



Three-Dimensional Angular Shear Deformation in Finite Elasticity

Federico Oyedeji Falope¹ · Luca Lanzoni¹ · Angelo Marcello Tarantino¹

Received: 3 April 2026 / Revised: 30 July 2026 / Accepted: 31 August 2026
© The Author(s) 2026

Abstract

Angular shear deformations are based solely on the angular variation of line elements and not on their extension. To describe kinematics, a material cube is considered that transforms into an oblique prism, where two opposite faces are equal and parallel. Using the displacement components of three vertices of the fundamental edges, positioned along the reference axes, the displacement field of the body has been obtained. Then, the deformation tensors have been determined and some general properties for this class of shear deformations have been highlighted. Using a Lagrangian analysis, the boundary value problem for the equilibrium of a cube subjected to a purely angular shear deformation has been formulated. Subsequently, Piola-Kirchhoff and Cauchy stress tensors, together with the normal and tangential Cauchy stress vector components acting on the inclined faces of the deformed cube, have been evaluated.

Keywords Finite elasticity · Hyperelasticity · Shear deformation · Pure shear · Simple shear · Compressible materials · Purely angular shear

Mathematics Subject Classification 74B20 · 74G75 · 74K10

1 Introduction

Finite angular shear deals with deformation mechanisms where the angles between line elements change significantly, meaning the linear approximations no longer hold. In these situations, parallel internal layers slide past one another, changing the shape and volume of an object.

In different physical contexts there are many practical applications characterized by large angular shear deformations. Three-dimensional shearing of crystal structures governs the

✉ Luca Lanzoni
luca.lanzoni@unimore.it

Federico Oyedeji Falope
federicooyedeji.falope@unimore.it

Angelo Marcello Tarantino
angelomarcello.tarantino@unimore.it

¹ DIEF, Università di Modena e Reggio Emilia, via P. Vivarelli 10, 41125, Modena, Italy

glide of layers of atoms, resulting in structural deformation,¹ lattice rotation, and potential phase transformations [1, 2] and [3]. This deformation mechanism is crucial in understanding plastic deformation, dislocation movement,² and the formation of shear bands³ in both atomic lattices (metals) [4] and colloidal crystals [5, 6].⁴

Structural geology measures the distortion of rocks (e.g., quartz-phyllite layers) to calculate finite shear strain, helping determine deformation mechanisms like boudinage, fault-slip, and folding [7]. Deformed fossil analysis is used in geological education and research to determine the amount of angular deformation in fossils, providing a deformed critter method to quantify rock shearing.

Elastography is a non-invasive medical imaging technique used to measure and map the elastic properties of soft tissues, where shear deformation (even large ones) influences the SNR imaging (signal-to-noise ratio) and elastic contrast [8, 9].

Shear models in finite elasticity are crucial for capturing nonlinear behavior in materials like polymers and rubber, involving large shear deformations, significant rotations, and nonlinear constitutive relations [10, 11]. Rivlin (1948) [12] was the first to propose simple shear even in the framework of finite elasticity. In simple shear, the edge of the body in the sliding direction does not change the length. This shear model has been studied for a long time and is illustrated by many scientific texts, for example, Truesdell and Noll [13], Green and Zerna [14], Ogden [15] and Gurtin, Fried and Anand [16]. Pure (or irrotational) shear is a flattening of a body, where one of the two dimensions orthogonal to the extension direction is preserved [17, 18]. There are many contributions in the literature for simple and pure shear (see, for example, Horgan and Murphy [19], Mihai and Goriely [20], Destrade, Murphy and Saccomandi [21] and Thiel et al. [22]).

Both simple and pure shear models generate (unlike the corresponding linearized elasticity models) kinematic effects due to both angular and extension deformations. This coupling can create uncertainties in the constitutive modeling and generate difficulties in the experimental investigation of the shear behavior of a material. To address these shortcomings, the authors recently proposed the purely angular shear model [23], which represents a direct extension of the linearized pure shear model to high strain levels. Unlike classical finite models, the purely angular shear model is based solely on the angular variation of line elements and not on their extension [23, 24].

Considering the angular distortions generated by a purely angular shear deformation, this paper develops an analysis to study hyperelastic bodies undergoing large three-dimensional rotational deformations. These rotational deformations are evaluated with respect to three mutually orthogonal line elements, which do not undergo extensional deformations. The lattice structure illustrated in Fig. 1, where a cube is composed of inextensible edges connected with spherical hinges, can be useful for visualizing the class of deformations that will be investigated in this paper.

¹3D crystal shearing is often modeled using methods like the Finite Element Method (FEM) to analyze deformation, or 3D crystal plasticity models to predict behavior under high-strain rates.

²3D shearing deformation is guided and accommodated by the motion (or glide) of dislocations, namely line defects in the crystal lattice, which separate areas of different inelastic shear displacement.

³In polycrystalline materials, particularly under high strain rates, shear localization leads to the formation of shear bands. Within these bands, grains undergo pronounced realignment.

⁴In colloidal crystals, shear can induce structural transitions, such as switching from a square to a triangular lattice or forming random hexagonal close-packed (RHCP) structures. High-amplitude shearing often reduces the quality of layer ordering, increasing the density of stacking faults.

Fig. 1 Cube-shaped lattice structure. Undeformed and deformed configurations

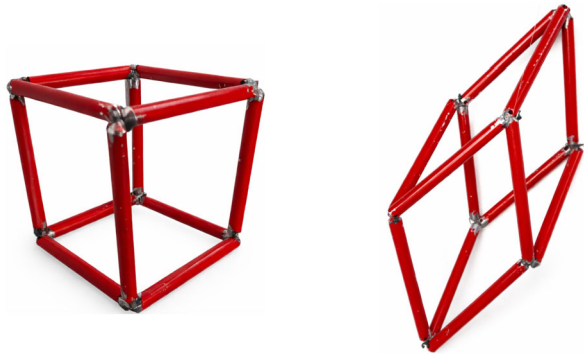
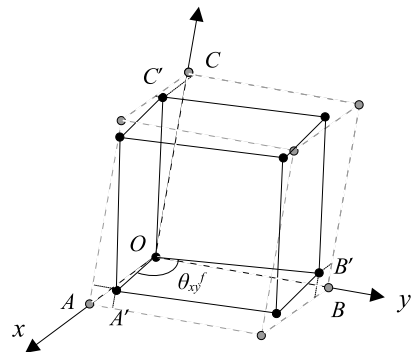


Fig. 2 Three-dimensional angular shear deformation in linearized elasticity. Note that point A moves in A' , that is, on the plane orthogonal to the X -axis that passes through A . The same thing happens for points B and C . The solid and the dashed lines indicate the deformed and undeformed configurations, respectively



In linearized elasticity, when only infinitesimal shear strains are taken into account, the symmetric infinitesimal strain tensor $\mathbf{E}$ takes the following form:⁵

$$[\mathbf{E}] = \begin{bmatrix} 0 & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & 0 & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & 0 \end{bmatrix}, \quad (1)$$

where $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$, and $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ are the infinitesimal shear strains, and u , v , and w denote displacement components. Note that, since all the angles generated by the above linearized deformation are very small, point A in Fig. 2, for example, moves to A' . The point A' belongs to the plane orthogonal to the x axis. The same thing happens to the other vertices B and C of the cube. The shear strain γ_{xy} measures the difference between the initial right angle in the xy plane and the final angle θ_{xy}^f formed by the two deformed line elements belonging to a different plane (cf. Fig. 2): $\gamma_{xy} = \frac{\pi}{2} - \theta_{xy}^f$ (expressed in radians). The shear strain γ_{xy} is positive when the initial angle decreases. The diagonal components of tensor $\mathbf{E}$ are zero because the line elements, along the reference axes, do not undergo length variations during deformation. The strain tensor $\mathbf{E}$, in form (1), with all diagonal components zero, is used to define a *state of pure shear strain*.

⁵Strain tensor $\mathbf{E}$ is set in the material configuration and reference is made to a Cartesian coordinate system $\{O, x, y, z\}$ (see, Fig. 2). The displacement components u , v , w are evaluated with respect to this coordinate system.

Substituting the strain tensor (1) into the linear elastic isotropic Navier constitutive law: $\mathbf{T} = 2\mu\mathbf{E} + \lambda\vartheta\mathbf{I}$, where μ and λ are the Lamé constants, $\mathbf{I}$ is the identity tensor, and being $\vartheta = \text{tr}\mathbf{E} = 0$ (i.e. absence of volume variation), the stress tensor $\mathbf{T}$ is immediately evaluated

$$[\mathbf{T}] = \begin{bmatrix} 0 & T_{xy} & T_{xz} \\ T_{xy} & 0 & T_{yz} \\ T_{xz} & T_{yz} & 0 \end{bmatrix}. \quad (2)$$

This stress tensor $\mathbf{T}$, with the tangential stress components non-zero and with the normal stress components all equal to zero, represents a *state of pure shear stress*.

Therefore, a state of pure shear stress corresponds to a state of pure shear strain and vice versa. As we will see below, this property does not hold in finite elasticity.

In this paper, the angular shear problem of the linearized elasticity, mentioned above, is extended to the fully nonlinear context of finite elasticity. The assumption of small deformations is removed, and the problem is formulated for large shear deformations. The paper is organized as follows. After some reminders and positions, Sect. 2 initially addresses the kinematics of three-dimensional angular shear. In detail, the displacement field, the displacement gradient, the right Cauchy-Green tensor, and the Green-St. Venant tensor are obtained. A Lagrangian analysis is then performed in order to formulate the boundary value problem for a material cube. The (first) Piola-Kirchhoff and Cauchy stress fields were subsequently determined. In particular, the Cauchy stress vector for the deformed faces of the cube is also evaluated. As a special case, Sect. 3 studies equitriaxial deformation, namely the deformation that transforms the cube into a rhombohedron, that is, a polyhedron with six equal rhombus-shaped faces. Section 4 closes the paper by summarizing the results obtained.

2 Purely Angular Shear Deformation in Finite Elasticity

Let us consider a hyperelastic body $\bar{\mathcal{B}}$ composed of a homogeneous, isotropic and compressible material under homogeneous deformation. Reference is made to a Cartesian coordinate system $\{O, X, Y, Z\}$ to identify the body with the closure of a regular domain of the three-dimensional Euclidean space $\mathcal{E}$.

The undeformed configuration $\bar{\mathcal{B}}$ of the body is assumed as the reference configuration, whereas the deformed configuration is given by the deformation $\mathbf{f}: \bar{\mathcal{B}} \rightarrow \mathcal{V}$.⁶ As usual, deformation $\mathbf{f}$ is a smooth enough, injective and orientation-preserving vector field (in the sense that $\det(\text{Grad } \mathbf{f}) > 0$). The deformation of a generic material point P can be expressed by the well-known relationship

$$\mathbf{f}(P) = \mathbf{s}(P) + \mathbf{id}(P), \quad (3)$$

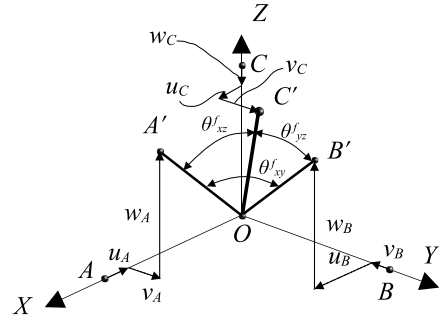
where $\mathbf{id}(P)$ is the position vector, which indicates the position of the material point P before the deformation, and

$$\mathbf{s}(P) = u(P)\mathbf{i} + v(P)\mathbf{j} + w(P)\mathbf{k}, \quad (4)$$

denotes the displacement vector. In expression (4), the functions $u(P)$, $v(P)$ and $w(P)$ are the scalar components of $\mathbf{s}(P)$, whereas $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors. The application of

⁶ $\mathcal{V}$ is the vector space associated with $\mathcal{E}$.

Fig. 3 Displacements of linear elements laid out along the reference axes



the material gradient operator $\text{Grad}(\cdot)$ to (3) gives

$$\mathbf{F} = \mathbf{H} + \mathbf{I}, \quad (5)$$

where $\mathbf{F}: \bar{\mathcal{B}} \rightarrow \text{Lin}^+$ and $\mathbf{H}: \bar{\mathcal{B}} \rightarrow \text{Lin}$ ⁽⁷⁾ are the deformation and displacement gradients, respectively.

Because homogeneous deformations will be considered, the shape of the body is irrelevant for the analysis. However, for convenience's sake, we assign the shape of a cube to the body $\mathcal{B}$

$$\mathcal{B} = \{(X, Y, Z) \mid 0 < X < a, 0 < Y < a, 0 < Z < a\},$$

where a is the length of the edge.⁸

2.1 General Kinematics

Due to angular shear deformation, a material cube transforms into an oblique prism with six faces, where the opposite faces are equal (congruent) and parallel. Since the edges along the three reference axes do not undergo axial deformation, the lateral faces of the cube transform from squares to rhombuses.

Let's first refer to the three edges of the cube arranged along the reference axes, for which the following vector representations are valid in the reference configuration (cf. Fig. 3):⁹

$$\begin{aligned} (A - O) &= a\mathbf{i} \\ (B - O) &= a\mathbf{j} \\ (C - O) &= a\mathbf{k} \end{aligned} \quad (6)$$

Assuming that the origin O remains fixed, after the deformation, the three vertices A , B , and C of the cube assume the positions A' , B' , and C' , shown in Fig. 3. Using the displacement components of the three vertices, the deformed edges can be written as

$$\begin{aligned} (A' - O) &= (a - u_A)\mathbf{i} + v_A\mathbf{j} + w_A\mathbf{k} \\ (B' - O) &= u_B\mathbf{i} + (a - v_B)\mathbf{j} + w_B\mathbf{k} \\ (C' - O) &= u_C\mathbf{i} + v_C\mathbf{j} + (a - w_C)\mathbf{k} \end{aligned} \quad (7)$$

⁷ Lin is the set of all (second order) tensors, whereas Lin^+ is the subset of tensors with positive determinant.

⁸If the edge of the cube is reduced to an infinitesimal, the entire analysis carried out in this paper becomes "local", that is, valid for a single material point even under conditions of non-homogeneous deformation.

⁹In the following notation, a vector is also given as the difference between two points.

In (7), the axial displacement components u_A , v_B , and w_C have been assumed to be negative, also in view of the conservation of the segment length, while all the other components have been assumed to be positive. The nine components that describe the displacements of the three vertices are linked by the following three relations derived on the basis of the Pythagorean theorem:

$$\begin{aligned}(a - u_A)^2 + v_A^2 + w_A^2 &= a^2 \\ u_B^2 + (a - v_B)^2 + w_B^2 &= a^2 \\ u_C^2 + v_C^2 + (a - w_C)^2 &= a^2\end{aligned}\quad (8)$$

The angles θ_{xy}^f , θ_{xz}^f and θ_{yz}^f subtended by the deformed edges can be determined by using the scalar product (cf. Fig. 3)

$$\begin{aligned}\cos \theta_{xy}^f &= \frac{(A'-O) \cdot (B'-O)}{|(A'-O)| |(B'-O)|} \\ \cos \theta_{xz}^f &= \frac{(A'-O) \cdot (C'-O)}{|(A'-O)| |(C'-O)|} \\ \cos \theta_{yz}^f &= \frac{(B'-O) \cdot (C'-O)}{|(B'-O)| |(C'-O)|}\end{aligned}\quad (9)$$

With (7) and being $\cos \theta_{ij}^f = \cos(\frac{\pi}{2} - \gamma_{ij}) = \sin \gamma_{ij}$, with $i, j = x, y$ and z , the previous expressions (9) transform into

$$\begin{aligned}\sin \gamma_{xy} &= \frac{(a - u_A)u_B + (a - v_B)v_A + w_A w_B}{a^2} \\ \sin \gamma_{xz} &= \frac{(a - u_A)u_C + v_A v_C + w_A (a - w_C)}{a^2} \\ \sin \gamma_{yz} &= \frac{u_B u_C + (a - v_B)v_C + w_B (a - w_C)}{a^2}\end{aligned}\quad (10)$$

Each edge is parallel to four edges of the cube, which remain parallel even after deformation. Therefore, based on the displacements of the three fundamental edges, positioned along the reference axes, we can derive the displacements of all twelve edges of the cube. Moreover, since the deformation is assumed to be homogeneous, the displacement field varies linearly in the material variables X , Y and Z . Thus, still using the displacements of the three vertices, the displacement field is

$$\begin{cases} u(X, Y, Z) = -u_A \frac{X}{a} + u_B \frac{Y}{a} + u_C \frac{Z}{a} \\ v(X, Y, Z) = v_A \frac{X}{a} - v_B \frac{Y}{a} + v_C \frac{Z}{a} \\ w(X, Y, Z) = w_A \frac{X}{a} + w_B \frac{Y}{a} - w_C \frac{Z}{a} \end{cases} \quad (11)$$

System (11) gives the displacements undergone by each point of the cube.

The geometric shape of the deformed cube is described by the Eulerian coordinates x, y, z , which are derived using (3) and the displacement field (11)

$$\begin{cases} x = X - u_A \frac{X}{a} + u_B \frac{Y}{a} + u_C \frac{Z}{a} \\ y = Y + v_A \frac{X}{a} - v_B \frac{Y}{a} + v_C \frac{Z}{a} \\ z = Z + w_A \frac{X}{a} + w_B \frac{Y}{a} - w_C \frac{Z}{a} \end{cases} \quad (12)$$

From the displacement field (11), the displacement gradient $\mathbf{H}$ and subsequently the deformation gradient $\mathbf{F} = \mathbf{H} + \mathbf{I}$ can be computed

$$[\mathbf{F}] = \frac{1}{a} \begin{bmatrix} (a - u_A) & u_B & u_C \\ v_A & (a - v_B) & v_C \\ w_A & w_B & (a - w_C) \end{bmatrix}. \quad (13)$$

Obviously, $\mathbf{F}$ is a constant tensor, which does not depend on the material variables.¹⁰ Once $\mathbf{F}$ is known, the (symmetric and positive definite) right Cauchy-Green tensor, $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, can be obtained

$$\begin{aligned} a^2 C_{11} &= (a - u_A)^2 + v_A^2 + w_A^2 \\ a^2 C_{12} &= (a - u_A) u_B + v_A (a - v_B) + w_B w_A \\ a^2 C_{13} &= (a - u_A) u_C + v_A v_C + w_A (a - w_C) \\ a^2 C_{21} &= u_B (a - u_A) + (a - v_B) v_A + w_B w_A \\ a^2 C_{22} &= u_B^2 + (a - v_B)^2 + w_B^2 \\ a^2 C_{23} &= u_B u_C + (a - v_B) v_C + w_B (a - w_C) \\ a^2 C_{31} &= u_C (a - u_A) + v_C v_A + (a - w_C) w_A \\ a^2 C_{32} &= u_C u_B + v_C (a - v_B) + (a - w_C) w_B \\ a^2 C_{33} &= u_C^2 + v_C^2 + (a - w_C)^2 \end{aligned} \quad (14)$$

Taking into account (8) and (10), tensor $\mathbf{C}$ can be rewritten in the following compact form:

$$[\mathbf{C}] = \begin{bmatrix} 1 & \sin \gamma_{xy} & \sin \gamma_{xz} \\ \sin \gamma_{xy} & 1 & \sin \gamma_{yz} \\ \sin \gamma_{xz} & \sin \gamma_{yz} & 1 \end{bmatrix}. \quad (15)$$

Given the polar decomposition of the deformation gradient, $\mathbf{C} = \mathbf{U}^2$, where $\mathbf{U}$ denotes the (symmetric and positive definite) right stretch tensor. The eigenvalues of $\mathbf{U}$ are the principal stretches λ_1 , λ_2 and λ_3 . Thus, the eigenvalues of $\mathbf{C}$ are the squares of the principal stretches. The expressions for the eigenvalues of $\mathbf{C}$ are given in the [Appendix](#). Obviously, in the principal reference system there are only extensions and no angular variations.

With the aim of eliminating the rigid part of the deformation, we move from the tensor $\mathbf{C}$ to Green-St. Venant tensor $\mathbf{E}$, defined as follows: $2\mathbf{E} = \mathbf{C} - \mathbf{I}$, which takes into account only the pure deformation

$$[\mathbf{E}] = \frac{1}{2} \begin{bmatrix} 0 & \sin \gamma_{xy} & \sin \gamma_{xz} \\ \sin \gamma_{xy} & 0 & \sin \gamma_{yz} \\ \sin \gamma_{xz} & \sin \gamma_{yz} & 0 \end{bmatrix}. \quad (16)$$

The normal components are all zero, because the edges of the cube do not undergo variations in length, while the off-diagonal components constitute a nonlinear measure of the shear deformations due exclusively to angular variations. Invoking the hypothesis of smallness of the displacement gradient $\mathbf{H}$ (which assumes that all nine (dimensionless) components of the tensor $\mathbf{H}$ satisfy the following inequality: $|\mathbf{H}_{ij}| \ll 1$, with $i, j = x, y$ and z , the linearization of the Green-St. Venant tensor provides the linearized strain tensor. Therefore, Green-St. Venant tensor $\mathbf{E}$ is suitable for describing the large/small transition that the shear deformation components undergo when passing from the nonlinear to linearized elasticity [25, 26]. For this reason, to denote the Green-St. Venant tensor we use the same symbol $\mathbf{E}$ as the strain tensor. In particular, under the hypothesis of small deformations, the components of $\mathbf{E}$, in the form (16), can be simplified being $\sin \gamma \simeq \gamma$. Thus, the Green-St. Venant tensor (16) reduces exactly to the infinitesimal strain tensor (1).

Another aspect needs to be emphasized. Through angular shear deformation, it is possible to define a *state of pure shear deformation in finite elasticity*, for which $\text{tr}(\mathbf{E}) = 0$, as is done in linearized elasticity.

¹⁰Since the deformation is homogeneous, the tensor $\mathbf{F}$ can be used to evaluate the transformation of even non-infinitesimal material fibres, for example, $(A' - O) = \mathbf{F}(A - O)$.

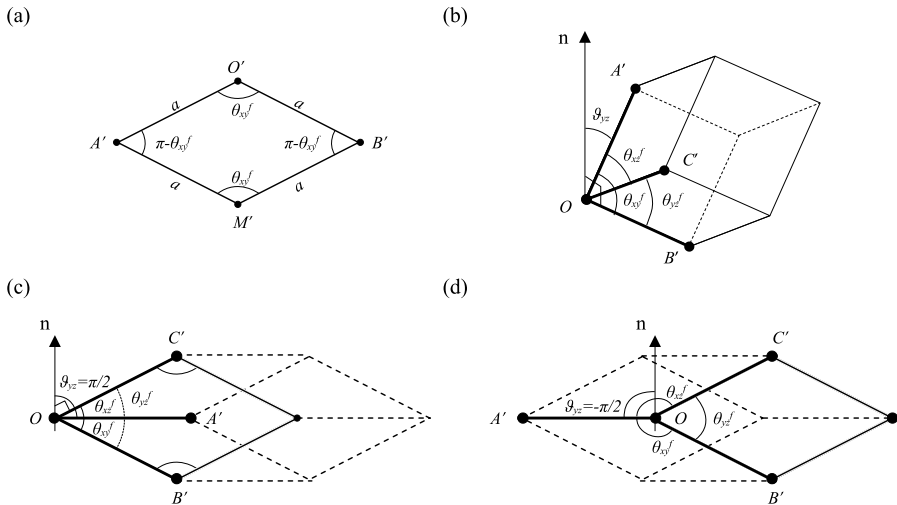


Fig. 4 Geometric details of the deformed cube. Internal angles of a face (a). Application of the scalar triple product to determine the signed volume (b). Two limit situations characterized by zero scalar triple product: $\vartheta_{yz} = \frac{\pi}{2}$ (c) and $\vartheta_{yz} = -\frac{\pi}{2}$ (d)

2.2 Transformation of Angles, Surfaces, and Volumes

Excluding the physically implausible case of the inversion of sign of the volume of the cube, as a result of the deformation, the following inequality must exist between the final angles at vertex O :

$$\theta_{xy}^f + \theta_{xz}^f + \theta_{yz}^f < 2\pi. \quad (17)$$

Equality holds in the unrealistic case where the cube collapses into a surface.

Figure 4a shows the shape that the base of the cube, initially belonging to the XY plane, assumes once deformed. As mentioned several times, it is a rhombus, that is, a polygon with four equal edges and congruent opposite angles. We are referring to this face, but similar considerations apply to the other five faces of the deformed cube. Angle θ_{xy}^f can present two limiting situations: $\theta_{xy}^f = 0$ and $\theta_{xy}^f = \pi$. Both represent (unrealistic) extreme deformations, whereby the rhombus is reduced to a segment of length $2a$. Therefore, to avoid the collapse of the rhombuses it must be

$$0 < \theta^f < \pi. \quad (18)$$

To determine the volume of the cube before and after deformation, v_i and v_f , the scalar triple product can be used

$$v_i = (A - O) \cdot (B - O) \wedge (C - O) = a^3, \quad (19)$$

$$v_f = (A' - O) \cdot (B' - O) \wedge (C' - O). \quad (20)$$

The ratio between the final and initial volume is equal to the determinant of the deformation gradient, $\det \mathbf{F} = v_f/v_i$, as can be easily checked in terms of displacement components using (6), (7) and (13).

The cross product in (20) gives

$$(B' - O) \wedge (C' - O) = (a^2 \sin \theta_{yz}^f) \mathbf{n}', \quad (21)$$

where $\mathbf{n}'$ denotes the unit vector defined in the deformed configuration

$$\mathbf{n}' = \frac{(B' - O) \wedge (C' - O)}{|(B' - O) \wedge (C' - O)|}, \quad (22)$$

whose direction is determined by the right-hand rule, while magnitude $(a^2 \sin \theta_{yz}^f)$ can be thought of as the base area. Since the scalar triple product remains unchanged under cyclic permutations of the vectors, other choices can be made for the base area. With (21) and calculating the dot product in (20), the final volume becomes

$$v_f = a^3 \cos \vartheta_{yz} \sin \theta_{yz}^f. \quad (23)$$

The angle ϑ_{yz} between the unit vector and the first vector $(A' - O)$ is shown in Fig. 4b. Two limiting situations can arise for angle ϑ_{yz} , in which $\det \mathbf{F}$ becomes zero and the three deformed vectors in (20) are coplanar. For $\vartheta_{yz} = \frac{\pi}{2}$, the angle θ_{yz}^f equals the sum of the other two vertex angles: $\theta_{yz}^f = \theta_{xz}^f + \theta_{xy}^f$ (cf. Fig. 4c). For $\vartheta_{yz} = -\frac{\pi}{2}$, the sum of the vertex angles equals the round angle: $\theta_{yz}^f + \theta_{xz}^f + \theta_{xy}^f = 2\pi$ (cf. Fig. 4d). Situation also excluded by (17). Therefore, v_f is positive when $-\frac{\pi}{2} < \vartheta_{yz} < \frac{\pi}{2}$ and zero for $\vartheta_{yz} = \pm \frac{\pi}{2}$. For $\vartheta_{yz} < -\frac{\pi}{2}$ and $\vartheta_{yz} > \frac{\pi}{2}$, v_f becomes negative and the orientation of the reference system changes. For $\vartheta_{yz} \leq -\frac{\pi}{2}$ and $\vartheta_{yz} \geq \frac{\pi}{2}$, deformations are physically inadmissible.

Given (23), it turns out

$$\det \mathbf{F} = \cos \vartheta_{yz} \sin \theta_{yz}^f, \quad (24)$$

and since it is the product of two trigonometric functions, it must necessarily be $\det \mathbf{F} \leq 1$. $\det \mathbf{F}$ is equal to one if $\vartheta_{yz} = 0$ and $\theta_{yz}^f = \frac{\pi}{2}$, that is, when the cube retains its shape and volume, namely, in the case of a rigid rotation. So ultimately, $0 < \det \mathbf{F} \leq 1$. From (24) it is also evident that the maximum volume is that of the cube in the initial configuration and then, due to the (pure) deformation, it always decreases.

2.3 Stresses and Boundary Value Problem

The constitutive properties of a hyperelastic material are described by the stored energy function ω . If the function ω is frame-indifferent, homogeneous and isotropic, then it depends only on the principal invariants I_i , with $i = 1, 2$ and 3 , of the Cauchy-Green deformation tensors. With these assumptions, the constitutive law ($\mathbf{T}_R = \partial \omega / \partial \mathbf{F}$) takes the following form [27, 28]:

$$\mathbf{T}_R = 2 \left(\frac{\partial \omega}{\partial I_1} + I_1 \frac{\partial \omega}{\partial I_2} \right) \mathbf{F} - 2 \frac{\partial \omega}{\partial I_2} \mathbf{B} \mathbf{F} + 2 I_3 \frac{\partial \omega}{\partial I_3} \mathbf{F}^{-T}, \quad (25)$$

where the tensor $\mathbf{T}_R$ denotes the (first) Piola-Kirchhoff stress tensor. In the above expression, $\mathbf{B} = \mathbf{F} \mathbf{F}^T$ is the left Cauchy-Green strain tensor and $\mathbf{B} \mathbf{F} = \mathbf{F} \mathbf{C}$, while the principal invariants of the Cauchy-Green tensors hold

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2, \\ I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2. \end{aligned} \quad (26)$$

Using (13) and (15), the Piola-Kirchhoff stresses generated by an angular shear deformation can be determined.

Performing a Lagrangian analysis, it is easy to formulate the boundary value problem. Being the deformation homogeneous, the stresses also do not vary. That is, these stresses are the same for every internal point of the cube and for its boundary. In particular, the stresses at the boundary can be interpreted as the external surface forces necessary to produce the displacement field (11).

In the undeformed configuration, equilibrium conditions require the following vectorial equation to be satisfied locally:

$$\text{Div } \mathbf{T}_R + \mathbf{b} = \mathbf{0}, \quad (27)$$

where the symbol $\mathbf{b}$ denotes the body forces. The boundary conditions, to be added to the previous field equation, are

$$\mathbf{t}_R = \mathbf{T}_R \mathbf{n}, \quad (28)$$

where $\mathbf{t}_R$ is the Piola-Kirchhoff stress vector and $\mathbf{n}$ the outward unit vector. These stress vectors are known and uniformly applied to the six faces of the cube, with opposite values in the three pairs of opposite faces. In this way, the global equilibrium of the cube in its undeformed configuration is guaranteed. For the three faces of the undeformed cube with positive normal, (28) gives

$$[\mathbf{t}_R] = \begin{bmatrix} T_{R_{xx}} \\ T_{R_{yx}} \\ T_{R_{zx}} \end{bmatrix}, [\mathbf{t}_R] = \begin{bmatrix} T_{R_{xy}} \\ T_{R_{yy}} \\ T_{R_{zy}} \end{bmatrix}, [\mathbf{t}_R] = \begin{bmatrix} T_{R_{xz}} \\ T_{R_{yz}} \\ T_{R_{zz}} \end{bmatrix}, \quad (29)$$

for $X = a$ and $0 \leq Y, Z \leq a$, $Y = a$ and $0 \leq X, Z \leq a$, $Z = a$ and $0 \leq X, Y \leq a$, respectively. For the three faces with negative normal, there are values opposite to the previous ones.

By setting $\mathbf{b} = \mathbf{0}$, the three scalar differential equations given by (27) are automatically satisfied, because Piola-Kirchhoff stresses are constant. While the boundary conditions (29) provide a coupled nonlinear system of nine equations to evaluate the parameters present in the displacement field (11).

From Piola-Kirchhoff stresses (to be understood as nominal) can be determined the Cauchy stresses, namely the corresponding stresses in the deformed configuration (to be understood as true). In fact, the Piola-Kirchhoff stress tensor $\mathbf{T}_R$ and Cauchy stress tensor $\mathbf{T}$ are related by the following well-known tensorial relationship:

$$\mathbf{T}_R = \mathbf{T} \mathbf{F}^*, \quad (30)$$

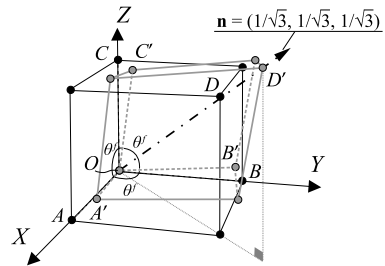
where $\mathbf{F}^* = (\det \mathbf{F}) \mathbf{F}^{-T}$ is the cofactor of the tensor $\mathbf{F}$.¹¹ A similar definition to (28) subsists for the Cauchy stress vector $\mathbf{t}$,

$$\mathbf{t} = \mathbf{T} \mathbf{n}', \quad (31)$$

where the unit vector $\mathbf{n}'$ is defined in the deformed configuration. With the Cauchy stress vector $\mathbf{t}$ it is possible to evaluate the stresses acting on the deformed faces of the cube. In

¹¹The tensor $\mathbf{F}$ is invertible by its definition.

Fig. 5 Equitriaxial angular shear deformation



fact, with reference to a deformed face, the unit vector $\mathbf{n}'$ is orthogonal to the face itself, and a second unit vector $\mathbf{m}'$, which is any unit vector parallel to the plane of the face, can be introduced. These two unit vectors are orthogonal to each other, that is $\mathbf{n}' \cdot \mathbf{m}' = 0$. Using (31), the normal and (global) tangential Cauchy stress vector components can be calculated

$$\mathbf{t}_{n'} = \mathbf{n}' \cdot \mathbf{T} \mathbf{n}', \quad (32)$$

$$\mathbf{t}_{n'm'} = \mathbf{m}' \cdot \mathbf{T} \mathbf{n}'. \quad (33)$$

The first observation that can be made is that, for a generic expression of the stored energy function ω , the symmetric Cauchy stress tensor $\mathbf{T}$ is full, that is, it has no zero normal components.¹² The Green-St. Venant tensor (16), with zero normal components, does not correspond to a Cauchy stress tensor with zero normal components. Therefore, for angular shear, *the state of pure shear deformation* (defined by the form of tensor (16)) *does not correspond to the state of pure shear stress*.¹³ This is different from what happens in linearized elasticity, where the two pure states are always associated with each other (cf. (1) and (2)). A second observation always concerns the differences with linearized elasticity. A shear strain in linearized elasticity is produced exclusively by tangential stresses. *This is not true for an angular shear deformation, which is produced by the simultaneous application of tangential and normal stresses* (cf. (32)).

3 Transformation of a Cube into a Rhombohedron (Equitriaxial Angular Shear Deformation Case)

Suppose the cube is deformed by lengthening or shortening the trisector¹⁴ and keeping the origin fixed. In this way, the deformed cube takes on the shape of a rhombohedron, characterized by six identical rhombus-shaped faces. This equitriaxial angular shear deformation is illustrated in Fig. 5. Note the angles at vertex O which are all equal. Under these conditions, the trisector is a principal direction of deformation. The symbol λ_1 will be used for the corresponding principal stretch. In a plane orthogonal to the trisector, the directions exiting

¹²Given the shape of the deformation gradient $\mathbf{F}$ (13), the Piola-Kirchhoff stress tensor (25) has no zero normal components. Through formula (30), $\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{T}_R \mathbf{F}^T$, the Cauchy stress tensor also has no zero normal components.

¹³From a different point of view, in [21] it was shown that in general a state of pure shear stress does not correspond to a state of pure shear deformation.

¹⁴The unit vector of the direction along the trisector is $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

the intersection point are also all principal. The stretch along these principal directions will be denoted by λ .

Given the axisymmetry around the trisector, the displacements of vertices A , B and C are

$$\begin{aligned} u_A &= v_B = w_C = s, \\ v_A &= w_A = u_B = w_B = u_C = v_C = r. \end{aligned} \quad (34)$$

With these displacements, the deformed edges (7) are simplified into

$$\begin{aligned} (A' - O) &= (a - s)\mathbf{i} + r\mathbf{j} + r\mathbf{k} \\ (B' - O) &= r\mathbf{i} + (a - s)\mathbf{j} + r\mathbf{k} \\ (C' - O) &= r\mathbf{i} + r\mathbf{j} + (a - s)\mathbf{k} \end{aligned} \quad (35)$$

Conservation of the length of the edges of the cube gives the following unique relation:

$$(a - s)^2 + 2r^2 = a^2. \quad (36)$$

Using (35), the following shear deformations are calculated:

$$\sin \gamma_{xy} = \sin \gamma_{xz} = \sin \gamma_{yz} = \sin \gamma = \frac{2r(a - s) + r^2}{a^2}. \quad (37)$$

With (34), the displacement field (11) becomes

$$\begin{cases} u(X, Y, Z) = -s\frac{X}{a} + r\frac{Y}{a} + r\frac{Z}{a} \\ v(X, Y, Z) = r\frac{X}{a} - s\frac{Y}{a} + r\frac{Z}{a} \\ w(X, Y, Z) = r\frac{X}{a} + r\frac{Y}{a} - s\frac{Z}{a} \end{cases} \quad (38)$$

From (38), the deformation gradient and the Green-St. Venant tensor are obtained

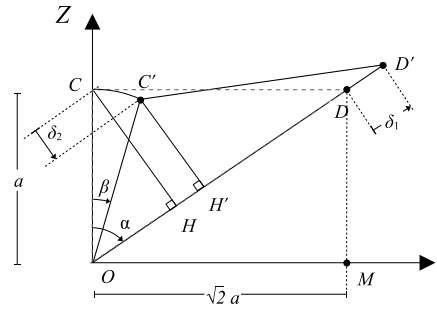
$$\begin{aligned} [\mathbf{F}] &= \frac{1}{a} \begin{bmatrix} (a - s) & r & r \\ r & (a - s) & r \\ r & r & (a - s) \end{bmatrix}, \\ [\mathbf{E}] &= \frac{1}{2} \begin{bmatrix} 0 & \sin \gamma & \sin \gamma \\ \sin \gamma & 0 & \sin \gamma \\ \sin \gamma & \sin \gamma & 0 \end{bmatrix}. \end{aligned} \quad (39)$$

From the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{R}$ is a proper orthogonal tensor and denotes the rotation tensor, we see that, since $\mathbf{F}$ is symmetric, $\mathbf{R} = \mathbf{I}$, namely absence of rigid deformations, and $\mathbf{F} = \mathbf{U}$. In the tensor $\mathbf{E}$ now only one deformation parameter appears.

The undeformed OCD triangle, shown in Fig. 6, has the following geometric dimensions: $OC = a$, $CD = a\sqrt{2}$, $OD = a\sqrt{3}$, $CH = a \sin \alpha$, and $OH = a \cos \alpha$, with $\sin \alpha = \sqrt{2/3}$ and $\cos \alpha = 1/\sqrt{3}$. After deformation, the diagonal OD elongates by the amount δ_1 , the height CH moves parallel to $C'H'$ (being a principal direction) and contracts by the amount δ_2 (cf. Fig. 6). Moreover, $OC = OC' = a$, $C'H' = \lambda a \sin \alpha$, $OH' = \lambda_1 a \cos \alpha$ and the angle at C' is no longer orthogonal. Between the displacements δ_1 , δ_2 and the principal stretches λ_1 , λ the following relations hold:

$$\delta_1 = a\sqrt{3}(\lambda_1 - 1), \quad \delta_2 = a(1 - \lambda)\sin \alpha. \quad (40)$$

Fig. 6 Deformation of the OCD triangle belonging to the diagonal cross-section of the cube



Applying the Pythagorean theorem to triangle $OC'H'$, the following relationship between the principal stretches λ_1 and λ is derived:

$$(\lambda_1^2 + 2\lambda^2) = 3. \quad (41)$$

Assigning λ_1 , the previous relation (41) provides λ . For the two-dimensional shear deformation, a formula equivalent to (41): $(\lambda_1^2 + \lambda_2^2) = 2$, was obtained in [23].

The squared principal stretches can be evaluated as eigenvalues of the right Cauchy-Green tensor $\mathbf{C}$. A third-degree characteristic equation is obtained, which, solved with Cardano's formula, admits three real roots, one of which is double. From these calculations, the following values of the principal stretches are obtained:

$$\lambda_1 = \sqrt{1 + 2 \sin \gamma}, \quad \lambda = \sqrt{1 - \sin \gamma}. \quad (42)$$

Note that they satisfy the relation (41).

The deformational effects of the purely angular shear have been illustrated previously using only the Green-St. Venant tensor $\mathbf{E}$. Obviously, other strain tensors can be used. An interesting alternative is the Biot strain tensor

$$\mathbf{E}_{Biot} = \mathbf{U} - \mathbf{I}. \quad (43)$$

In terms of principal stretches λ_i and their corresponding principal directions $\mathbf{n}_i^p$, the Biot tensor (43) can be represented by its spectral decomposition as

$$\mathbf{E}_{Biot} = (\lambda_i - 1) \mathbf{n}_i^p \otimes \mathbf{n}_i^p, \quad (44)$$

where $i = 1, 2, 3$, and the repeated index convention has been used. In the equitriaxial angular shear, $\mathbf{F} = \mathbf{U}$ and the reference system adopted is not principal.¹⁵ As mentioned above, the principal directions coincide with the direction of the trisector and with any two directions orthogonal to each other and parallel to a plane orthogonal to the trisector (the characteristic equation has a double root). The principal stretches are indicated by (42). Obviously, in the diagonal representation (44), a state of pure shear deformation is less apparent (because it occurs for a different reference system), however (44) has the advantage of being able to be used in a stretch-based formulation, which allows to define the constitutive behavior of a material directly using principal stretches rather than their squared values or strain tensors (see, for example, [29–33], and [34]).

¹⁵The tensor $\mathbf{F} = \mathbf{U}$ contains several pieces of information. In particular, $\|\mathbf{U}_i\| = \|\mathbf{U}_j\| = \|\mathbf{U}_k\| = 1$ and $(\mathbf{U}_i) \cdot (\mathbf{U}_j) = (\mathbf{U}_i) \cdot (\mathbf{U}_k) = (\mathbf{U}_j) \cdot (\mathbf{U}_k) = \sin \gamma$. Specifically, the length of the edges of the cube does not change and the deformation is a purely angular equitriaxial shear deformation.

For small deformations, the Biot strain tensor linearly approximates the infinitesimal strain tensor. In fact, with the linearization of (39)₁, being $s \simeq 0$ and $\frac{r}{a} \simeq \frac{1}{2}\gamma$, (43) provides

$$[\mathbf{E}_{Biot}] \simeq \begin{bmatrix} 0 & \frac{1}{2}\gamma & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & \frac{1}{2}\gamma & 0 \end{bmatrix}, \quad (45)$$

namely the strain tensor (1) for the equitriaxial case.

Let's now calculate the stresses. In the case of the equitriaxial angular shear, the principal invariants of the Cauchy-Green tensors (26) become

$$\begin{aligned} I_1 &= 3, \\ I_2 &= 3(1 - \sin^2 \gamma), \\ I_3 &= 1 - 3 \sin^2 \gamma + 2 \sin^3 \gamma. \end{aligned} \quad (46)$$

From the deformation gradient (39)₁ the following components, useful for the computation of the Piola-Kirchhoff stresses using (25), are obtained:

$$\begin{aligned} [\mathbf{BF}]_{XX} &= [\mathbf{BF}]_{YY} = [\mathbf{BF}]_{ZZ} = \frac{1}{a}[(a-s) + 2r \sin \gamma], \\ [\mathbf{BF}]_{XY} &= [\mathbf{BF}]_{XZ} = [\mathbf{BF}]_{YX} = [\mathbf{BF}]_{YZ} = [\mathbf{BF}]_{ZX} = [\mathbf{BF}]_{ZY} = \\ &= \frac{1}{a}[(a-s) \sin \gamma + r(1 + \sin \gamma)], \end{aligned} \quad (47)$$

and

$$\begin{aligned} [\mathbf{F}^{-1}]_{XX} &= [\mathbf{F}^{-1}]_{YY} = [\mathbf{F}^{-1}]_{ZZ} = \frac{1}{a^2 \det \mathbf{F}}[(a-s)^2 - r^2], \\ [\mathbf{F}^{-1}]_{XY} &= [\mathbf{F}^{-1}]_{XZ} = [\mathbf{F}^{-1}]_{YX} = [\mathbf{F}^{-1}]_{YZ} = [\mathbf{F}^{-1}]_{ZX} = [\mathbf{F}^{-1}]_{ZY} = \\ &= \frac{1}{a^2 \det \mathbf{F}}[r^2 - r(a-s)], \end{aligned} \quad (48)$$

with $\det \mathbf{F} = \frac{1}{a^3}[(a-s)^3 - 3r^2(a-s) + 2r^3] = (1 - \sin \gamma) \sqrt{1 + 2 \sin \gamma}$.

From (39)₁, (47) and (48), it is evident that the Piola-Kirchhoff stress tensor, independently from the shape of the stored energy function, is characterized by normal components which are all equal and by tangential components which are also all equal. There is no reason to suppose that the normal components can be zero.

To evaluate the Piola-Kirchhoff stresses using the constitutive law (25), it should be assigned a specific law to the stored energy function ω . For it the compressible Mooney-Rivlin form is assumed¹⁶

$$\omega(I_1, I_2, I_3) = \tilde{a} I_1 + \tilde{b} I_2 + \tilde{c} I_3 - \left(\tilde{a} + 2\tilde{b} + \tilde{c} \right) \ln I_3, \quad (49)$$

where the constants $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are strictly positive quantities. It is well known that the above stored energy function, which depends on all three deformation invariants, describes properly the constitutive behavior of rubbers and rubber-like materials.¹⁷ From (49) the following

¹⁶This function is polyconvex and satisfies the growth conditions: $\omega \rightarrow \infty$ as $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$. It was used, for example, in [25, 26, 35] and [36].

¹⁷For an extensive discussion of the terms related to the volume change in the stored energy function see the recent paper by M. Pellicciari et al. [35].

derivatives can be computed:

$$\omega_{,1} = \tilde{a}, \quad \omega_{,2} = \tilde{b}, \quad \omega_{,3} = \tilde{c} - \frac{\tilde{a} + 2\tilde{b} + \tilde{c}}{I_3}, \quad (50)$$

where $\omega_{,i} = \frac{\partial \omega}{\partial I_i}$, for $i = 1, 2$ and 3 .

Using (25), (39)₁, (46), (47), (48), and (50), the Piola-Kirchhoff stresses are determined

$$\begin{aligned} [\mathbf{T}_R]_{XX} &= [\mathbf{T}_R]_{YY} = [\mathbf{T}_R]_{ZZ} = \\ &2 \left(\tilde{a} + 3\tilde{b} \right) \frac{1}{a} (a-s) - 2\tilde{b} \frac{1}{a} [(a-s) + 2r \sin \gamma] + \\ &2 \left[\tilde{c} (1 - 3 \sin^2 \gamma + 2 \sin^3 \gamma) - (\tilde{a} + 2\tilde{b} + \tilde{c}) \right] \frac{1}{a^2 \det \mathbf{F}} [(a-s)^2 - r^2], \\ [\mathbf{T}_R]_{XY} &= [\mathbf{T}_R]_{XZ} = [\mathbf{T}_R]_{YX} = [\mathbf{T}_R]_{YZ} = [\mathbf{T}_R]_{ZX} = [\mathbf{T}_R]_{ZY} = \\ &2 \left(\tilde{a} + 3\tilde{b} \right) \frac{1}{a} r - 2\tilde{b} \frac{1}{a} [(a-s) \sin \gamma + r (1 + \sin \gamma)] + \\ &2 \left[\tilde{c} (1 - 3 \sin^2 \gamma + 2 \sin^3 \gamma) - (\tilde{a} + 2\tilde{b} + \tilde{c}) \right] \frac{1}{a^2 \det \mathbf{F}} [r^2 - r(a-s)]. \end{aligned} \quad (51)$$

From the Piola-Kirchhoff stresses (51) the corresponding Cauchy stresses can be evaluated through (30)

$$\begin{aligned} (\det \mathbf{F}) [\mathbf{T}]_{xx} &= (\det \mathbf{F}) [\mathbf{T}]_{yy} = (\det \mathbf{F}) [\mathbf{T}]_{zz} = \\ &-4\tilde{b} \sin^2 \gamma + 2 \left[\tilde{c} (-3 \sin^2 \gamma + 2 \sin^3 \gamma) \right], \\ (\det \mathbf{F}) [\mathbf{T}]_{xy} &= (\det \mathbf{F}) [\mathbf{T}]_{xz} = (\det \mathbf{F}) [\mathbf{T}]_{yz} = \\ &2 \left(\tilde{a} + \tilde{b} \right) \sin \gamma - 2\tilde{b} \sin^2 \gamma. \end{aligned} \quad (52)$$

To obtain (52), (36) and (37) were used. The Cauchy stress tensor is also characterized by equal normal components and equal tangential components. In the absence of deformation ($\sin \gamma = r = s = 0$, and $\det \mathbf{F} = 1$), all Piola-Kirchhoff and Cauchy stress components are zero.

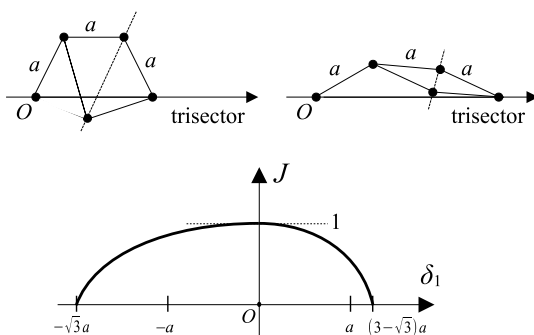
Assuming that the angle γ is small, through the linearization of (52) ($\sin \gamma \simeq \gamma$, $\det \mathbf{F} \simeq 1$, and $\mu \simeq 2 \left(\tilde{a} + \tilde{b} \right)$ [25]), it is immediate to derive the stress components of the linearized elasticity (cf. Eq. (2))

$$\begin{aligned} [\mathbf{T}]_{xx} &= [\mathbf{T}]_{yy} = [\mathbf{T}]_{zz} \simeq 0, \\ [\mathbf{T}]_{xy} &= [\mathbf{T}]_{xz} = [\mathbf{T}]_{yz} \simeq \mu \gamma. \end{aligned} \quad (53)$$

3.1 Application

For the equitriaxial angular shear deformation, the change in volume, $J = \det \mathbf{F} = \lambda_1 \lambda^2$, is evaluated as the displacement δ_1 , applied along the trisector of the cube, varies. The displacement δ_1 plays the role of independent variable. Assigned δ_1 , from (40)₁ we obtain λ_1 and then λ from (41). Once the stretches have been calculated, J is known: $J = 1 - \delta_1^2 \left(9a + \sqrt{3} \delta_1 \right) / (18a^3)$. δ_1 varies in the following range: $-\sqrt{3}a < \delta_1 < 3a - \sqrt{3}a$. In the limiting situation $\delta_1 \rightarrow \left(3a - \sqrt{3}a \right)$ the volume of the cube collapses into a segment of length $3a$. For $\delta_1 \rightarrow \left(-\sqrt{3}a \right)$, the volume collapses into a surface with the shape of a regular hexagon. In Fig. 7 the function J is plotted.

Fig. 7 Graph of $J = \det \mathbf{F} = \lambda_1 \lambda_2^2$ as a function of δ_1



4 Conclusions

In this paper, the angular shear deformation of hyperelastic and isotropic materials has been investigated in the framework of finite elasticity. Under purely angular shear deformation, the angle between two line elements undergoes large changes, while the length of the two line elements is preserved. This differs from the two classic shear models: the simple and the pure shear, where only one line element maintains its length while the other changes it (in a way that depends on the model itself). Consequently, the two classical shear models generate combined kinematic effects due to both angular distortions and extension deformations. The purely angular shear model, on the other hand, allows the shear behavior of materials to be evaluated based exclusively on angular distortions.

The direct extension of the two-dimensional model for the angular shear (proposed by the authors in [23]) is not immediate. In fact, the planar case depends on a single angular variation (cf. Fig. 4 of [23]). While in the three-dimensional case, the shape of the deformed configuration of the solid depends on three different angular variations, and determining the displacement field becomes more complicated. To overcome this difficulty in the three-dimensional case, in this paper, a kinematic analysis was developed considering a cube, and the displacements experienced by the three vertices of the edges lying along the axes of the reference system. Then, using the condition that the length of these edges does not vary and that the deformation is homogeneous, the displacement field was obtained.

From the displacement field, deformation gradient and Green-St. Venant tensor have been determined. In particular, the Green-St. Venant tensor has shear components ($E_{xy} = E_{yx} = \frac{1}{2} \sin \gamma_{xy}$, $E_{xz} = E_{zx} = \frac{1}{2} \sin \gamma_{xz}$ and $E_{yz} = E_{zy} = \frac{1}{2} \sin \gamma_{yz}$) different from zero, and all normal components equal to zero. For small deformation, $\sin \gamma_{ij} \simeq \gamma_{ij}$, with $i, j = x, y, z$, so the linearization of the Green-St. Venant tensor leads directly to the infinitesimal shear strain tensor.

Therefore, based on the form assumed by the Green-St. Venant tensor, with zero diagonal components, it is possible, for the three-dimensional angular shear deformation, to define a state of pure shear deformation in finite elasticity, as is usually done in linearized elasticity. This generally does not happen for the two classical shear models, since they have non-zero diagonal components, caused by the extension of the line elements. But this does not happen even in the two-dimensional model of angular shear, where an orthogonal axial deformation is present (cf. Eq. (38) of [23]).

As an alternative to the Green-St. Venant tensor, the use of the Biot tensor has been discussed, which allows the so-called stretch-based formulations to describe the constitutive behavior of a material.

For the three-dimensional angular shear deformation, explicit expressions for transformations of angles, surfaces and volumes have been derived.

Obviously, the kinematics exposed for the three-dimensional angular shear is also valid for anisotropic materials.

Using a Lagrangian analysis, the boundary value problem for the equilibrium of a cube subjected to purely angular shear deformation has been formulated. Leaving completely unspecified the stored energy function, Piola-Kirchhoff and Cauchy stress tensors were evaluated.

Typically, the Cauchy stress tensor has no zero diagonal components. Thus, for the angular shear deformation, it is not possible to define a state of pure shear stress. Consequently, a state of pure shear deformation is not associated with a state of pure shear stress, as instead happens in linearized elasticity.

The normal and tangential Cauchy stress vector components on the inclined faces of the deformed cube were calculated, demonstrating that the simultaneous application of tangential and normal stresses is necessary to produce an angular shear deformation. This is another substantial difference from linearized elasticity, where a shear strain is generated exclusively by tangential stresses.

The equitriaxial angular shear deformation case, where the deformed cube takes on the shape of a rhombohedron (that is, a regular polyhedron with six equal faces), has been examined in detail. In this case, the cube's trisector can be lengthened or shortened, and the three angular shear deformations are all the same. By assigning the law of compressible Mooney-Rivlin materials to the stored energy function, the Piola-Kirchhoff and Cauchy stresses were determined. In both cases, these stress tensors are characterized by equal normal components and equal tangential components.

Appendix: Eigenvalues of the Right Cauchy-Green Tensor $\mathbf{C}$

In the general case, the right Cauchy-Green tensor $\mathbf{C}$ for the angular shear deformation takes the following form:

$$[\mathbf{C}] = \begin{bmatrix} 1 & e & f \\ e & 1 & g \\ f & g & 1 \end{bmatrix}, \quad (54)$$

where $e = |\sin \gamma_{xy}| \leq 1$, $f = |\sin \gamma_{xz}| \leq 1$, and $g = |\sin \gamma_{yz}| \leq 1$. Imposing the condition

$$\det \begin{bmatrix} (1 - \Lambda) & e & f \\ e & (1 - \Lambda) & g \\ f & g & (1 - \Lambda) \end{bmatrix} = 0, \quad (55)$$

the expressions of the three eigenvalues in closed form are found

$$\Lambda = 1 - \frac{\left(-9efg + \frac{1}{3}\sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}} + \frac{-e^2 - f^2 - g^2}{\left(-27efg + \sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}}}, \quad (56)$$

$$\Lambda = 1 + \frac{\left(-\frac{1}{3}\right)^{\frac{1}{3}} (e^2 + f^2 + g^2)}{\left(-9efg + \frac{1}{3}\sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}}} + \quad (57)$$

$$\frac{1}{6} (1 - i\sqrt{3}) \left(-27efg + \sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}},$$

$$\Lambda = 1 - \frac{\frac{1}{3}(-3)^{\frac{2}{3}} (e^2 + f^2 + g^2)}{\left(-9efg + \frac{1}{3}\sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}}} + \quad (58)$$

$$\frac{1}{6} (1 + i\sqrt{3}) \left(-27efg + \sqrt{729e^2f^2g^2 - 27(e^2 + f^2 + g^2)^3}\right)^{\frac{1}{3}}.$$

These eigenvalues represent λ_1^2 , λ_2^2 , and λ_3^2 . When we extract the roots, the imaginary parts cancel out, and we have three distinct real roots. The eigenvalues are positive since the tensor $\mathbf{C}$ is positive definite.

In the particular case $a = b$, the eigenvalues assume the following compact form:

$$\Lambda_1 = 1 - c, \quad 2\Lambda_2 = 2 + c - \sqrt{8a^2 + c^2}, \quad \text{and} \quad 2\Lambda_3 = 2 + c + \sqrt{8a^2 + c^2}. \quad (59)$$

For $a = b = c$, the previous eigenvalues become

$$\Lambda_1 = 1 - c, \quad \Lambda_2 = 1 - c \quad \text{and} \quad \Lambda_3 = 1 + 2c, \quad (60)$$

and setting $c = \sin \gamma$, the eigenvalues (42) of the triaxial angular shear are found again.

Acknowledgements Financial support from the Italian Ministry of University and Research (MUR) through the research project FISA-2022-00183 “Earth-Tech” - Implementation of new Shot-Earth technology in the construction industry, CUP: E93C24000250001, is gratefully acknowledged. This work was also supported by the research grant FIS 2 “Large and offshore deformations of everyday life: from seismic isolators to person’s health”, project FIS-2023-01661, CUP E53C25000450001 and the research grant FIS 2 “PNC-Gen” (project n. FIS-2023-02409; CUP E53C25000440001). Financial support by the National Group of Mathematical Physics (GNFM-INdAM) is also acknowledged.

Author Contributions F.O.F.: Theoretical conceptualization, investigation, software, validation. L.L.: data reduction, investigation, writing – review. A.M.T.: visualization, supervision, project administration, funding acquisition.

Funding Information Open access funding provided by Università degli Studi di Modena e Reggio Emilia within the CRUI-CARE Agreement.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Harren, S., Lowe, T.C., Asaro, R.J., Needleman, A.: Analysis of large-strain shear in rate-dependent face-centred cubic polycrystals: correlation of micro- and macromechanics. *Philos. Trans. R. Soc. A* **328**, 443–500 (1989)
2. Tvergaard, V., Needleman, A.: Shear band development in polycrystals. *Proc. Roy. Soc. A* **443**, 547–562 (1993)
3. Wang, X., Cazes, F., Li, J., Hocini, A., Ameyama, K., Dirras, G.: A 3D crystal plasticity model of monotonic and cyclic simple shear deformation for commercial-purity polycrystalline Ti with a harmonic structure. *Mech. Mater.* **128**, 117–128 (2019)
4. Yadav, S., Sagapuram, D.: In situ analysis of shear bands and boundary layer formation in metals. *Proc. R. Soc. A* **476**, 20190519 (2020)
5. Besseling, T.H., Hermes, M., Fortini, A., Dijkstra, M., Imhof, A., van Blaaderen, A.: Oscillatory shear-induced 3D crystalline order in colloidal hard-sphere fluids. *Soft Matter* **8**, 6931–6939 (2012)
6. Ling Wu, Y., Derks, D., van Blaaderen, A., Imhof, A.: Melting and crystallization of colloidal hard-sphere suspensions under shear. *Proc. Natl. Acad. Sci. USA* **106**, 10564–10569 (2009)
7. Poirier, J.P.: Shear localization and shear instability in materials in the ductile field. *J. Struct. Geol.* **2**, 135–142 (1980)
8. Kasban, H., El-Bendary, M.A.M., Salama, D.H.: A comparative study of medical imaging techniques. *Int. J. Inf. Sci. Intell. Syst.* **4**, 37–58 (2015)
9. Ormachea, J., Parker, K.J.: Elastography imaging: the 30 year perspective. *Phys. Med. Biol.* **65**, 24TR06 (2020)
10. Truesdell, C., Noll, W.: *The Non-linear Field Theories of Mechanics*, 2nd edn. Springer, Berlin (1992)
11. Boulanger, Ph., Hayes, M.: On the finite shear. *Arch. Ration. Mech. Anal.* **151**, 125–185 (2000)
12. Rivlin, R.S.: Large elastic deformation of isotropic materials. IV. Further developments of the general theory. *Philos. Trans. R. Soc. A* **241**, 379–397 (1948)
13. Truesdell, C.: *The Elements of Continuum Mechanics*. Springer Verlag, Heidelberg (1966)
14. Green, A.E., Zerna, W.: *Theoretical Elasticity*, 2nd edn. Clarendon, Oxford (1968)
15. Ogden, R.W.: *Nonlinear Elastic Deformations*. Ellis Horwood, Chichester (1984). Reprinted by Dover, New York, 1997
16. Gurtin, M.E., Fried, E., Anand, L.: *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, Cambridge (2010)
17. Treloar, L.R.G.: Stress-strain data for vulcanised rubber under various types of deformation. *Trans. Faraday Soc.* **40**, 59–70 (1944)
18. Rivlin, R.S., Saunders, D.W.: Large elastic deformations of isotropic materials VII. Experiments on the deformation of rubber. *Philos. Trans. R. Soc. A* **243**, 251–288 (1951)
19. Horgan, C.O., Murphy, J.G.: Simple shearing of incompressible and slightly compressible isotropic non-linearly elastic materials. *J. Elast.* **98**, 205–221 (2010)
20. Mihai, L.A., Goriely, A.: Numerical simulation of shear and Poynting effects by finite element method: an application of the generalised empirical inequalities in non-linear elasticity. *Int. J. Non-Linear Mech.* **49**, 1–14 (2013)
21. Destrade, M., Murphy, J.G., Saccomandi, G.: Simple shear is not so simple. *Int. J. Non-Linear Mech.* **47**, 210–214 (2012)
22. Thiel, C., Voss, J., Martin, R.J., Neff, P.: Shear, pure and simple. *Int. J. Non-Linear Mech.* **112**, 57–72 (2019)
23. Falope, F.O., Lanzoni, L., Tarantino, A.M.: Shear models in finite elasticity. *J. Elast.* **158** 1–27 (2026).
24. Falope, F.O., Lanzoni, L., Tarantino, A.M.: Large deformations generated by the purely angular shear. *Int. J. Eng. Sci.* **226**, 104563 (2026)
25. Lanzoni, L., Tarantino, A.M.: Finite anticlastic bending of hyperelastic solids and beams. *J. Elast.* **131**, 137–170 (2018)

26. Lanzoni, L., Tarantino, A.M.: The bending of beams in finite elasticity. *J. Elast.* **139**, 91–121 (2020)
27. Piola, G.: Nuova analisi per tutte le questioni della meccanica molecolare. Tipografia camerale, Modena (1835)
28. Rivlin, R.S., Ericksen, J.L.: Stress-deformation relations for isotropic materials. *J. Ration. Mech. Anal.* **4**, 323–425 (1955)
29. Vitral, E., Hanna, J.A.: Quadratic-stretch elasticity. *Math. Mech. Solids* **27**, 462–473 (2022)
30. Vitral, E.: Stretch formulations and the Poynting effect in nonlinear elasticity. *Int. J. Non-Linear Mech.* **148**, 104293 (2023)
31. Firouzi, N., Misra, A.: New insight into large deformation analysis of stretch-based and invariant-based rubber-like hyperelastic elastomers. *Thin-Walled Struct.* **192**, 111162 (2023)
32. Ellmer, N., Ortigosa, R., Martínez-Frutos, J., Poya, R., Sienz, J., Gil, A.J.: Stretch-based hyperelastic constitutive metamodels via gradient enhanced Gaussian predictors. *Comput. Methods Appl. Mech. Eng.* **432**, 117408 (2024)
33. Horgan, C.O.: Pure torsion for stretch-based constitutive models for incompressible isotropic hyperelastic soft materials. *J. Elast.* **156**, 237–254 (2024)
34. Yang, J., Chen, K., Yu, C., Zhou, K., Kang, G.: A hyperelastic constitutive model for soft elastomers considering the entanglement-dependent finite extensibility. *J. Mech. Phys. Solids* **196**, 106000 (2025)
35. Pellicciari, M., Sirotti, S., Tarantino, A.M.: A strain energy function for large deformations of compressible elastomers. *J. Mech. Phys. Solids* **176**, 105308 (2023)
36. Falope, F.O., Lanzoni, L., Tarantino, A.M.: Energetic exhaustiveness for the direct characterization of energy forms of hyperelastic isotropic materials. *J. Mech. Phys. Solids* **193**, 105885 (2024)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.