theta) F_0[f](\theta) \\
&\quad - \chi_{[0,\infty)}(\mu) R_+(0) m_+(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} \text{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \\
&= \chi_{[0,\infty)}(\mu) (T(0) - 1) m_+(\mu, 0) (\mathcal{H}f)(-\mu) - \chi_{[0,\infty)}(\mu) R_+(0) m_+(\mu, 0) (\mathcal{H}f)(\mu).
\end{aligned}$$

We have:

Lemma 3.3. $U_{12} \in B(L^p, L^p)$ for all $p \in [1, \infty]$ if and only if

$$(1) \quad T(0) - 1 + R_+(0) = 0.$$

Proof. We have $m_+(\mu, 0) \rightarrow 1$ for $\mu \nearrow \infty$ if $q \in \ell^{1,1}$. We have $(\mathcal{H}f)(-\mu) = (\mathcal{H}f(-\cdot))(\mu)$. Set $\widehat{\chi} = F_0^*(\chi)$. Then $U_{12} \in B(L^p, L^p)$ for $p = 1, \infty$ exactly if

$$(2) \quad \chi_{\mathbb{N}}(\mu) (T(0) - 1 + R_+(0)) \mathcal{H}(\widehat{\chi} * f)(\mu) \in \ell^p \text{ for all } f \text{ even in } \ell^p$$

$$(3) \quad \chi_{\mathbb{N}}(\mu) (T(0) - 1 - R_+(0)) \mathcal{H}(\widehat{\chi} * f)(\mu) \in \ell^p \text{ for all } f \text{ odd in } \ell^p.$$

We show that (2) requires (1). We have $\widehat{\chi} * \chi_{\{0\}} = \widehat{\chi}$ and

$$(\mathcal{H}\widehat{\chi})(\mu) = \frac{2i}{\pi\mu} \sum_{\nu \in \mu+2\mathbb{Z}+1} \widehat{\chi}(\nu) - \frac{2i}{\pi} \sum_{\nu \in \mu+2\mathbb{Z}+1} \left[\frac{1}{\mu} - \frac{1}{\mu - \nu} \right] \widehat{\chi}(\nu).$$

The second term on the right is in $\ell^1([1, \infty))$ but the first is $i \frac{\sqrt{2}}{\sqrt{\pi}\mu}$, which is not in $\ell^1([1, \infty))$. Hence we need equality (1). So (2) requires (1). We now show that (3) occurs always. It is enough to prove $\mathcal{H}f \in \ell^p$ for all f odd. We have

$$\sum_{\nu \in \mu+2\mathbb{Z}+1} \frac{1}{\mu - \nu} f(\nu) = 2 \sum_{\nu \in \mu+2\mathbb{Z}+1}^{\nu > 0} \frac{\nu}{\mu^2 - \nu^2} f(\nu).$$

So

$$\|\mathcal{H}f\|_{\ell^1} \lesssim \sum_{\nu > 0} |f(\nu)| \sum_{\mu \in \nu+2\mathbb{Z}+1} \frac{\nu}{|\mu^2 - \nu^2|} \leq C \|f\|_{\ell^1}$$

for a fixed $C < \infty$.

Our next step is to show in Lemma 3.4 that $U_{111} \in B(L^p, L^p)$ for all $p \in [1, \infty]$. In Lemma 3.5 that $U_{112} \in B(L^p, L^p)$ for all $p \in [1, \infty]$. Hence $U_1 \in B(L^p, L^p)$ for all $p \in [1, \infty]$ exactly if $U_{12} \in B(L^p, L^p)$ for all $p \in [1, \infty]$.

Lemma 3.4. *Let $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$. Then $U_{111} \in B(L^p, L^p)$ for all $p \in [1, \infty]$.*

Proof. If for $g = (R_+(\theta) - R_+(0)) \text{sign}(\theta) \chi(\theta)$ and $f = (T(\theta) - T(0)) \text{sign}(\theta) \chi(\theta)$ we have $F_0^* f$ and $F_0^* g \in \ell^1$, then by $|m_+(\mu, 0)| \leq C$ for all $\mu \geq 0$, we get Lemma 3.3. Here consider only $F_0^* f$ only, since the proof for $F_0^* g$ is similar. We have for $\tilde{\chi}(\theta)$ another even smooth cutoff function in \mathbb{T} with $\tilde{\chi} = 1$ on the support of χ and $\tilde{\chi} = 0$ near π ,

$$\chi(\theta)T(\theta) = -2i \frac{\chi(\theta) \sin(\theta)}{\tilde{\chi}(\theta)W(\theta)}.$$

By Lemma 2.3 we have $F_0^* W \in \ell^{1,1+\sigma}$. By the argument in Lemma 5.10 [CT] we have $F_0^* \left[\frac{W(\theta)}{\sin(\theta)} \right] \in \ell^{1,\sigma}$. Then $F_0^* [\chi(\theta)T(\theta)] \in \ell^{1,\sigma}$ by Wiener's Lemma: case $\sigma = 0$ is stated in 11.6 [R]; for $\sigma > 0$ one can provide $\ell^{1,\sigma}$ with a structure of commutative Banach algebra (changing the norm to an equivalent one, 10.2 [R]) and then repeat the argument in 11.6 [R].

Consider now $A(\theta) = (T(\theta) - T(0)) \chi(\theta)$. We have $F_0^* [A] \in \ell^{1,\sigma}$ and $A(0) = A(\pi) = 0$. We have

$$\hat{f}(\nu) = \frac{2i}{\pi} \sum_{\mu \in \nu + 2\mathbb{Z} + 1} \frac{1}{\nu - \mu} \hat{A}(\mu).$$

We consider

$$\sum_{\nu \in \mathbb{Z}} |\hat{f}(\nu)| \leq I + II + III$$

with

$$I = \sum_{\nu \in \mathbb{Z}} \left| \sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\hat{A}(\mu)}{\nu - \mu} \right|,$$

$$II = \sum_{\nu \in \mathbb{Z}} \sum_{|\nu|/2 \leq |\mu| \leq 2|\nu|} \frac{|\hat{A}(\mu)|}{\langle \nu - \mu \rangle}, \quad III = \sum_{\nu \in \mathbb{Z}} \sum_{|\mu| \geq 2|\nu|} \frac{|\hat{A}(\mu)|}{\langle \nu - \mu \rangle}.$$

We see immediately that

$$III \lesssim \|\hat{A}\|_{\ell^{1,\sigma}} \sum_{\nu \in \mathbb{Z}} \langle \nu \rangle^{-1-\sigma} < \infty.$$

We have

$$II \lesssim \sum_{\mu \in \mathbb{Z}} \langle \mu \rangle^\sigma |\hat{A}(\mu)| \sum_{|\nu| \leq 2|\mu|} \langle \nu - \mu \rangle^{-1} \langle \mu \rangle^{-\sigma} \lesssim \sum_{\mu \in \mathbb{Z}} \langle \mu \rangle^\sigma |\hat{A}(\mu)| < \infty.$$

We write

$$\sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\hat{A}(\mu)}{\nu - \mu} = \sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\hat{A}(\mu)}{\nu} +$$

$$\sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\mu}{(\nu - \mu)\nu} \hat{A}(\mu).$$

Notice

$$\sum_{\nu \in \mathbb{Z}} \sum_{|\mu| \leq |\nu|/2} \frac{|\mu \hat{A}(\mu)|}{\langle \nu - \mu \rangle \langle \nu \rangle} \lesssim \sum_{\mu \in \mathbb{Z}} |\mu \hat{A}(\mu)| \sum_{|\nu| \geq 2|\mu|} \langle \nu \rangle^{-2} \lesssim \|\hat{A}\|_{\ell^1} < \infty.$$

The fact that $A(0) = 0$ implies $\sum \hat{A}(\mu) = 0$. The fact that $A(\pi) = 0$ implies $\sum (-1)^\mu \hat{A}(\mu) = 0$. Hence

$$\sum_{\mu \in 2\mathbb{Z}} \hat{A}(\mu) = \sum_{\mu \in 2\mathbb{Z}+1} \hat{A}(\mu) = 0.$$

This implies that

$$\sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z}+1} \hat{A}(\mu) = - \sum_{|\mu| > |\nu|/2, \mu \in \nu + 2\mathbb{Z}+1} \hat{A}(\mu).$$

Then

$$\sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left| \sum_{|\mu| \leq |\nu|/2, \mu \in \nu + 2\mathbb{Z}+1} \frac{\hat{A}(\mu)}{\nu} \right| = \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left| \sum_{|\mu| > |\nu|/2, \mu \in \nu + 2\mathbb{Z}+1} \frac{\hat{A}(\mu)}{\nu} \right|.$$

This can be bounded with the same argument of *III*. Hence we have shown $\hat{f} \in \ell^1$.

Lemma 3.5. *Let $q \in \ell^{1,1+\sigma}$ with $\sigma > 0$. Then $U_{112} \in B(L^p, L^p)$ for all $p \in [1, \infty]$.*

Proof. The proof is similar to the previous one. Let $g(\mu, \theta) = A(\mu, \theta) \text{sign}(\theta)$ with $A(\mu, \theta) = (n_+(\mu, \theta) - n_+(\mu, 0)) \chi(\theta)$. Set $\hat{g}(\mu, \cdot) = F^*[g(\mu, \cdot)]$ and $\hat{A}(\mu, \cdot) = F^*[A(\mu, \cdot)]$. It is enough to show that there exists $b(\nu)$ in ℓ^1 such that $|\hat{g}(\mu, \nu)| \leq b(\nu)$ for all $\mu \geq 0$ and all $\nu \in \mathbb{Z}$. Notice that $F^*[n_+(\mu, \cdot) - n_+(\mu, 0)](\nu) = \chi_{(0, \infty)}(\nu) B_+(\mu, \nu)$ for $\nu \neq 0$ and $= -n_+(\mu, 0)$ for $\nu = 0$. By Lemma 2.1 we have $|B_+(\mu, \nu)| \leq e^{\gamma(0)} \chi_{(0, \infty)}(\nu) \eta(\nu)$. Hence $|\hat{A}(\mu, \nu)| \leq h(\nu)$ for all $\mu \geq 0$ and all $\nu \in \mathbb{Z}$, with $h \in \ell^{1, \sigma}$.

We have

$$\hat{g}(\mu, \nu) = \frac{2i}{\pi} \sum_{\nu' - \nu \in 2\mathbb{Z}+1} \frac{1}{\nu - \nu'} \hat{A}(\mu, \nu') = \frac{2i}{\pi} (I + II + III)$$

with

$$\begin{aligned} I &= \sum_{|\nu'| \leq |\nu|/2, \nu' \in \nu + 2\mathbb{Z}+1} \frac{\hat{A}(\mu, \nu')}{\nu - \nu'}, \\ II &= \sum_{|\nu|/2 < |\nu'| \leq 2|\nu|, \nu' \in \nu + 2\mathbb{Z}+1} \frac{\hat{A}(\mu, \nu')}{\nu - \nu'}, \\ III &= \sum_{|\nu'| > 2|\nu|, \nu' \in \nu + 2\mathbb{Z}+1} \frac{\hat{A}(\mu, \nu')}{\nu - \nu'}. \end{aligned}$$

We have

$$|III(\mu, \nu)| \lesssim \|h\|_{\ell^{1,\sigma}} \langle \nu \rangle^{-1-\sigma}.$$

We have

$$|II(\mu, \nu)| \lesssim \alpha(\nu) := \sum_{|\nu|/2 < |\nu'| \leq 2|\nu|} \frac{|h(\nu')|}{\langle \nu - \nu' \rangle}.$$

We write

$$\begin{aligned} \sum_{|\nu'| \leq |\nu|/2, \nu' \in \nu + 2\mathbb{Z} + 1} \frac{\widehat{A}(\mu, \nu')}{\nu - \nu'} &= I_1 + I_2 \\ I_1 &= \frac{1}{\nu} \sum_{|\nu'| \leq |\nu|/2, \nu' \in \nu + 2\mathbb{Z} + 1} \widehat{A}(\mu, \nu'), \quad I_2 = \sum_{|\nu'| \leq |\nu|/2, \nu' \in \nu + 2\mathbb{Z} + 1} \frac{\nu'}{(\nu - \nu')\nu} \widehat{A}(\mu, \nu'). \end{aligned}$$

We have

$$I_1(\mu, \nu) = -\frac{1}{\nu} \sum_{|\nu'| > |\nu|/2, \nu' \in \nu + 2\mathbb{Z} + 1} \widehat{A}(\mu, \nu')$$

and so

$$|I_1(\mu, \nu)| \lesssim \|h\|_{\ell^{1,\sigma}} \langle \nu \rangle^{-1-\sigma}.$$

Finally

$$|I_2(\mu, \nu)| \lesssim \beta(\nu) := \sum_{|\nu'| \leq |\nu|/2} \frac{\langle \nu' \rangle}{\langle \nu - \nu' \rangle \langle \nu \rangle} h(\nu')$$

Then there is a function $b(\nu)$ in ℓ^1 such that $|\widehat{g}(\mu, \nu)| \leq b(\nu)$ of the form $b(\nu) = C(\alpha(\nu) + \beta(\nu) + \langle \nu \rangle^{-1-\sigma})$.

By repeating the previous arguments one has:

Lemma 3.6. *For $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$ the operator W extends into a bounded operator in ℓ^p for $p = 1, \infty$ when operators (3.1)–(3.4) are bounded. Here (3.1) has been defined above while (3.2)–(3.4) are defined as follows, for $\chi + \chi_1$ a smooth partition of unity in \mathbb{T} with $\chi = 1$ near 0 and $\chi = 0$ near π :*

$$\begin{aligned} (3.2) \quad V_2 f(\mu) &= \chi_{[0,\infty)}(\mu) (T(\pi) - 1) m_+(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} \text{sign}(\theta) \chi_1(\theta) F_0[f](\theta) \\ &\quad - \chi_{[0,\infty)}(\mu) R_+(\pi) m_+(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} \text{sign}(\theta) \chi_1(\theta) F_0[f](\theta) d\theta. \end{aligned}$$

$$\begin{aligned} (3.3) \quad V_3 f(\mu) &= \chi_{(-\infty,0)}(\mu) (1 - T(0)) m_-(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} \text{sign}(\theta) \chi(\theta) F_0[f](\theta) \\ &\quad + \chi_{(-\infty,0)}(\mu) R_-(0) m_-(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} \text{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta. \end{aligned}$$

$$\begin{aligned}
(3.4) \quad V_4 f(\mu) &= \chi_{(-\infty, 0)}(\mu) (1 - T(0)) m_-(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} \text{sign}(\theta) \chi_1(\theta) F_0[f](\theta) \\
&+ \chi_{(-\infty, 0)}(\mu) R_-(0) m_-(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} \text{sign}(\theta) \chi_1(\theta) F_0[f](\theta) d\theta.
\end{aligned}$$

We have:

Lemma 3.7. $W \in B(\ell^p, \ell^p)$ for $p = 1, \infty$ exactly when $T(0) = T(\pi) = 1$.

Proof. If $T(0) = T(\pi) = 1$ we have $V_j = 0$ for all j . Then $W \in B(\ell^p, \ell^p)$ for $p = 1, \infty$. Viceversa $W \in B(\ell^1, \ell^1)$ implies $V_j \in B(\ell^1, \ell^1)$ for all j . If $V_3 \in B(\ell^1, \ell^1)$ then, proceeding as in Lemma 3.3,

$$1 - T(0) - R_-(0) = 1 - T(0) + R_+(0) = 0.$$

This together with (1) in Lemma 3.3 implies $T(0) = 1$. The implication $T(\pi) = 1$ is obtained similarly.

§A APPENDIX: FINITE NUMBER OF EIGENVALUES

We will prove:

Lemma A.1. If $q \in \ell^{1,1}$ the total number of eigenvalues of H is $\leq 4 + \|\nu q(\nu)\|_{\ell^1}$.

Let $q_-(\nu) = \min(0, q(\nu))$. We recall that if we have $(-\Delta + q)u = \lambda u$, then if we define v by $v(\nu) = (-1)^\nu u(\nu)$ we have $(-\Delta - q)v = (4 - \lambda)v$. Hence Lemma 6.1 is a consequence of:

Lemma A.2. If $q \in \ell^{1,1}$ the total number of eigenvalues of H inside $(-\infty, 0)$ is $\leq 2 + \|\nu q_-(\nu)\|_{\ell^1}$.

Proof. For $\lambda \leq 0$ we set $u(\nu, \lambda) = f_+(\nu, \theta)$, where $\lambda = 2(1 - \cos(\theta))$. Notice that $u(\nu, \lambda) \in \mathbb{R}$. We denote by $X(\lambda)$ the set of those ν such that either $u(\nu, \lambda) = 0$ or $u(\nu, \lambda)u(\nu + 1, \lambda) < 0$. We denote by $N(\lambda)$ the cardinality of $X(\lambda)$. Notice that by the min-max principle the operator $\tilde{H} = -\Delta - q_-$ has at least as many negative eigenvalues as H . So, to prove our Lemma 6.2 it is not restrictive to assume $q(\nu) = q_-(\nu) = -|q(\nu)|$ for all ν in Lemma A.3 below. We have:

Lemma 6.3. We have $N(0) \leq 2 + \|\nu q_-(\nu)\|_{\ell^1}$.

Proof. We assume $N(0) > 1$. Let $\nu_0, \nu_1 \in X(0)$ be two consecutive elements, with $\nu_0 < \nu_1$. For $u(\nu) = u(\nu, 0)$ we have

$$u(\nu) = u(\nu_0) + (u(\nu_0 + 1) - u(\nu_0))(\nu - \nu_0) - \sum_{j=\nu_0}^{\nu-1} (j - \nu_0) |q(j)| u(j).$$

It is not restrictive below to assume $A := u(\nu_0 + 1) - u(\nu_0) > 0$. Then $u(\nu_1 + 1) < 0$ or $u(\nu_1) = 0$. In the first case, we have

$$0 > u(\nu_0 + 1) - u(\nu_1 + 1) = A(\nu_1 - \nu_0) \left(1 - \sum_{j=\nu_0}^{\nu_1} (j - \nu_0) |q(j)| \right).$$

This implies

$$(1) \quad \sum_{j=\nu_0+1}^{\nu_1} (j - \nu_0) |q(j)| \geq 1. \text{ By a similar argument } \sum_{j=\nu_0}^{\nu_1-1} (\nu_1 - j) |q(j)| \geq 1.$$

(1) holds also if $u(\nu_1) = 0$. So for $\nu_0 < \nu_1 < \dots < \nu_n$ consecutive elements in $X(0)$,

$$\text{we have } \sum_{j=\nu_0+1}^{\nu_n} (j - \nu_0) |q(j)| \geq n \text{ and } \sum_{j=\nu_0}^{\nu_n-1} (\nu_n - j) |q(j)| \geq n.$$

Then $q \in \ell^{1,1}$ implies $N(0) < \infty$. If $X(0)$ is formed by

$$\nu_0 < \dots < \nu_n (< 0 \leq) \mu_0 < \dots < \mu_m$$

then

$$n \leq \sum_{j=\nu_0}^{\nu_n-1} (\nu_n - j) |q(j)| \leq \sum_{j=\nu_0}^{\nu_n-1} |j| |q(j)|$$

and

$$m \leq \sum_{j=\mu_0+1}^{\mu_m} (j - \mu_0) |q(j)| \leq \sum_{j=\mu_0+1}^{\mu_m} |j| |q(j)|.$$

So $n + m \leq \|\nu q(\nu)\|_{\ell^1}$. Then $N(0) \leq 2 + \|\nu q(\nu)\|_{\ell^1}$. This yields Lemma 6.2.

Notice that

$$\langle Hu, u \rangle = \sum_{\nu \in \mathbb{Z}} |u(\nu + 1) - u(\nu)|^2 + \sum_{\nu \in \mathbb{Z}} q(\nu) |u(\nu)|^2.$$

If H has negative eigenvalues, there is a minimal one λ_0 . Then we have $u(\nu, \lambda_0) = |u(\nu, \lambda_0)| > 0$ for all ν by the min-max principle and by the fact that $u(\nu, \lambda_0) = e^{i\nu\theta} m_+(\nu, \theta_0)$ where $m_+(\nu, \theta) \rightarrow 1$ for $|\nu| \nearrow \infty$ by (1) Lemma 5.1 [CT]. Notice that by this argument it is easy to conclude that $N(\lambda) < \infty$ for any $\lambda < 0$.

Next we have the following discrete version of the Sturm oscillation theorem, see Lemma 4.4 [T].

Lemma A.4. *$N(\lambda)$ is increasing for $\lambda \leq 0$.*

Lemmas A.4 and A.3 yield Lemma A.2.

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