PERGAMON Computers and Mathematics with Applications 46 (2003) 685–694

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Feynman Integrals with Point Interactions

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(Received January 2001; revised and accepted November 2002)

Abstract—A Feynman-Kac formula for Schrödinger operators including a one-center point interaction in R^3 plus a bounded potential is proved. Functional integration methods on similar Kac's averages with point interactions allow us to construct bounded self-adjoint semigroups in $L^2(R^3)$, with bounded below Schrödinger generators, when $V^+ \in L^2_{\text{loc}}$ and V^- belongs to a large class of $L^2 + L^\infty$ potentials. Moreover, a pointwise bound on the range of the semigroup is given. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Feynman-Kac formula, Markov property, Schrödinger semigroup, Point estimates, Delta-type interaction.

1. INTRODUCTION

Point interactions in the Feynman-Kac formula are an interesting object of study, as they are considered in various cases in the literature on path integrals [1].

A description of the one-center point interactions of quantum mechanics, in R^3, can be obtained [2] from the Laplacian operator $-(1/2)\Delta$ with domain $C_\circ^\infty(R^3 - \{O\})$: it consists of the one-parameter family of self-adjoint extensions $\{H(\alpha)\}_{\alpha\in R}$. The integral kernels $K_{t,\alpha}(x,y)$ of the associated semigroups $\{e^{-tH(\alpha)}\}_{t\geq 0}$ are explicitly known [3, p. 228]. By means of probability measures $\mu_{x,\alpha}$ induced on path space by such kernels, a Feynman-Kac formula including point interactions and continuous bounded potentials is proved (Section 2). Then, from Kac's averages with point interactions,

$$P_t^V f(x) = \int_\Omega f(\omega(t)) \exp\left[-\int_0^t V(\omega(s))\,ds\right] d\mu_{x,\alpha}(\omega) \tag{1.1}$$

bounded self-adjoint semigroups are constructed in L^2 under the following conditions: $V^+ \in L^2_{\text{loc}}$, $V^-\chi_B \in L^2$, $V^-(1-\chi_B) \in L^\infty$ for some measurable and bounded set B containing a neighbourhood of $O \in R^3$ (the indicator function of B is denoted by χ_B).

In particular, two results for such unbounded potentials are remarkable: an estimate of functions in the range of the semigroup (1.1) holds in a general form (Theorem 3.7 and Corollary 3.10), and the fact that each spectrum of these (Schrödinger) generators is bounded below

This work has been partially supported by GNFM, CNR.

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi: 10.1016/S0898-1221(03)00277-3

(Theorem 3.9). While the case of a point interaction plus a Coulomb potential in R^3 is a known solvable model [2], here the construction is able to cover both potentials V different from the Coulomb one, and potentials with (several) L^2-singularities which can be different from the center of the point interaction.

2. THE FEYNMAN-KAC FORMULA WITH ONE-CENTER POINT INTERACTIONS

For each $\alpha \in R$ there is a self-adjoint extension $H(\alpha)$ of $-(1/2)\Delta$ on $C_o^\infty(R^3 - \{O\})$. The parameter α corresponds to a coupling constant of $(1/\alpha)$, i.e., $H(\alpha) = -(1/2)\Delta + (1/\alpha)\delta(x)$, or a scattering length of $-(4\pi\alpha)^{-1}$. The associated Schrödinger semigroup is characterized [3, p. 228] by

$$e^{-tH(\alpha)}g(x) = \int_{R^3} K_{t,\alpha}(x,y)g(y)\,dy, \tag{2.1}$$

where $K_{t,\alpha} = p_t Y_{t,\alpha}$, $p_t = (2\pi t)^{-3/2}\exp\{-|x-y|^2/2t\}$, and

$$Y_{t,\alpha}(x,y) = 1 + U_t(x,y) + V_{t,\alpha}(x,y), \tag{2.2}$$

$$U_t = \frac{t\exp\{-(1/t)(x\cdot y + |x||y|)\}}{|x||y|}, \qquad V_{t,\alpha} = -\frac{4\pi\alpha t}{|x||y|}\int_0^\infty \frac{p_t(u+|x|+|y|)}{p_t(|x-y|)}e^{-4\pi\alpha u}\,du. \tag{2.3}$$

DEFINITION 2.1. *For $t > 0$, let $\omega : [0,t] \to R^3$ be a continuous path. For $h = 0,1,\dots,n$, and $x_h = \omega(ht/n)$, we define*

$$\Phi^n_{t,\alpha}[\omega] := \prod_{k=1}^n Y_{t/n,\alpha}(x_{k-1},x_k) \tag{2.4}$$

with the conventional value $+\infty$ if some divisor is zero.

THEOREM 2.2. *Let $H(\alpha) = -(1/2)\Delta_\alpha$, with $\alpha \in R$. For each $x \neq O$, denoting by W_x the Wiener measure on paths starting from x and by E_x the associated expectation, then $\Phi^n_{t,\alpha}$ is W_x-a.e. finite and*

$$\forall f \in L^2(R^3), \qquad e^{-tH(\alpha)}f(x) = E_x\left\{\Phi^n_{t,\alpha}[\omega]f(\omega(t))\right\}. \tag{2.5}$$

Moreover, let $\Omega = \prod_{t\geq 0}\dot{R}^3$ be the space of all paths with values in the one-point compactification of $\dot{R}^3$, endowed with the product topology. For each $x \in R^3 - \{O\}$, there is a measure $\mu_{x,\alpha}$, with $\mu_{x,\alpha}(\Omega) < +\infty$, such that

$$e^{-tH(\alpha)}f(x) = \int_\Omega f(\omega(t))\,d\mu_{x,\alpha}(\omega), \qquad \text{a.e. in } R^3. \tag{2.6}$$

PROOF.

(A) Let $x_o \neq O$ and $n \in N$. Then, by (2.1),

$$\begin{aligned} e^{-tH(\alpha)}f(x_o) &= \left(e^{-tH(\alpha)/n}\right)^n f(x_o) \\ &= \int_{R^{3n}} \prod_{k=1}^n p_{t/n}(x_{k-1},x_k)Y_{t/n,\alpha}(x_{k-1},x_k)f(x_n)\,dx_1\cdots dx_n. \end{aligned} \tag{2.7}$$

Now for Brownian motion, the joint density of $\omega(t/n), \omega(2t/n), \dots, \omega(t)$, with the prescription $\omega(0) = x_o$, is given by $p_{t/n}(x_o,x_1)\cdots p_{t/n}(x_{n-1},x_n)$. Thus,

$$E_{x_o}\left\{\prod_{k=1}^n Y_{t/n,\alpha}\left[\omega\left(\frac{(k-1)t}{n}\right),\omega\left(\frac{kt}{n}\right)\right]f[\omega(t)]\right\} \equiv E_{x_o}\left\{\Phi^n_{t,\alpha}[\omega]f[\omega(t)]\right\} \tag{2.8}$$

is finite and equal to (2.7).

(B) The product space Ω is compact by the Tychonoff theorem. Let $C_{\text{fin}}(\Omega)$ be the set of functions ϕ on Ω which continuously depend only on a finite number of coordinates: $\phi(\omega) = F(\omega(t_1), \dots, \omega(t_m))$ for fixed $0 \equiv t_\circ < t_1 \leq \cdots \leq t_m \equiv t$ and for some continuous function F on $(\dot{R})^{3m}$. For each $x_\circ \neq O$, define

$$L_{x_\circ,\alpha}(F) = \int_{R^{3m}} F(x_1, \dots, x_m) \prod_{h=1}^{m} K_{t_h - t_{h-1}}(x_{h-1}, x_h)\, dx_1, \dots, dx_m, \qquad \forall F \in C_{\text{fin}}(\Omega). \quad (2.9)$$

Now (2.9) determines a positive linear functional $L_{x_\circ} : C_{\text{fin}}(\Omega) \to R$. To this end, we have to show that $L_{x_\circ,\alpha}\phi$ is independent of the representation of ϕ, as $\phi \equiv F$ may be independent of some x_h. Now the semigroup property of the kernel in (2.1) implies

$$\int_{R^3} K_{t_h - t_{h-1}}(x_{h-1}, x_h) K_{t_{h+1} - t_h}(x_h, x_{h+1})\, dx_h = K_{t_{h+1} - t_{h-1}}(x_{h-1}, x_{h+1}). \quad (2.10)$$

For example, $\forall f \in C_\circ^\infty$,

$$\int_{R^6} K_{\tau_1}(x_\circ, x_1) K_{\tau_2}(x_1, x_2)\, dx_1 f(x_2)\, dx_2 = \left[e^{-(\tau_1+\tau_2)H} f\right](x_\circ)$$

$$\Longrightarrow \int K_{\tau_1}(x_\circ, x_1) K_{\tau_2}(x_1, x_2)\, dx_1 = K_{\tau_1+\tau_2}(x_\circ, x_2).$$

As a consequence, $L_{x_\circ,\alpha}\phi$ is well defined. Moreover, $L_{x_\circ,\alpha}(1) < +\infty$ and $L_{x_\circ,\alpha}\phi \geq 0$ if $\phi > 0$ since the kernel $K_{t,\alpha}$ is positive. By density of $C_{\text{fin}}(\Omega)$ in $C(\Omega)$, $L_{x_\circ,\alpha}$ has a unique bounded extension with $\|L_{x_\circ,\alpha}\| = L_{x_\circ,\alpha}(1)$. By the Riesz-Markov theorem, there is a unique regular Borel measure $d\mu_{x_\circ,\alpha}$ on Ω such that $L_{x_\circ,\alpha}f = \int f[\omega(t)]\, d\mu_{x_\circ,\alpha}(\omega)$. Moreover, $\mu_{x_\circ,\alpha}(\Omega) = L_{x_\circ,\alpha}(1) < +\infty$. By (2.9), when $F = f(x_m)$, $e^{-tH(\alpha)}f(x_\circ) = L_{x_\circ,\alpha}(f)$, and the theorem is proved. ∎

By the above theorem, each one-center point interaction in R^3, with parameter α, is represented in functional integration by means of a positive measure $\mu_{x,\alpha}$ on path space. The following theorem gives a Feynman-Kac formula in the presence of a point interaction plus a potential of a suitable class.

THEOREM 2.3. *Let $H(\alpha)$ and $d\mu_{x,\alpha}$ be taken as above. If $V \in L^\infty(R^3)$, then for almost all $x \in R^3 - \{O\}$,*

$$e^{-t(H(\alpha)+V)} f(x) = \int_\Omega G(\omega; V, t) f[\omega(t)]\, d\mu_{x,\alpha}. \quad (2.11)$$

Here the function G on path space has the expression $G(\omega; V, t) = \exp[-\int_0^t V(\omega(s))\, ds]$ for any ω such that $V \circ \omega$ is integrable.

PROOF. Let us denote $\mu_{x,\alpha}$ by μ_x for simplicity. $H(\alpha)$ is bounded from below and V is bounded, so by the Trotter product formula [4,5]

$$e^{-t(H(\alpha)+V)} f = \lim_{n\to\infty} \left(e^{-tH(\alpha)/n} e^{-tV/n}\right)^n f. \quad (2.12)$$

By passing to a subsequence, still denoted by n, (2.12) can be read as an a.e. limit on R^3. By density in L^2, it is enough to consider $f \in C_\circ^\infty(R^3)$. The sequence of functions on path space

$$\omega \to G_n(\omega; V, t) f[\omega(t)] = \exp\left\{-\frac{t}{n} \sum_{k=1}^{n} V\left[\omega\left(\frac{kt}{n}\right)\right]\right\} f[\omega(t)] \quad (2.13)$$

obeys Arzela-Ascoli conditions. Indeed, since V is bounded, the set of images $\{G_n(\omega; V, t) : n \in N\}$, for fixed $\omega \in \Omega$, is bounded in R; moreover, the G_ns are equicontinuous at each point of Ω, since V is continuous and bounded. Therefore, in $C(\Omega)$ there exists the limit $G(\omega; V, t)$ of some subsequence $\{G_{n_i}(\omega; V, t))\}_{i \in N}$. On the other hand, the $G_n f$s belong to $C_{\text{fin}}(\Omega)$ (in the notation of the above proof) and, according to (2.9), they are transformed into

$$
\begin{aligned}
L_{x_o}(G_n f) &= \int_{(R^3)^n} \exp\left\{-\frac{t}{n}\sum_{h=1}^{n} V(x_h)\right\} f(x_n) \prod_{h=1}^{n} K_{t/n,\alpha}(x_{h-1}, x_h)\, dx_1 \cdots dx_n \\
&= \left(e^{-tH(\alpha)/n} e^{-tV/n}\right)^n f(x_o)
\end{aligned}
\tag{2.14}
$$

by the positive functional L_{x_o}. But the convergence of $G_{n_i} f$ is dominated, since $|f(\omega(t)) \times G_n(\omega; V, t)| \le \exp(t\|V\|_\infty)\|f\|_\infty$, and hence,

$$
L_{x_o}(G_{n_i} f) \equiv \int_\Omega G_{n_i}(\omega; V, t) f(\omega(t))\, d\mu_{x_o} \to \int_\Omega G(\omega; V; t) f(\omega(t))\, d\mu_{x_o}, \tag{2.15}
$$

as $i \to \infty$. Now (2.14) converges for almost all x_o by the Trotter product formula, and the two limits of (2.14) and (2.15) must be equal: so formula (2.11) follows in the case of continuous and bounded Vs.

The result is extended to any $V \in L^\infty$ by use of a suitable sequence V_m of continuous potentials such that $V_m \to V$ pointwise a.e.: for the details, see the analogous extension performed in [6, Theorem 6.2], the first part of the proof. Thus the statement is proved. ∎

PROPOSITION 2.4. *Fixing α, let $E_x^\mu\{\cdot\}$ denote the integration with respect to $d\mu_{x,\alpha}$. For each $t > 0$, let r_t be the path space transformation*

$$
[r_t\omega](s) = \begin{cases} \omega(t-s), & \text{if } 0 \le s \le t, \\ \omega(0), & \text{if } s > t. \end{cases} \tag{2.16}
$$

Then

$$
\int_{R^3} E_x^\mu\{F\}\, dx = \int_{R^3} E_x^\mu\{F \circ r_t\}\, dx, \tag{2.17}
$$

for any positive Σ_t-random variable (where Σ_t is the minimal σ-algebra with respect to which the coordinate functions $\omega \to \omega_s$ are measurable $\forall\, s \le t$).

PROOF. Let $\omega' = r_t\omega$. Since $\omega'((k-1)t/n) = \omega((n-k+1)t/n)$, setting $h = n-k+1$, we have

$$
\Phi_{t,\alpha}^n(\omega') = \prod_{h=1}^{n} Y_{t/n,\alpha}\left(\omega\left(\frac{(h-1)t}{n}\right), \omega\left(\frac{ht}{n}\right)\right) = \Phi_{t,\alpha}^n[\omega]. \tag{2.18}
$$

In the notation of the proof of Theorem 2.2, for each $F \in C_{\text{fin}}(\Omega)$,

$$
\begin{aligned}
E_{x_o}^\mu\{F\} &= \int_{R^{3m}} F(x_1, \dots, x_m) \prod_{h=1}^{m} K_{t_h - t_{h-1}}(x_{h-1}, x_h) dx_1, \dots, dx_m \\
&= E_{x_o}\left\{\Phi_t^m(\omega) F\left[\omega\left(\frac{t}{m}\right), \dots, \omega(t)\right]\right\},
\end{aligned}
$$

so

$$
E_{x_o}^\mu\{F \circ r_t\} = E_{x_o}\left\{\Phi_t^m(\omega') F\left[\omega\left(\frac{(m-1)t}{m}\right), \dots, \omega(0)\right]\right\}. \tag{2.19}
$$

Integrating in x_o and using (2.18), the assertion comes down to the well-known fact

$$
\int_{R^3} E_x\{F\}\, dx = \int_{R^3} E_x\{F \circ r_t\}\, dx, \tag{2.20}
$$

which is true for the Brownian motion expectations E_x. ∎

3. POINT INTERACTIONS PLUS UNBOUNDED POTENTIALS

Starting from Kac's averages like (2.11) including unbounded potentials, we want to construct Schrödinger semigroups and get properties on their generators (e.g., properties of eigenfunctions). First, we present the (simpler) case of bounded below potentials.

PROPOSITION 3.1. *Let E_x^μ denote the $d\mu_{x,\alpha}$-integration, for each $x \in R^3 - \{O\}$ and fixed $\alpha \in R$. Let V be real-valued and measurable with* $\inf V = -\bar{\lambda} > -\infty$. *Then the formula*

$$P_t^V f(x) = E_x^\mu \left\{ f(\omega(t)) \exp\left[-\int_0^t V(\omega(s))\, ds \right] \right\} \tag{3.1}$$

determines a self-adjoint bounded semigroup of operators in L^2.

PROOF. By Theorem 2.2, $\mu_x(\Omega) < \infty$. Thus, if V is assumed measurable and bounded below, the right-hand side of (3.1) is clearly absolutely convergent. Although μ_x is not a probability, for any $\psi \in L^2(\Omega, \mu_x)$ and any sub-σ-algebra $\Sigma_\epsilon = \sigma\{\omega(t),\, t \le \epsilon\}$, we denote $E_x^\mu\{\psi(\omega) \mid \Sigma_\epsilon\}$ the orthogonal projection of ψ on the subspace of Σ_ϵ-measurable functions. Under such notation, and denoting $\theta_\epsilon(\omega)(t) = \omega(t+\epsilon)$, a Markov property $E_x^\mu\{\psi \circ \theta_\epsilon \mid \Sigma_\epsilon\} = E_{\omega(\epsilon)}^\mu\{\psi\}$ is induced just by the semigroup $e^{-tH(\alpha)}$. By using such Markov property, the semigroup property is verified even when $V \neq 0$: indeed, fixing V and α,

$$\begin{aligned} P_s P_t f(x) &= E_x^\mu \left\{ e^{-\int_0^s V(\omega_r)\, dr} (P_t f)(\omega_s) \right\} \\ &= E_x^\mu \left\{ e^{-\int_0^s V(\omega_r)\, dr} E_{\omega(s)}^\mu \left\{ e^{-\int_0^t V(\omega_u)\, du} f(\omega_t) \right\} \right\} \\ &= E_x^\mu \left\{ e^{-\int_0^s V(\omega_r)\, dr} E_x^\mu \left\{ e^{-\int_0^t V(\omega_{u+s})\, du} f(\omega_{t+s}) \mid \Sigma_s \right\} \right\} \\ &= E_x^\mu \left\{ e^{-\int_0^s V(\omega_r)\, dr} e^{-\int_0^t V(\omega_{u+s})\, du} f(\omega_{t+s}) \right\} \\ &= E_x^\mu \left\{ e^{-\int_0^{t+s} V(\omega_r)\, dr} f(\omega_{t+s}) \right\} = (P_{s+t} f)(x). \end{aligned} \tag{3.2}$$

Furthermore, each P_t is self-adjoint by virtue of Proposition 2.4. Indeed, for positive f and g,

$$\begin{aligned} \int_{R^3} [P_t f](x) g(x)\, dx &= \int_{R^3} E_x^\mu \left\{ f(\omega_t) g(\omega_o) \exp\left[-\int_0^t V(\omega_s)\, ds \right] \right\} dx \\ &= \int_{R^3} E_x^\mu \left\{ f(\omega_o) g(\omega_t) \exp\left[-\int_0^t V(\omega_s)\, ds \right] \right\} dx, \end{aligned} \tag{3.3}$$

and the extension to general f and g is obvious. Finally, since $V \ge -\bar{\lambda}$,

$$\|P_t f\|_2 \le e^{\bar{\lambda} t} \|E\mu.\{f(\omega_t)\}\|_2 = e^{t\bar{\lambda}} \left\| e^{-tH(\alpha)} f \right\|_2. \tag{3.4}$$

By (3.4), P_t is bounded in L^2 since $H(\alpha)$ is bounded below [2], and the proposition is proved. ∎

COROLLARY 3.2. *Under the assumptions of Proposition 3.1 and if, moreover, $V \in L^2_{\text{loc}}$, the infinitesimal generator $-A$ of P_t^V in L^2 satisfies*

$$\begin{aligned} &D(A) \subset \{ u \in D(H(\alpha)) : H(\alpha)u + Vu \in L^2 \} \quad \textit{and} \\ &u \in D(A) \implies Au = H(\alpha)u + Vu. \end{aligned} \tag{3.5}$$

PROOF. By Proposition 3.1, we can consider the infinitesimal generator $-A$ of P_t^V in L^2. Let us fix $u \in D(A)$ and define

$$V_k(x) = \min\{V(x), k\}, \qquad \forall\, k \in N. \tag{3.6}$$

V_k is bounded and satisfies the Feynman-Kac formula (2.11), and we want to infer properties on P_t^V from properties of $P_t^{V_k}$ according to a standard approximation argument [7]. First,

$$\left\|P_t^{V_k}\right\|_2 \leq ce^{tE} \tag{3.7}$$

with E independent of k. For example, we can choose $E = (1/2)(4\pi\alpha)^2 - \inf V$, on the basis of the well-known spectrum of $H(\alpha)$ [2]. The generator of $P_t^{V_k}$ is just $-(H(\alpha)+V_k)$ by Theorem 2.3. If $\lambda > E$ is fixed and if $v \in L^2$ is defined by

$$u = (\lambda + A)^{-1}v, \tag{3.8}$$

we set

$$u_k = (\lambda + H(\alpha) + V_k)^{-1}v, \qquad k \in N. \tag{3.9}$$

For each $k \geq 1$, $u_k \in D(H(\alpha) + V_k)$, and since V_k is bounded, $D(H(\alpha) + V_k) = D(H(\alpha))$, which implies

$$H(\alpha)u_k = -V_k u_k - \lambda u_k + v. \tag{3.10}$$

Now the proof depends on the convergence $u_k \to u$ in L^2: if such convergence holds, then the right-hand side of (3.9) converges in L^1_{loc} to the function $-Vu - \lambda u + v$, while the left-hand side converges (weakly) to $H(\alpha)u$. By this fact, together with (3.8), (3.5) is proved. As for the claimed convergence $u_k \to u$ in L^2, the essential point is the monotone convergence

$$\lim_{k\to\infty} P_t^{V_k}|\varphi|(x) = P_t^V|\varphi|(x), \qquad \forall \varphi \in L^2, \tag{3.11}$$

when $V_k \uparrow V$ (both a.e. and in L^2 sense since $\varphi \in L^2$): these facts are achieved by use of monotone and dominated convergence theorems, as in [7, p. 267]. Thus, the corollary is proved. ∎

When the potential V is unbounded below, one of the problems is to choose a type of controlled singularity by which $H(\alpha)+V$ is acceptable as a Schrödinger operator. A possible way is to achieve sufficient conditions under which $g(\omega_t)\exp[-\int_0^t V(\omega_s)\,ds]$ is μ_x-integrable on the path space for $g \in L^2$. The estimates are in terms of the "unperturbed" semigroup $e^{-tH(\alpha)}$, which is the reason of the following two propositions.

PROPOSITION 3.3. *Let B be a measurable bounded set containing some neighbourhood of the origin in R^3. For any f such that $f\chi_B \in L^2$ and $f(1-\chi_B) \in L^\infty$, let us define*

$$\|f\|_+ := \|f\chi_B\|_2 + \|f(1-\chi_B)\|_\infty. \tag{3.12}$$

Then

$$\|fg\|_+ \leq \|f\|_+\|g\|_+; \qquad \|fg\|_2 \leq \|f\|_+\|g\|_2. \tag{3.13}$$

PROOF. The first inequality in (3.13): $\|fg\chi_B\|_2 + \|fg(1-\chi_B)\|_\infty \leq \|f\chi_B\|_2\|g\chi_B\|_2 + \|f(1-\chi_B)\|_\infty\|g(1-\chi_B)\|_\infty$, and this is less than $\|f\|_+\|g\|_+$. The second inequality is obvious and the proposition is proved. ∎

PROPOSITION 3.4.

(i) *Let $\mu(\alpha) = 8(\pi\alpha)^2$ if $\alpha < 0$, $\mu(\alpha) = 0$ elsewhere. Then*

$$\left\|e^{-tH(\alpha)}g\right\|_2 \leq e^{\mu(\alpha)t}\|g\|_2, \qquad \forall g \in L^2. \tag{3.14}$$

(ii) *Let $g \in L^p(R^3)$, with $p > 3/2$. There are $b = b(p) > 0$, and $w_t = w_t(x)$ such that*

$$\left|e^{-tH(\alpha)}g(x)\right| \leq \|g\|_p b(p) t^{-3/(2p)} w_t(x), \qquad \forall x \neq O, \quad t > 0, \quad \text{with } \|w_t\|_+ < +\infty. \tag{3.15}$$

In particular, if $g \in L^2$, $\exists b > 0 : \forall t > 0$, $\forall x \neq O$,

$$\left| e^{-tH(\alpha)} g(x) \right| \leq b \|g\|_2 t^{-3/4} w_t(x), \tag{3.16a}$$

where $w_s \leq w_t$ for $s \leq t$ and, for some $c(t) > 0$, for any $\delta > 0$,

$$w_t(x) \leq c(t) \left(1 + \frac{1}{|x|} \right); \quad \|w_t\|_+ = o\left(e^{\delta t}\right), \qquad \text{as } t \to +\infty. \tag{3.16b}$$

PROOF. Only (ii) has to be proved, since (i) depends on the spectral properties of $H(\alpha)$ [2]. Let $\alpha = 0$, since the cases $\alpha \neq 0$ admit a similar treatment. Let $1/p + 1/q = 1$, so that $q < 3$. In view of (1.1), $|e^{-tH(0)} g(x)|$ is less than

$$\begin{aligned}
&(2\pi t)^{-3/2} \left[\|g\|_p \left(\int e^{-q|x-y|^2/2t} \, dy \right)^{1/q} + \int e^{-(x^2+y^2)/2t} \frac{t}{|x||y|} |g(y)| \, dy \right] \\
&\leq \|g\|_p c(p) t^{-3/(2p)} + \frac{t e^{-x^2/2t}}{|x|} (2\pi t)^{-3/2} \left(\int e^{-qy^2/2t} |y|^{-q} \, dy \right)^{1/q} \|g\|_p \\
&\leq \|g\|_p c(p) t^{-3/(2p)} + \frac{e^{-x^2/2t}}{|x|} t (2\pi t)^{-3/2} t^{3/2q} t^{-1/2} \|g\|_p,
\end{aligned} \tag{3.17}$$

where the integral in y converges since $q < 3$. Thus, $w_t(x) = 1 + t^{1/2} \exp(-x^2/2t) |x|^{-1}$ and $\|w_t\|_+ \leq c_\circ t^{3/4}$ for some $c_\circ > 0$ depending on the choice of B. For $\alpha \neq 0$, the bound is analogous and the statements are proved. ∎

THEOREM 3.5. *Let the potential V satisfy $V\chi_B \in L^2$ and $V(1 - \chi_B) \in L^\infty$, for some bounded set B containing a neighbourhood of $O \in R^3$. Let $0 < s < t$ and $f \in L^2$. Then the following pointwise and L^2 estimates hold:*

$$|E_x^\mu \{V(\omega_s) f(\omega_t)\}| \leq s^{-3/4} w_t(x) \|V\|_+ e^{(t-s)\mu(\alpha)} \|f\|_2, \tag{3.18}$$

$$\|E^\mu \{V(\omega_s) f(\omega_t)\}\|_2 \leq e^{t\mu(\alpha)} \|V\|_+ \|f\|_2. \tag{3.19}$$

PROOF. By the Markov property, the l.h.s. of (3.18) is

$$\begin{aligned}
|E_x^\mu \{V(\omega_s) E_x^\mu \{f(\omega_t) \mid \Sigma_s\}\}| &= \left| E_x^\mu \{V(\omega_s) E_{\omega_s}^\mu \{f(\omega_{t-s})\}\} \right| \\
&\equiv |E_x^\mu \{V(\omega_s) g(\omega_s)\}|,
\end{aligned} \tag{3.20}$$

where $g(z) \equiv E_z^\mu \{f(\omega_{t-s})\}$ is equal to $e^{-(t-s)H(\alpha)} f(z)$ by Theorem 2.2. Then, again by Theorem 2.2, (3.20) is equal to $e^{-sH(\alpha)}(Vg)$. Now (3.18) follows as a pointwise estimate of (3.20),

$$= \left| e^{-sH(\alpha)} (Vg)(x) \right| \leq s^{-3/4} w_t(x) \|Vg\|_2 \leq s^{-3/4} w_t(x) \|V\|_+ \|g\|_2 \tag{3.21}$$

(here we have applied (3.16a) and (3.13b))

$$\leq w_t(x) s^{-3/4} \|V\|_+ e^{(t-s)\mu(\alpha)} \|f\|_2 \tag{3.22}$$

(here we have applied (3.14)). On the other hand, (3.19) is an L^2 estimate of (3.20),

$$\left\| e^{-sH(\alpha)} (Vg) \right\|_2 \leq e^{s\mu} \|Vg\|_2 \leq e^{s\mu} \|V\|_+ \left\| e^{-(t-s)H(\alpha)} f \right\|_2 \leq e^{t\mu} \|V\|_+ \|f\|_2, \tag{3.23}$$

by virtue of (3.14) and (3.13b). So the theorem is proved. ∎

The above theorem can be generalized as follows.

THEOREM 3.6. *Let the potential V satisfy $V\chi_B \in L^2$ and $V(1-\chi_B) \in L^\infty$, for some bounded B containing a neighbourhood of $O \in R^3$. Let $0 < s_1 < s_2 < \cdots < s_k < t$ and let $f \in L^2$. Then*

$$|E_x^\mu \{V(\omega_{s_1}) \cdots V(\omega_{s_k}) f(\omega_t)\}| \le w_t(x) \|V\|_+^k s_1^{-3/4} e^{(t-s_1)\mu(\alpha)} \|f\|_2, \qquad \forall f \in L^2. \tag{3.24}$$

PROOF. Choosing $k = 2$ and proceeding as in the proof of Theorem 3.5,

$$E_x^\mu \{V(\omega_{s_1}) E_x^\mu \{V(\omega_{s_2}) f(\omega_t) \mid \Sigma_{s_1}\}\} = E_x^\mu \left\{V(\omega_{s_1}) E_{\omega_{s_1}}^\mu \{V(\omega_{s_2-s_1}) f(\omega_{t-s_1})\}\right\}. \tag{3.25}$$

Now, setting $g(z) \equiv E_z^\mu\{V(\omega_{s_2-s_1}) f(\omega_{t-s_1})\}$, (3.25) is

$$\left|e^{-s_1 H(\alpha)}(Vg)(x)\right| \le w_t(x) s_1^{-3/4} \|Vg\|_2 \le w_t(x) s_1^{-3/4} \|V\|_+ \|g\|_2. \tag{3.26}$$

Here we have applied (3.16a) and (3.13b). By Theorem 3.5, we get a bound for $\|g\|_2$, if only we replace t by $t - s_1$ and s by $s_2 - s_1$,

$$\|g\|_2 \le e^{(t-s_1)\mu(\alpha)} \|V\|_+ \|f\|_2. \tag{3.27}$$

Finally, by combining (3.26) and (3.27), the assertion is proved for $k = 2$. By iterating the argument, the inequality is verified for all $k \in N$ and the theorem is proved. ∎

THEOREM 3.7. *Let $f \in L^2$ and let the potential V satisfy $V\chi_B \in L^2$ and $V(1-\chi_B) \in L^\infty$, for some bounded B containing a neighbourhood of $O \in R^3$. Then, $\forall t > 0$,*

$$\left|E_x^\mu \left\{e^{\int_0^t V(\omega_s)\,ds} f(\omega_t)\right\}\right| \le w_t(x) K(V,t) e^{t\mu(\alpha)} \|f\|_2, \tag{3.28}$$

where, for some $c_o > 0$,

$$K = K(V,t) = c_o t^{1/4} \|V\|_+ \exp\left(t\|V\|_+\right). \tag{3.29}$$

PROOF. As in [7, p. 264], the exponential is obtained by expansion starting from the identity

$$\left(\int_0^t V(\omega_s)\,ds\right)^k = k! \int_0^t \int_{s_1}^t \cdots \int_{s_{k-1}}^t V(\omega_{s_1}) \cdots V(\omega_{s_k})\,ds_1 \cdots ds_k. \tag{3.30}$$

For any $x \ne O$,

$$\begin{aligned} G_k(x) &\equiv E_x^\mu \left\{\left(\int_0^t V(\omega_s)\,ds\right)^k f(\omega_t)\right\} \\ &= k! \int_0^t \int_{s_1}^t \cdots \int_{s_{k-1}}^t E_x^\mu \{V(\omega s_1) \cdots V(\omega_{s_k}) f(\omega_t)\}\,ds_1 \cdots ds_k. \end{aligned} \tag{3.31}$$

Thus, by Theorem 3.6, $|G_k(x)|$ is less than

$$w_t(x) k! \|V\|_+^k \|f\|_2 e^{t\mu(\alpha)} \int_0^t s_1^{-3/4} \int_{s_1}^t \cdots \int_{s_{k-1}}^t ds_1 \cdots ds_k, \tag{3.32}$$

where, $\forall k \in N$,

$$\int_0^t s_1^{-3/4} \int_{s_1}^t \cdots \int_{s_{k-1}}^t ds_1 \cdots ds_k = \int_0^t s_1^{-3/4}\,ds_1 \frac{(t-s_1)^{k-1}}{(k-1)!} \le 4t^{1/4} \frac{t^{k-1}}{(k-1)!}. \tag{3.33}$$

After expanding the exponential, by interchanging the expectation and the expansion, we finally check (3.28) and the theorem is proved. ∎

REMARK 3.8. Theorem 3.7 implies the main result of this section, i.e., the boundedness below of Schrödinger-type operators A containing a point interaction plus a negative $L^2 + L^\infty$ potential of a fairly general type. As we can see in the following proof, a lower bound of the spectrum will depend on $\|V^-\|_+$: $A \ge -E \equiv -\delta - \|V^-\|_+ - \mu(\alpha)$. Since $\|\cdot\|_+$ depends on the bounded set $B \subset R^3$, for fixed V^-, one can choose the set B so that $\|V^-\|_+$ is minimal.

THEOREM 3.9. *Let $V = V^+ - V^-$ with $V^+ \in L^2_{\text{loc}}$ and $V^-\chi_B \in L^2$, $V^-(1-\chi_B) \in L^\infty$, for some bounded domain B containing the origin of R^3. The operator $f \to P_t^V f$ defined by*

$$\left(P_t^V f\right)(x) = E_x \left\{ f(\omega_t) \exp\left[-\int_0^t V(\omega_s)\, ds \right] \right\} \tag{3.34}$$

satisfies

$$\left(f, P_t^V f\right) \le (\|f\|_2)^2 D e^{Et}, \qquad \forall f \in L^2, \quad \forall t > 0, \tag{3.35}$$

for some $D > 0$, $E > \mu(\alpha)$ independent of f and t. Hence $P_t^V = e^{-At}$, $\forall t \ge 0$, for some self-adjoint and bounded below operator A in $L^2(R^3)$.

PROOF. Bounded below potentials in L^2_{loc} have already been treated in Proposition 3.1 and do not affect the validity of (3.35), so let us take $V = -V^-$. By combining Theorem 3.7, (3.16b), and (3.13b), we have

$$\left(f, P_t^{(-V^-)} f\right) \le \|f\|_2 \|w_t\|_+ K\left(V^-, t\right) e^{t\mu(\alpha)} \|f\|_2 \le c_1 e^{\delta t} e^{t\|V^-\|_+} e^{t\mu(\alpha)} (\|f\|_2)^2. \tag{3.36}$$

Therefore, $f \to (f, P_t^V f)$ is a bounded quadratic form, and it determines a bounded self-adjoint semigroup $\{P_t^V : t \ge 0\}$ in L^2 (self-adjointness and the semigroup property can be verified as in Proposition 3.1). So the theorem is proved. ∎

Another result contained in Theorem 3.7 is an estimate near the origin (and actually everywhere) of all functions in the range of $\{P_t^V : t \ge 0\}$. We notice that such an estimate corresponds to the known behaviour in the case of a solvable model (point interaction plus a Coulomb potential with the same center, see [2]). Here the generalization regards both potentials different from the Coulomb one, and L^2-singularities different from the center of the point interaction. The following corollary states (3.29) in terms of the generators.

COROLLARY 3.10. *If $V = V^+ - V^-$ with $V^+ \in L^2_{\text{loc}}$ and $V^-\chi_B \in L^2$, $V^-(1-\chi_B) \in L^\infty$, the infinitesimal generator $-A$ of P_t^V in L^2 satisfies*

$$D(A) \subset \left\{ u \in D(H(\alpha)) : H(\alpha)u + Vu \in L^2 \right\} \quad \text{and} \quad u \in D(A) \implies Au = H(\alpha)u + Vu, \tag{3.37}$$

and, for some D and $\lambda > 0$, $\forall t \ge 0$,

$$\left| e^{-At} f(x) \right| \le D\left(1 + \frac{1}{|x|}\right) e^{t\lambda} \|f\|_2. \tag{3.38}$$

PROOF. For each $k \in N$, we define the truncated potential function

$$V_k(x) = \max\{V(x), -k\}. \tag{3.39}$$

Now we choose a positive constant E which is independent of k and which satisfies (3.35) for all the semigroups $\{P_t^{V_k} : t \ge 0\}$. If $-A_k$ denotes the infinitesimal generator of the semigroup $\{P_t^{V_k} : t \ge 0\}$ acting in L^2, and if λ is fixed so that $\lambda > E$, $-\lambda$ is in the resolvent set of A and all the A_k. In these conditions, $(\lambda + A_k)^{-1}$ is strongly convergent to $(\lambda + A)^{-1}$ as $k \to \infty$. Indeed, the essential point is the monotone convergence $P_t^{V_k}|\varphi|(x) \uparrow P_t^V|\varphi|(x)$, $\forall \varphi \in L^2$, whenever $V_k \downarrow V$ as $k \to \infty$; and such convergence occurs both a.e. and in L^2 sense: these facts are verified by use of monotone and dominated convergence theorems. Now the proof is completed as above in Corollary 3.2. Namely, let us fix $u \in D(A)$ and define v and u_k by (3.8) and (3.9). Then the limit as $k \to \infty$ of (3.10) still holds because V_k is bounded below and since we already proved (3.37) for such potential functions (Corollary 3.2). Finally, in view of these facts, the last assertion is contained in (3.28) and (3.16b), and the theorem is proved. ∎

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