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Inversive Planes, Minkowski Planes and Regular Sets of Points

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New examples of regular sets of points for the Miquelian inversive planes of order q, q a prime power, $q \geq 7$, are found and connections between such planes and certain Minkowski planes of order q^2 are presented.

1. INTRODUCTION

A permutation group G on a finite set E has a natural action on the power-set $\mathcal{P}(E)$. To find orbits of maximal length for G acting on $\mathcal{P}(E)$ was a problem studied in the past by many authors [4, 5, 10]. The search is aided if G is looked at as the full automorphism group of an incidence structure. Then a regular set for an incidence structure is a subset of the point-set, such that the unique automorphism of G mapping the set onto itself is the identity. A regular set containing t points is called a regular t-set.

The search for regular sets for an incidence structure has an intrinsic value as it gives information and properties concerning the structure itself. For example, the search for particular regular sets in the known finite Minkowski planes allows one to characterize those planes which are associated to sharply 3-transitive permutation groups [20].

The problem of finding regular sets has been studied by many authors: see [6, 9, 12–16, 20]. In particular, in [16], Key, Siemons and Wagner give examples of regular sets for the group $P\Gamma L(2, q)$ (q a prime power) acting on the points of the projective line in the usual way. More precisely, they prove the existence of regular 5-sets when $q > 27$, together with the existence of regular 6-sets when $q \in \{13, 17, 25, 27\}$. Furthermore, they prove the non-existence of regular sets when either $q \leq 11$ or $q = 16$. In [14] the authors observe that each finite Miquelian inversive plane $M(q)$, q a prime power, $q \geq 5$, contains regular sets. This result follows directly from [16]; in fact $GF(q^2) \cup \{\infty\}$ is the point-set of $M(q)$ and $P\Gamma L(2, q^2)$ is its full automorphism group.

In this paper, bearing in mind the Miquelian inversive plane of order q, $M(q)$, is embeddable in each known Minkowski plane of order q^2 (see [19]), we intend to find new examples of regular sets for $M(q)$ and we intend to give further information on the existence of certain regular sets for the known Minkowski planes.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let n be a positive integer, $n \geq 2$. A finite inversive plane of order n is a $3-(n^2+1, n+1, 1)$-design; that is, an incidence structure of point-set $\bar{\mathcal{P}}$, block-set $\bar{\mathcal{C}}$, such that $|\bar{\mathcal{P}}| = n^2 + 1$, each block is incident with $n + 1$ points, and any three points are together incident with a unique block. Let q be a prime power; if Θ is an ovoid in the projective space $PG(3, q)$, the incidence structure with points the points of Θ and blocks the secant plane sections of Θ is an example of an inversive plane of order q, called an egg-like inversive plane.

The ovoid Θ is an elliptic quadric if and only if the inversive plane is Miquelian, (i.e., it satisfies the configurational proposition known as Miquel's theorem, see [7]). Thus far, all known finite inversive planes are egg-like; moreover, all known finite inversive planes of odd order are Miquelian.

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We will denote by $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ the Miquelian inversive plane of order q.
Let n be a positive integer, $n \geq 3$. A Minkowski plane of order n is an incidence structure $\mathcal{M} = (\mathcal{P}, \mathcal{B}, \mathcal{G}_1, \mathcal{G}_2)$, where $\mathcal{P}$ is a set of $(n+1)^2$ elements called points, $\mathcal{B}$ and $\mathcal{G}_i$, $i = 1, 2$, are non-empty sets of subsets of $\mathcal{P}$ called, respectively, blocks and generators, such that:

(1) The two classes $\mathcal{G}_1$ and $\mathcal{G}_2$ partition the point-set $\mathcal{P}$. Two generators of different classes intersect at exactly one point.
(2) Each block contains $n + 1$ points and intersects each generator of each class at exactly one point.
(3) For any three points, no two of which lie on the same generator, there is exactly one block containing them.

A set of points, no two of which lie on a same generator, will be termed independent points.
Let p^m be a prime power and let $\mathcal{H}$ be a hyperbolic quadric in the projective space $PG(3, p^m)$. The incidence structure with points the points of $\mathcal{H}$, generators the generating lines of $\mathcal{H}$ and blocks the non-trivial plane sections of $\mathcal{H}$ (that do not contain a generating line), is an example of a Minkowski plane of order p^m.
Other examples of Minkowski planes are obtained as follows.
Consider the group $P\Gamma L(2, p^m)$. Denote by $\bar{1}$ the identity of $\mathrm{Aut}(GF(p^m))$ and let $\chi : P\Gamma L(2, p^m) \longrightarrow \{1, -1\}$ and $\Phi : P\Gamma L(2, p^m) \longrightarrow \mathrm{Aut}(GF(p^m))$ be the group homomorphisms mapping each element of $P\Gamma L(2, p^m)$ onto its quadratic character and its associated field automorphism, respectively.
Let $\sigma \in \mathrm{Aut}(GF(p^m))$ and define

$$G(p^m, \sigma) = \{g \in P\Gamma L(2, p^m) \mid \chi(g) = 1, \Phi(g) = \bar{1}\} \cup \{g \in P\Gamma L(2, p^m) \mid \chi(g) = -1, \Phi(g) = \sigma\}.$$

If $p = 2$, then $G(p^m, \sigma) = PSL(2, p^m) = PGL(2, p^m)$.
If $p \neq 2$, then $G(p^m, \sigma) = PSL(2, p^m) \cup PSL(2, p^m)'\sigma$, where we denote by $PSL(2, p^m)'$ the set $PGL(2, p^m) - PSL(2, p^m)$. The permutation set $G(p^m, \sigma)$ is sharply 3-transitive on $GF(p^m) \cup \{\infty\}$ and it is a subgroup of $P\Gamma L(2, p^m)$ if and only if $\sigma^2 = \bar{1}$ [17]. Furthermore, the set $G(p^m, \bar{1})$ is the projective general linear group $PGL(2, p^m)$.
Let $E = GF(p^m) \cup \{\infty\}$ and observe that each element of $G(p^m, \sigma)$ is identified with a special subset of the Cartesian product $E \times E$. More precisely, the element $g \in G(p^m, \sigma)$ is identified with the set $\{(x, g(x)) \mid x \in E\}$. A Minkowski plane of order p^m can be constructed as follows (see [1, 2, 11, 17, 18]): $\mathcal{P} = E \times E$ is the point-set. The set $\mathcal{B} = G(p^m, \sigma)$ is the block-set. If a is an element of E we define: $g_a = \{(a, x) \mid x \in E\}$, $h_a = \{(x, a) \mid x \in E\}$. Then, $\mathcal{G}_1 = \{g_a \mid a \in E\}$ and $\mathcal{G}_2 = \{h_a \mid a \in E\}$ are the sets of generators. We will denote this plane by $\mathcal{M}(p^m, \sigma)$.
Thus far, these are the only known finite Minkowski planes. The classical model obtained by a hyperbolic quadric corresponds to the plane $\mathcal{M}(p^m, \bar{1})$.
Suppose m to be even and let $q^2 = p^m$. Let $\bar{\mathcal{P}} = \{(x, x^q) \mid x \in GF(q^2) \cup \{\infty\}\}$. $\bar{\mathcal{P}}$ is a set of independent points. Let $\sigma \in \mathrm{Aut}(GF(q^2))$, denote by $\mathcal{C}$ the set of blocks of $\mathcal{M}(q^2, \sigma)$ that contain at least three points of $\bar{\mathcal{P}}$. For every $B \in \mathcal{C}$ let $\bar{B} = B \cap \bar{\mathcal{P}}$ and let $\bar{\mathcal{C}} = \{\bar{B} \mid B \in \mathcal{C}\}$. It was proved in [19] that each block of $\mathcal{C}$ is an element of $PSL(2, q^2)$ with $q + 1$ points in $\bar{\mathcal{P}}$ and $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ is a Miquelian inversive plane of order q embedded in $\mathcal{M}(q^2, \sigma)$.
Denote by $\mathcal{A}(q^2, \sigma)$ the automorphism group of $\mathcal{M}(q^2, \sigma)$. If $f \in \mathcal{A}(q^2, \sigma)$, then f is either a type 1 automorphism, i.e., $f(\mathcal{G}_1) = \mathcal{G}_1$ and $f(\mathcal{G}_2) = \mathcal{G}_2$, or a type 2 automorphism, i.e., $f(\mathcal{G}_1) = \mathcal{G}_2$ and $f(\mathcal{G}_2) = \mathcal{G}_1$ [3, 8]. Let $\mathcal{A}^+(q^2, \sigma)$ and $\mathcal{A}^-(q^2, \sigma)$ denote, respectively,

the set of automorphisms of type 1 and 2. If $\sigma^2 = \bar{1}$, then:

$$\mathcal{A}^+(q^2,\sigma) = \{(x,y) \to (\delta_1\tau(x), \delta_2\tau(y)) \mid \delta_1,\delta_2 \in G(q^2,\sigma),\ \tau \in \mathrm{Aut}GF(q^2)\}$$
$$\mathcal{A}^-(q^2,\sigma) = \{(x,y) \to (\delta_1\tau(y), \delta_2\tau(x)) \mid \delta_1,\delta_2 \in G(q^2,\sigma),\ \tau \in \mathrm{Aut}GF(q^2)\}.$$

On the other hand, if $\sigma^2 \neq \bar{1}$, then:

$$\begin{aligned}\mathcal{A}^+(q^2,\sigma) = \{(x,y) \to (g_1(x), g_2(y)) | g_1, g_2 \in P\Gamma L(2,q^2),\\ \Phi(g_1) = \Phi(g_2),\ \chi(g_1) = \chi(g_2)\}.\\ \mathcal{A}^-(q^2,\sigma) = \{(x,y) \to (g_1(y), g_2(x)) | g_1, g_2 \in P\Gamma L(2,q^2),\\ \Phi(g_2) = \sigma\Phi(g_1),\ \chi(g_2) = -\chi(g_1)\},\end{aligned}$$

see [8].

If we consider $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ embedded in $\mathcal{M}(q^2,\sigma)$, then the automorphism group of $M(q)$ is identified with a subgroup of $\mathcal{A}(q^2,\sigma)$. This subgroup is isomorphic to $P\Gamma L(2,q^2)$ and its elements map $\bar{\mathcal{P}}$ onto itself.

More precisely, let $\alpha : x \longrightarrow \frac{ax+b}{cx+d}$ be an element of $PGL(2,q^2)$. Denote by α_q the element of $PGL(2,q^2)$ defined by: $\alpha_q : x \longrightarrow \frac{a^q x+b^q}{c^q x+d^q}$. It is easy to show that the automorphism group of $M(q)$ is the subgroup $A(q) < \mathcal{A}(q^2,\sigma)$ defined by:

$$A(q) = \{(x,y) \longrightarrow (\alpha\tau(x), \alpha_q\tau(y))\ |\alpha \in PGL(2,q^2),\ \tau \in \mathrm{Aut}(GF(q^2))\}.$$

Furthermore, if $\sigma^2 = \bar{1}$, each automorphism $(x,y) \longrightarrow (\alpha\tau(x), \alpha_q\tau(y))$ coincides on $\bar{\mathcal{P}}$ with the automorphism $(x,y) \longrightarrow (\alpha\theta(y), \alpha_q\theta(x))$, $\theta \in \mathrm{Aut}(GF(q^2))$ and $\theta(x^q) = \tau(x)$, for every $x \in GF(q^2)$. Therefore, if $\sigma^2 = \bar{1}$, we also have:

$$A(q) = \{(x,y) \longrightarrow (\alpha\theta(y), \alpha_q\theta(x)) \mid \alpha \in PGL(2,q^2), \theta \in \mathrm{Aut}(GF(q^2))\}.$$

If $\sigma^2 \neq \bar{1}$, it is easy to prove that no type 2 automorphism of $\mathcal{M}(q^2,\sigma)$ maps $\bar{\mathcal{P}}$ onto itself.

3. Regular Sets of Points in $M(q)$

Consider the Miquelian inversive plane of order q, $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$, embedded in each known finite Minkowski plane of order q^2 as described in the previous section. A subset $R \subset \bar{\mathcal{P}}$ is a set of independent points of each $\mathcal{M}(q^2,\sigma)$. Is there a link between the regularity property of R in $M(q)$ and those in $\mathcal{M}(q^2,\sigma)$? The following propositions answer this question.

Proposition 1. *Let $R \subset \bar{\mathcal{P}}$, $|R| \geq 3$. The set R is a regular set for $M(q)$ if and only if it is a regular set for each plane $\mathcal{M}(q^2,\sigma)$, with $\sigma \in Aut(GF(q^2))$, $\sigma^2 \neq \bar{1}$.*

Proof. Let $\sigma \in \mathrm{Aut}(GF(q^2))$, $\sigma^2 \neq \bar{1}$. Suppose R is regular for $\mathcal{M}(q^2,\sigma)$. Then R is regular for $M(q)$ since each automorphism of $M(q)$ is the restriction of an automorphism of $\mathcal{M}(q^2,\sigma)$. Conversely, let R be a regular set in $M(q)$ with $R = \{(x_1, x_1^q), \ldots, (x_r, x_r^q) \mid x_i \in GF(q^2) \cup \{\infty\}\}$. Let $\phi \in \mathcal{A}(q^2,\sigma)$ with $\phi(R) = R$. If ϕ is a type 1 automorphism, then $\phi : (x,y) \longrightarrow (\alpha\tau(x), \beta\tau(y))$, $\alpha, \beta \in PGL(2,q^2)$, $\chi(\alpha) = \chi(\beta)$, $\tau \in \mathrm{Aut}(GF(q^2))$, and $\beta\tau(x_i^q) = [\alpha\tau(x_i)]^q = \alpha_q\tau(x_i^q)$ for each $i \in \{1,\ldots,r\}$. As $|R| = r \geq 3$ and $PGL(2,q^2)$ is sharply 3-transitive, we obtain $\beta = \alpha_q$ and $\phi \in A(q)$. The regularity of R in $M(q)$ implies $\phi = id$, (id denotes the identical automorphism). Suppose now ϕ is a type 2 automorphism with $\phi : (x,y) \longrightarrow (\alpha\tau(y), \beta\sigma\tau(x))$, $\alpha, \beta \in PGL(2,q^2)$, $\tau \in \mathrm{Aut}(GF(q^2))$, $\chi(\alpha) \neq \chi(\beta)$.

Then we have $[\alpha\tau(x_i^q)]^q = \alpha_q\tau(x_i) = \beta\sigma\tau(x_i)$ and $\sigma^{-1}\beta^{-1}\alpha_q\tau(x_i) = \tau(x_i)$, for every $i \in \{1, \dots, r\}$. This is a contradiction: the element $\sigma^{-1}\beta^{-1}\alpha_q \in PSL'(2, q^2)\sigma^{-1}$ and it fixes at least r elements of $GF(q^2) \cup \{\infty\}$, with $r \geq 3$. □

PROPOSITION 2. *Each subset $R \subset \bar{\mathcal{P}}$ is not a regular set for $\mathcal{M}(q^2, \sigma)$ when $\sigma^2 = \bar{1}$.*

PROOF. The type 2 automorphism $\phi : (x, y) \longrightarrow (y^q, x^q)$ maps each subset $R \subset \bar{\mathcal{P}}$ onto itself and $\phi \neq id$. □

Let us consider $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ embedded in $\mathcal{M}(q^2, \sigma)$. This can aid the search for regular sets. To show this, in the following proposition we give new examples of such regular sets. In the proof we will use the above notation considering $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ embedded in $\mathcal{M}(q^2, \bar{1})$.

PROPOSITION 3. *The Miquelian inversive plane $M(q)$, $q \geq 7$, contains regular t-sets for every t, with $9 \leq t \leq q + 4$.*

PROOF. Let r be a positive integer with $5 \leq r \leq q$ and let ϵ be a primitive element of $GF(q^2)$. Consider the Minkowski plane $\mathcal{M}(q^2, \bar{1})$ and let R be a subset of its point-set defined by

$$R = \{(\infty, \infty), (0, 0), (1, 1), (x_3, x_3), \dots, (x_r, x_r)\} \\ \cup \{(\epsilon, \epsilon^q), (\epsilon^{q+2}, \epsilon^{2q+1}), (\epsilon^{2q+3}, \epsilon^{3q+2})\},$$

with the x_i distinct elements in $GF(q)$, ϵ primitive element of $GF(q^2)$ and $x_i \neq \epsilon^{3(q+1)}$ for every i, (this is possible as $q \geq 7$). Observe that $R \subset \bar{\mathcal{P}}$ and $|R| = t = r + 4$, thus $9 \leq |R| \leq q + 4$. Let $\phi \in A(q)$ with $\phi : (x, y) \longrightarrow (\alpha\tau(x), \alpha_q\tau(y))$, $\alpha \in PGL(2, q^2)$, $\tau \in \mathrm{Aut}(GF(q^2))$ and $\phi(R) = R$. The set R has exactly three points outside the identity block $Id = \{(x, x) \mid x \in GF(q^2) \cup \{\infty\}\}$ and at least six points on Id. Thus ϕ maps at least three points of Id onto points of Id. This implies $\phi(Id) = Id$ and $\phi(\{(\epsilon, \epsilon^q), (\epsilon^{q+2}, \epsilon^{2q+1}), (\epsilon^{2q+3}, \epsilon^{3q+2})\}) = \{(\epsilon, \epsilon^q), (\epsilon^{q+2}, \epsilon^{2q+1}), (\epsilon^{2q+3}, \epsilon^{3q+2})\}$. The relation $\phi(Id) = Id$ implies $\alpha_q\tau(x) = \alpha\tau(x)$, $\forall x \in GF(q^2) \cup \{\infty\}$ and then $\alpha \in PGL(2, q)$.

The points (ϵ, ϵ^q), $(\epsilon^{q+2}, \epsilon^{2q+1})$, $(\epsilon^{2q+3}, \epsilon^{3q+2})$ are on the block $\beta = \{(x, \epsilon^{q-1}x) \mid x \in GF(q^2) \cup \{\infty\}\}$, therefore $\phi(\beta) = \beta$. Observe that $\beta \cap Id = \{(\infty, \infty), (0, 0)\}$ and $\phi(\beta \cap Id) = \beta \cap Id$.

This yields two possibilities:

(1) $\phi(\infty, \infty) = (\infty, \infty)$, $\phi(0, 0) = (0, 0)$.
(2) $\phi(\infty, \infty) = (0, 0)$, $\phi(0, 0) = (\infty, \infty)$.

Suppose the first possibility holds.

We obtain $\alpha : x \longrightarrow ax$, $a \in GF(q)$, and $\phi(x, \epsilon^{q-1}x) = (a\tau(x), a\tau(\epsilon^{q-1}x))$ with $a\tau(\epsilon^{q-1})\tau(x) = \epsilon^{q-1}a\tau(x)$, $\forall x \in GF(q^2) \cup \{\infty\}$. This implies $\tau(\epsilon^{q-1}) = \epsilon^{q-1}$. Let $q = p^m$ and $\tau : x \longrightarrow x^{p^i}$, $0 < i \leq 2m$, then $(p^i - 1)(q - 1) = k(q^2 - 1)$, with k a positive integer, i.e., $(p^i - 1) = k(q + 1)$. This implies $i = m + t$, $0 < t \leq m$ and $p^{m+t} - 1 = k(p^m + 1)$, i.e., $p^m(p^t - k) = k + 1$.

Let $k + 1 = hp^m + r$, $0 \leq r < p^m$, then $p^m(p^t - hp^m - r + 1) = hp^m + r$, i.e., $p^{m+t} - hp^{2m} - rp^m + p^m = hp^m + r$.

Thus either $r = 0$ or p^m divides r. The condition $r < p^m$ implies $r = 0$. This yields $p^{m+t} - hp^{2m} + p^m = hp^m$ which implies: $p^{m+t} + p^m = h(p^{2m} + p^m)$. We have $0 < t \leq m$, from which it follows $t = m$ and $h = 1$. The condition $t = m$ implies $\tau = id$.

Thus $\phi : (x, y) \longrightarrow (ax, ay)$ and $a\epsilon \in \{\epsilon, \epsilon^{q+2}, \epsilon^{2q+3}\}$. This yields three possibilities. The possibility $a = 1$ implies ϕ is the identical automorphism. The possibility $a = \epsilon^{q+1}$ implies $\epsilon^{3q+4} = \epsilon$; this gives a contradiction as $q \geq 7$. The possibility $a = \epsilon^{2q+2}$ implies either $\epsilon^{3q+4} = \epsilon$ or $\epsilon^{3q+4} = \epsilon^{q+2}$, and in both cases we have a contradiction.

Suppose now $\phi(\infty, \infty) = (0, 0)$ and $\phi(0, 0) = (\infty, \infty)$. This implies:

$\phi : (x, y) \longrightarrow \left(\frac{a}{\tau(x)}, \frac{a}{\tau(y)}\right)$, $a \in GF(q)$, and $\phi(x, \epsilon^{q-1}x) = \left(\frac{a}{\tau(x)}, \frac{a}{\tau(\epsilon^{q-1})\tau(x)}\right)$ with $\frac{\epsilon^{q-1}a}{\tau(x)} = \frac{a}{\tau(\epsilon^{q-1})\tau(x)}$, $\forall x \in GF(q^2) \cup \{\infty\}$. Therefore $\epsilon^{q-1}\tau(\epsilon^{q-1}) = 1$.

Let $q = p^m$ and let $\tau : x \longrightarrow x^{p^i}$, $0 < i \leq 2m$. Then $\epsilon^{q-1}\epsilon^{p^i(q-1)} = 1$, and $(p^i + 1)(q - 1) = k(q^2 - 1)$, with k a positive integer. Therefore $i = m + t$, $0 \leq t \leq m$ and $p^{m+t} + 1 = k(p^m + 1)$, i.e., $p^m(p^t - k) = k - 1$.

It follows that $t = 0$. In fact if we suppose $t \neq 0$, we also have $k-1 \neq 0$ and $k-1 = hp^m+r$, $0 \leq r < p^m$, then $p^m(p^t - hp^m - r - 1) = hp^m + r$; $r < p^m$ implies $r = 0$ and $p^{m+t} - p^m = h(p^{2m} + p^m)$: a contradiction.

Therefore $\phi : (x, y) \longrightarrow \left(\frac{a}{x^q}, \frac{a}{y^q}\right)$. Furthermore, in this case we have three possibilities. The first one: $\frac{a}{\epsilon^q} = \epsilon$ implies either $\frac{\epsilon^{q+1}}{\epsilon^{2q+1}} = \epsilon^{q+2}$ or $\frac{\epsilon^{q+1}}{\epsilon^{3q+2}} = \epsilon^{q+2}$ and both give a contradiction. The second possibility is $\frac{a}{\epsilon^q} = \epsilon^{q+2}$, which implies either $\frac{\epsilon^{2q+2}}{\epsilon^{2q+1}} = \epsilon$ or $\frac{\epsilon^{2q+2}}{\epsilon^{3q+2}} = \epsilon^{2q+3}$. The second case is not possible as $q \geq 7$; the first one also implies $\frac{\epsilon^{2q+2}}{\epsilon^{3q+2}} = \epsilon^{2q+3}$ and again a contradiction. The third possibility is $\frac{a}{\epsilon^q} = \epsilon^{2q+3}$ which implies $a = \epsilon^{3q+3}$ and $\phi(1, 1) = (\epsilon^{3(q+1)}, \epsilon^{3(q+1)}) \in R$: this is a contradiction. □

If R is a regular set for $M(q)$, then $\bar{\mathcal{P}} - R$ is also regular set. In Proposition 3 we give examples of regular t-sets for every t, with $9 \leq t \leq q + 4$. Therefore, we also have examples of regular t-sets for every t with $q^2 - q - 3 \leq k \leq q^2 - 8$. These regular sets are different from those of [16]. In fact examples of regular 5-sets are presented in this paper.

REMARK 1. Let $\sigma \in \mathrm{Aut}(GF(q^2))$, $\sigma^2 \neq \bar{1}$. The existence of a regular set S for the Minkowski plane $\mathcal{M}(q^2, \sigma)$, with S contained in a block, is equivalent to the existence of a regular set $\bar{S}$ for the group $P\Gamma L(2, q^2)$ acting on $GF(q^2) \cup \{\infty\}$ (see [20]). Furthermore, as the automorphism group of $\mathcal{M}(q^2, \sigma)$ is transitive on the set of triples of independent points, we can suppose S to be contained in the identity block Id, i.e., $S = \{(x_1, x_1), \ldots, (x_t, x_t)\}$, $x_i \in GF(q^2)$, and then we obtain $\bar{S} = \{x_1, \ldots, x_t\}$ [20].

Consider $M(q) = (\bar{\mathcal{P}}, \bar{\mathcal{C}})$ embedded in $\mathcal{M}(q^2, \sigma)$. The set $\bar{S}$ is thus identified with the set $S' = \{(x_1, x_1^q), \ldots, (x_t, x_t^q)\} \subset \bar{\mathcal{P}}$ and S is regular for $\mathcal{M}(q^2, \sigma)$ if and only if S' is regular for $M(q)$.

Therefore, Proposition 3 gives new examples of regular sets for the Minkowski planes $\mathcal{M}(q^2, \sigma)$ $\sigma^2 \in \mathrm{Aut}GF(q^2)$, $\sigma^2 \neq \bar{1}$. These examples are new because those found in [8] and [20] were not subsets of blocks.

REMARK 2. The Minkowski plane $\mathcal{M}(p^m, \bar{1})$ is a proper subplane of $\mathcal{M}(p^n, \bar{1})$ whenever $GF(p^m)$ is a proper subfield of $GF(p^n)$. Each subset R of the point-set of $\mathcal{M}(p^m, \bar{1})$ is not a regular set for $\mathcal{M}(p^n, \bar{1})$. In fact, the automorphism $f : (x, y) \longrightarrow (x^{p^m}, y^{p^m})$ maps R onto itself, but it is not the identity of $\mathcal{A}(p^n, \bar{1})$.

REMARK 3. Let p be a prime, $p \neq 2$, and consider the Galois field $GF(p^m)$. Let $n = mh$, with h a positive odd integer, $h \neq 1$. An element $a \in GF(p^m)$ is a square in $GF(p^m)$ if and only if it is a square in $GF(p^n)$. Let $\sigma \in \mathrm{Aut}(GF(p^n))$, $\sigma \neq \bar{1}$, thus the restriction of σ to $GF(p^m)$ is an automorphism of $GF(p^m)$. It is easy to show that the Minkowski plane $\mathcal{M}(p^m, \sigma)$ is a subplane of $\mathcal{M}(p^n, \sigma)$. In fact, the point-set of $\mathcal{M}(p^m, \sigma)$ is a subset of the point-set of $\mathcal{M}(p^n, \sigma)$. Now let $\mathcal{B}$ and $\mathcal{B}'$ be the block-sets of $\mathcal{M}(p^n, \sigma)$ and $\mathcal{M}(p^m, \sigma)$, respectively, and let $\mathcal{P}'$ be the point-set of $\mathcal{M}(p^m, \sigma)$. Therefore, for every $B' \in \mathcal{B}'$, there exists $B \in \mathcal{B}$ such that $B' = B \cap \mathcal{P}'$.

Furthermore, in this case, each subset R of the point-set of $\mathcal{M}(p^m, \sigma)$ is not a regular set for $\mathcal{M}(p^n, \sigma)$. This follows as in the previous remark.

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