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Modulus-based matrix splitting algorithms for generalized complex-valued horizontal linear complementarity problems

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ABSTRACT

In this paper, we introduce the complex-valued horizontal linear complementarity problem (CHLCP), we provide two equivalent real-valued reformulations, and study modulus-based matrix splitting algorithms for solving the CHLCP. This latter point is motivated by the recent introduction of modulus-based matrix splitting methods for (non-horizontal) complex linear complementarity problems (CLCPs), which we generalize. We study the convergence of the proposed algorithms. Whenever possible, we seek convergence conditions that are directly based on the form of the real and imaginary parts of the matrices of the CHLCP in its complex form. This makes the convergence easier to evaluate than in existing convergence analyses. Finally, we study the numerical properties of the proposed algorithms by solving several CHLCPs. In this context, we also revisit results on the CLCP under the larger CHLCP framework, providing new numerical insights on the efficiency of existing algorithms for the CLCP.

1. Introduction

We define the complex-valued horizontal linear complementarity problem (CHLCP) as

$$\tilde{A}\tilde{\boldsymbol{z}} - \tilde{B}\tilde{\boldsymbol{w}} = \tilde{\boldsymbol{q}}; \quad |arg(\tilde{\boldsymbol{z}})| \leq \boldsymbol{\gamma}; \quad |arg(\tilde{\boldsymbol{w}})| \leq \frac{\pi}{2}\boldsymbol{e} - \boldsymbol{\gamma}; \quad Re(\tilde{\boldsymbol{z}}^*\tilde{\boldsymbol{w}}) = 0, \tag{1.1}$$

where $\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}} \in \mathbb{C}^n$ are complex unknown vectors, $\tilde{A}, \tilde{B} \in \mathbb{C}^{n\times n}$ are given complex coefficient matrices, $\boldsymbol{\gamma} \in \mathbb{R}^n$ is a prescribed vector of values $\gamma_i \in [0, \frac{\pi}{2}]$, and $\tilde{\boldsymbol{q}} \in \mathbb{C}^n$ is a known term. In the adopted notation, a tilde sign above vector and matrices means that they have complex entries. Vectors and matrices without the tilde sign are intended to be real. Furthermore, given a complex number $\tilde{a} \in \mathbb{C}$, $\tilde{a}^*$ denotes the complex conjugate and $arg(\tilde{a})$ denotes the argument, which is angle between the real axis and the line that joins $\tilde{a}$ with the origin in the complex plane. These definitions are applied component-wise to vectors of complex entries. Finally, $\boldsymbol{e}$ denotes the unit vector and inequalities are intended to act component-wise.

The problem (1.1) reduces to the complex-valued linear complementarity problem (CLCP) when $\tilde{B} = I$. The CLCP was studied after it was demonstrated that the duality theory of linear programming holds also in the complex space [1]. Thus, the complex linear and quadratic programming were unified in the CLCP by [2], where an existence theory of the CLCP was also provided. Then, in [3] a solution strategy for the CLCP was first proposed, based on rewriting the n-dimensional CLCP to an equivalent real linear complementarity problem (LCP) of dimension $2n$. In this context, it was noticed that being able to solve the CLCP implies the ability to solve complex linear and quadratic problems.

With this motivation, recently Li et al. [4] proposed a modulus-based method for solving the CLCP. In particular, the problem was first reformulated in the real space in the same way as in [3]. Then, the equivalent problem was written in a modulus-based

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formulation analogous to [5]. Finally, a modulus-based solution strategy was proposed to solve the problem. The Ref. [4] can then be contextualized in the rich literature of modulus-based algorithms for complementarity problems, which originated from [5] and from earlier works on the modulus-based formulation [6–8]. Notable extensions of this procedure beyond the "standard" real-valued LCP include generalizations to the nonlinear complementarity problem [9–11], to the horizontal linear complementarity problem (HLCP) [12,13], to the vertical linear complementarity problem [14], and to other generalizations [15–18]. Alternative modulus-based strategies that do not employ an auxiliary variable [19–21] and non-splitting modulus-based methods [22] exist as well.

The aim of this paper is to further study modulus-based algorithms for the CLCP and to generalize the analysis to the CHLCP (1.1). In particular, we introduce the following novelties:

- We formulate the CHLCP, which generalizes the CLCP. Such generalization is conceptually similar to the one provided by the HLCP in the real space.
- We provide two real-valued reformulations of the CHLCP. One reformulation can be applied when the matrices of the problem satisfy some commutativity properties, while the other works for general CHLCPs.
- We reformulate the real-valued equivalent of the CHLCP in modulus-based form and we study its solution by standard modulus-based matrix splitting algorithms [12] and by generalizing the algorithm proposed in [4]. In both cases, the solution of the horizontal problem is addressed directly, which is computationally advantageous with respect to transforming the CHLCP into an equivalent CLCP.
- We prove the convergence of the algorithms [12] under conditions tailored on the nature of the complex problem. In particular, many of the proposed conditions refer to the real and imaginary parts of $\tilde{A}$ and $\tilde{B}$, instead of using the matrices of the equivalent real HLCP, which are larger and not immediately available in (1.1).
- To carry out the convergence analysis, we improve the convergence theory of modulus-based AOR methods for HLCPs [12], in both their standard and accelerated forms. These results are valid even for standard, real-valued HLCPs.
- We use the analysis performed on the CHLCP to further study the modulus-based solution of the CLCP. In particular, while [4] mostly considered problems with $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, we provide new theoretical and numerical results for a general selection of $\boldsymbol{\gamma}$. Furthermore, we show numerically that standard accelerated modulus-based methods can be faster than the algorithm in [4], which can also be more sensitive to variations of the starting iterate.

First, in Section 2 we provide two equivalent real-valued reformulations of the CHLCP and present our modulus-based matrix splitting algorithms. Then, in Section 3 we study the convergence of the methods. Modulus-based accelerated over-relaxation (AOR) methods are particularly analyzed and convergence conditions tailored on the form of the matrices of the CHLCP are provided. In Section 4 several numerical experiments are solved and analyzed. In particular, we revisit some results on the CLCP and we solve examples of HLCPs for both the considered real-valued reformulations. Some concluding remarks are then provided in Section 5. Finally, a list of the used acronyms is reported at the end of the manuscript.

In our analysis we use the following names and notation. We say that a matrix $A \in \mathbb{R}^{n\times n}$ is a Z-matrix if all its off-diagonal entries are non-positive and we say that A is an M−matrix if it is a Z-matrix with $A^{-1} \geq 0$ [23]. We say that A is an H-matrix if its comparison matrix $\langle A\rangle$ (i.e., the matrix of entries $|a_{ii}|$ along the main diagonal and $-|a_{ij}|$ off-diagonal, for $i,j = 1,\dots,n$ with $i \neq j$) is an M-matrix [23]. A is an H_+-matrix if it is an H-matrix and has positive diagonal entries [24]. Finally, we say that the splitting $A = M - N$ is an M-splitting if M is an M-matrix and $N \geq 0$ [25].

2. Real-valued formulations of the CHLCP

2.1. Equivalent problem in the real space

Let us first transform (1.1) into an equivalent real-valued HLCP. In this regard, let us first separate the real and the imaginary parts by writing

$$\tilde{B} = B_R + iB_I; \quad \tilde{A} = A_R + iA_I; \quad \tilde{\boldsymbol{z}} = \boldsymbol{z}_R + i\boldsymbol{z}_I; \quad \tilde{\boldsymbol{w}} = \boldsymbol{w}_R + i\boldsymbol{w}_I; \quad \tilde{\boldsymbol{q}} = \boldsymbol{q}_R + i\boldsymbol{q}_I,$$

where $B_R, B_I, A_R, A_I \in \mathbb{R}^{n\times n}$, $\boldsymbol{z}_R, \boldsymbol{z}_I, \boldsymbol{w}_R, \boldsymbol{w}_I, \boldsymbol{q}_R, \boldsymbol{q}_I \in \mathbb{R}^n$, and $i = \sqrt{-1}$. Then, the complex-valued system of Eq. (1.1) can be written as

$$(A_R + iA_I)(\boldsymbol{z}_R + i\boldsymbol{z}_I) - (B_R + iB_I)(\boldsymbol{w}_R + i\boldsymbol{w}_I) = \boldsymbol{q}_R + i\boldsymbol{q}_I,$$

which is

$$A_R\boldsymbol{z}_R + iA_R\boldsymbol{z}_I + iA_I\boldsymbol{z}_R - A_I\boldsymbol{z}_I - B_R\boldsymbol{w}_R - iB_R\boldsymbol{w}_I - iB_I\boldsymbol{w}_R + B_I\boldsymbol{w}_I = \boldsymbol{q}_R + i\boldsymbol{q}_I,$$

This implies that the real and imaginary parts of the vectors $\boldsymbol{z}, \boldsymbol{w}$ that satisfy the complex-valued $n \times n$ system (1.1) must solve the real-valued $2n \times 2n$ system

$$\begin{cases} A_R\boldsymbol{z}_R - A_I\boldsymbol{z}_I - B_R\boldsymbol{w}_R + B_I\boldsymbol{w}_I = \boldsymbol{q}_R \\ A_I\boldsymbol{z}_R + A_R\boldsymbol{z}_I - B_I\boldsymbol{w}_R - B_R\boldsymbol{w}_I = \boldsymbol{q}_I \end{cases}.$$

Setting

$$A_{RI} := \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} \quad B_{RI} := \begin{pmatrix} B_R & -B_I \\ B_I & B_R \end{pmatrix},$$

such system can equivalently be written as

$$A_{RI} \begin{pmatrix} z_R \\ z_I \end{pmatrix} - B_{RI} \begin{pmatrix} w_R \\ w_I \end{pmatrix} = \begin{pmatrix} q_R \\ q_I \end{pmatrix}. \tag{2.1}$$

Next, the conditions on the arguments of $\boldsymbol{z}$, $\boldsymbol{w}$ required by (1.1) are satisfied if

$$\begin{pmatrix} z_L \\ z_U \end{pmatrix} = \begin{pmatrix} \tan(\boldsymbol{\gamma}) & I \\ \tan(\boldsymbol{\gamma}) & -I \end{pmatrix} \begin{pmatrix} z_R \\ z_I \end{pmatrix} \geq \mathbf{0} \qquad \begin{pmatrix} w_L \\ w_U \end{pmatrix} = \begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & I \\ \tan(\boldsymbol{\gamma})^{-1} & -I \end{pmatrix} \begin{pmatrix} w_R \\ w_I \end{pmatrix} \geq \mathbf{0}, \tag{2.2}$$

with $z_L^T w_L + z_U^T w_U = 0$, as can be demonstrated by proceeding as in [3, Section 3]. In the previous relations, $\tan(\boldsymbol{\gamma})$ denotes the $n\times n$ diagonal matrix whose ith diagonal entry is $\tan(\gamma_i)$. In the following, for simplicity and in accordance with the literature [3,4], we limit our analysis to $0 < \gamma_i < \frac{\pi}{2}$ for $i = 1, \dots, n$, for which $\tan(\boldsymbol{\gamma})$ is a positive diagonal matrix. The special cases where some γ_i are zero or $\frac{\pi}{2}$ can be dealt with by treating individually the indices where these values occur [3].

Under this assumption, we change the variables of the problem to z_L, z_U, w_L, w_U by inverting the matrices of (2.2), which can be easily computed as

$$T_1 := \frac{1}{2} \begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & \tan(\boldsymbol{\gamma})^{-1} \\ I & -I \end{pmatrix}; \qquad T_2 := \frac{1}{2} \begin{pmatrix} \tan(\boldsymbol{\gamma}) & \tan(\boldsymbol{\gamma}) \\ I & -I \end{pmatrix}.$$

Thus, the CHLCP (1.1) can equivalently be written as determining vectors z_L, z_U, w_L, w_U that satisfy

$$\begin{aligned} & A_{RI} T_1 \begin{pmatrix} z_L \\ z_U \end{pmatrix} - B_{RI} T_2 \begin{pmatrix} w_L \\ w_U \end{pmatrix} = \begin{pmatrix} q_R \\ q_I \end{pmatrix} \\ & z_L, z_U, w_L, w_U \geq \mathbf{0}; \quad z_L^T w_L + z_U^T w_U = 0. \end{aligned} \tag{2.3}$$

This result generalizes to horizontal problems the equivalent reformulation provided in [3] for the CLCP. In particular, if we multiply both sides by T_2^{-1}, it reduces to the form [3] when $B_R = I$ and $B_I = 0$. By simple renaming of the terms, the same applies when $A_R = I$ and $A_I = 0$.

2.1.1. General, non-commuting matrices

Notice that the problem (2.3) is a real-valued HLCP, which can be solved by any solution algorithm for such problems. Nonetheless, it is useful to provide particular analyses that allow a direct application of the methods and of their convergence theory. In this regard, let us first consider a general case, that applies when no special properties are available to simplify the problem (2.3). In this case, for the following analysis it is convenient to change sign to the bottom blocks of the system in (2.3), obtaining

$$\begin{aligned} & \begin{pmatrix} A_R & -A_I \\ -A_I & -A_R \end{pmatrix} T_1 \begin{pmatrix} z_L \\ z_U \end{pmatrix} - \begin{pmatrix} B_R & -B_I \\ -B_I & -B_R \end{pmatrix} T_2 \begin{pmatrix} w_L \\ w_U \end{pmatrix} = \begin{pmatrix} q_R \\ -q_I \end{pmatrix} \\ & z_L, z_U, w_L, w_U \geq \mathbf{0}; \quad z_L^T w_L + z_U^T w_U = 0. \end{aligned} \tag{2.4}$$

At this point, for compactness, let us set

$$\begin{aligned} & A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_R & -A_I \\ -A_I & -A_R \end{pmatrix} \begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & \tan(\boldsymbol{\gamma})^{-1} \\ I & -I \end{pmatrix} \qquad z = \begin{pmatrix} z_L \\ z_U \end{pmatrix} \\ & B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_R & -B_I \\ -B_I & -B_R \end{pmatrix} \begin{pmatrix} \tan(\boldsymbol{\gamma}) & \tan(\boldsymbol{\gamma}) \\ I & -I \end{pmatrix} \qquad w = \begin{pmatrix} w_L \\ w_U \end{pmatrix} \qquad q = 2 \begin{pmatrix} q_R \\ -q_I \end{pmatrix}. \end{aligned} \tag{2.5}$$

The real-valued equivalent of the CHLCP (1.1) is then given by

$$Az - Bw = q \text{ with } z \geq \mathbf{0},\ w \geq \mathbf{0},\ z^T w = 0. \tag{2.6}$$

2.1.2. Commuting B_{RI} and T_2

An interesting special case arises when B_{RI} and T_2 commute. Notice that this situation includes the CLCP, which makes this special case particularly relevant. Nonetheless, it is still more general than the CLCP, as B_{RI} and T_2 can commute even when $\tilde{B} \neq I$. Indeed, by definition of B_{RI} and T_2, the commutativity requires

$$\begin{cases} B_R \tan(\boldsymbol{\gamma}) - B_I = \tan(\boldsymbol{\gamma}) B_R + \tan(\boldsymbol{\gamma}) B_I \\ B_R \tan(\boldsymbol{\gamma}) + B_I = \tan(\boldsymbol{\gamma}) B_R - \tan(\boldsymbol{\gamma}) B_I \\ B_R + B_I \tan(\boldsymbol{\gamma}) = B_R - B_I \\ -B_R + B_I \tan(\boldsymbol{\gamma}) = -B_R - B_I. \end{cases}$$

As $\tan(\boldsymbol{\gamma})$ is a positive diagonal matrix, it is easy to notice that a necessary and sufficient condition for commutativity is $B_I = 0$ and $B_R \tan(\boldsymbol{\gamma}) = \tan(\boldsymbol{\gamma}) B_R$. Beyond the CLCP case, this is verified, for instance, when $B_I = 0$ and B_R is a real diagonal matrix or when $\gamma_1 = \gamma_2 = \cdots = \gamma_n$ with symmetric B_R and with $B_I = 0$. In all these cases, B_{RI} and T_2 will commute.

Exploiting the above conditions of commutativity and multiplying both sides by T_2^{-1} (which is easy to compute in closed form), Eq. (2.3) can equivalently be written as

$$T_2^{-1}\begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} T_1 \begin{pmatrix} z_L \\ z_U \end{pmatrix} - \begin{pmatrix} B_R & 0 \\ 0 & B_R \end{pmatrix}\begin{pmatrix} \boldsymbol{w}_L \\ \boldsymbol{w}_U \end{pmatrix} = T_2^{-1}\begin{pmatrix} \boldsymbol{q}_R \\ \boldsymbol{q}_I \end{pmatrix} \\ \boldsymbol{z}_L, \boldsymbol{z}_U, \boldsymbol{w}_L, \boldsymbol{w}_U \geq \boldsymbol{0}; \quad \boldsymbol{z}_L^T \boldsymbol{w}_L + \boldsymbol{z}_U^T \boldsymbol{w}_U = 0. \tag{2.7}$$

Hence, we can write the HLCP (2.7) in a way that is formally identical to (2.6) by setting

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & I \\ \tan(\boldsymbol{\gamma})^{-1} & -I \end{pmatrix}\begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix}\begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & \tan(\boldsymbol{\gamma})^{-1} \\ I & -I \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_L \\ \boldsymbol{z}_U \end{pmatrix} \\ B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_R & 0 \\ 0 & B_R \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} \boldsymbol{w}_L \\ \boldsymbol{w}_U \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} \tan(\boldsymbol{\gamma})^{-1} & I \\ \tan(\boldsymbol{\gamma})^{-1} & -I \end{pmatrix}\begin{pmatrix} \boldsymbol{q}_R \\ \boldsymbol{q}_I \end{pmatrix}. \tag{2.8}$$

Evidently, this latter formulation is less general than (2.5)–(2.6), which can be applied to both commuting and non-commuting matrices. Nonetheless, it is useful for the analysis of the convergence of the algorithms when commutativity holds. For instance, a standard assumption in the convergence analysis of modulus-based methods for HLCPs is that the matrices A, B are H_+-matrices. However, for a same CHLCP (1.1) with commuting matrices, A, B formulated as in (2.8) might be H_+-matrices when A, B as in (2.5) are not. This is apparent for the CLCP case, where we would get $B = T_2$ in (2.5) and $B = I$ in (2.8). Setting, for instance, $\tan(\boldsymbol{\gamma}) = I$, it is easy to notice that $\langle T_2 \rangle$ is singular, while I is a special instance of a non-singular M-matrix.

2.2. Modulus-based formulation and algorithms for the CHLCP

In this subsection, we provide modulus-based formulations to the equivalent real-valued HLCP (2.6) and we summarize corresponding modulus-based solution algorithms.

2.2.1. Generalization of the method in [4]

As regards the algorithm proposed in [4] for the CLCP, we can generalize it to the CHLCP by defining

$$\boldsymbol{w}_L = \Omega_1(|\boldsymbol{x}| - \boldsymbol{x}) \qquad \boldsymbol{z}_L = \Omega_2(|\boldsymbol{x}| + \boldsymbol{x}) \\ \boldsymbol{w}_U = \Omega_1(|\boldsymbol{y}| - \boldsymbol{y}) \qquad \boldsymbol{z}_U = \Omega_2(|\boldsymbol{y}| + \boldsymbol{y}), \tag{2.9}$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Using the definitions of $A, B, \boldsymbol{q}$ as in (2.5) or in (2.8), the CHLCP can be equivalently formulated as the problem that consists in finding $\boldsymbol{x}, \boldsymbol{y}$ that satisfy

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \Omega_2(|\boldsymbol{x}| + \boldsymbol{x}) \\ \Omega_2(|\boldsymbol{y}| + \boldsymbol{y}) \end{pmatrix} - \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \Omega_1(|\boldsymbol{x}| - \boldsymbol{x}) \\ \Omega_1(|\boldsymbol{y}| - \boldsymbol{y}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{pmatrix}, \tag{2.10}$$

where $\boldsymbol{q}_1$ is the vector that contains the first n components of $\boldsymbol{q}$ and $\boldsymbol{q}_2$ contains the last n components of $\boldsymbol{q}$. The system (2.10) can equivalently be written as

$$\begin{cases} A_{11}\Omega_2(|\boldsymbol{x}| + \boldsymbol{x}) + A_{12}\Omega_2(|\boldsymbol{y}| + \boldsymbol{y}) - B_{11}\Omega_1(|\boldsymbol{x}| - \boldsymbol{x}) - B_{12}\Omega_1(|\boldsymbol{y}| - \boldsymbol{y}) = \boldsymbol{q}_1 \\ A_{21}\Omega_2(|\boldsymbol{x}| + \boldsymbol{x}) + A_{22}\Omega_2(|\boldsymbol{y}| + \boldsymbol{y}) - B_{21}\Omega_1(|\boldsymbol{x}| - \boldsymbol{x}) - B_{22}\Omega_1(|\boldsymbol{y}| - \boldsymbol{y}) = \boldsymbol{q}_2, \end{cases}$$

which, reordering terms, can be written as

$$\begin{cases} (B_{11}\Omega_1 + A_{11}\Omega_2)\boldsymbol{x} = (B_{11}\Omega_1 - A_{11}\Omega_2)|\boldsymbol{x}| + (B_{12}\Omega_1 - A_{12}\Omega_2)|\boldsymbol{y}| - (B_{12}\Omega_1 + A_{12}\Omega_2)\boldsymbol{y} + \boldsymbol{q}_1 \\ (B_{22}\Omega_1 + A_{22}\Omega_2)\boldsymbol{y} = (B_{22}\Omega_1 - A_{22}\Omega_2)|\boldsymbol{y}| + (B_{21}\Omega_1 - A_{21}\Omega_2)|\boldsymbol{x}| - (B_{21}\Omega_1 + A_{21}\Omega_2)\boldsymbol{x} + \boldsymbol{q}_2. \end{cases} \tag{2.11}$$

The equivalence between the problems (2.4) or (2.7) and the modulus-based form (2.11) can be demonstrated similarly to [12, Theorem 2] with minor renaming of terms and exploiting the block structure of A, B. At this point, we can generalize to the CHLCP the modulus-based matrix splitting method for the CLPC in [4] as follows. In this regard, we name such algorithm *block-diagonal modulus-based matrix splitting method for CHLCPs*, as, at each iteration, it solves two sub-problems defined by splittings of the upper and lower diagonal blocks of A and B.

Method 1 (*Block-Diagonal Modulus-Based Matrix Splitting Methods for CHLCPs*). *Let $A, B \in \mathbb{R}^{2n\times 2n}$ and $\boldsymbol{q} \in \mathbb{R}^{2n}$ be defined as in (2.5) or in (2.8). Furthermore, let $A_{11} = M_{A_{11}} - N_{A_{11}}$, $A_{22} = M_{A_{22}} - N_{A_{22}}$, $B_{11} = M_{B_{11}} - N_{B_{11}}$, and $B_{22} = M_{B_{22}} - N_{B_{22}}$ and let Ω_1, Ω_2 be positive diagonal matrices. Then, starting from an initial guess $\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}$, the block-diagonal modulus-based matrix splitting method for CHLCPs computes the iterates $\boldsymbol{x}^{(k+1)}, \boldsymbol{y}^{(k+1)}$ by solving*

$$\begin{cases} (M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2)\boldsymbol{x}^{(k+1)} = (N_{B_{11}}\Omega_1 + N_{A_{11}}\Omega_2)\boldsymbol{x}^{(k)} \\ \quad +(B_{11}\Omega_1 - A_{11}\Omega_2)|\boldsymbol{x}^{(k)}| + (B_{12}\Omega_1 - A_{12}\Omega_2)|\boldsymbol{y}^{(k)}| - (B_{12}\Omega_1 + A_{12}\Omega_2)\boldsymbol{y}^{(k)} + \boldsymbol{q}_1 \\ (M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2)\boldsymbol{y}^{(k+1)} = (N_{B_{22}}\Omega_1 + N_{A_{22}}\Omega_2)\boldsymbol{y}^{(k)} \\ \quad +(B_{22}\Omega_1 - A_{22}\Omega_2)|\boldsymbol{y}^{(k)}| + (B_{21}\Omega_1 - A_{21}\Omega_2)|\boldsymbol{x}^{(k+1)}| - (B_{21}\Omega_1 + A_{21}\Omega_2)\boldsymbol{x}^{(k+1)} + \boldsymbol{q}_2 \end{cases}$$

for $k = 0, 1, \ldots$. The complementarity vectors at each iteration of the algorithms can be computed by

$$\boldsymbol{w}_L^{(k+1)} = \Omega_1(|\boldsymbol{x}^{(k+1)}| - \boldsymbol{x}^{(k+1)}) \qquad \boldsymbol{z}_L^{(k+1)} = \Omega_2(|\boldsymbol{x}^{(k+1)}| + \boldsymbol{x}^{(k+1)}) \\ \boldsymbol{w}_U^{(k+1)} = \Omega_1(|\boldsymbol{y}^{(k+1)}| - \boldsymbol{y}^{(k+1)}) \qquad \boldsymbol{z}_U^{(k+1)} = \Omega_2(|\boldsymbol{y}^{(k+1)}| + \boldsymbol{y}^{(k+1)}), \tag{2.12}$$

and the corresponding vectors that, at convergence, satisfy the CHLCP (1.1) can be computed by

$$\begin{pmatrix} z_R^{(k+1)} \\ z_I^{(k+1)} \end{pmatrix} = T_1 \begin{pmatrix} z_L^{(k+1)} \\ z_U^{(k+1)} \end{pmatrix} \qquad \begin{pmatrix} w_R^{(k+1)} \\ w_I^{(k+1)} \end{pmatrix} = T_2 \begin{pmatrix} w_L^{(k+1)} \\ w_U^{(k+1)} \end{pmatrix}.$$

It is easy to verify that Method 1 generalizes [4, Method 3.1] (which was proposed for the CLCP) to the CHLCP. Furthermore, in the special case of a (non-horizontal) CLCP (i.e., $B_R = I$, $B_I = 0$), Method 1 reduces (up to a scaling) to [4, Method 3.1] when A, B are defined as in (2.8) with $B_R = I$ and when, for prescribed $r \in \mathbb{R}^+$ and positive diagonal matrix $\Omega \in \mathbb{R}^n$, $\Omega_1 = \frac{1}{2r}\Omega$, $\Omega_2 = \frac{1}{r}I$, and $M_{B_{11}} = M_{B_{22}} = I$. Different choices of the splittings and of the parametric matrices produce new algorithms not included in [4] even for the CLCP.

Notice that whenever $\tilde{B}$ of (1.1) is not the identity matrix, reformulating the original problem into a CLCP in order to apply the algorithms in [4] requires inverting the matrix $\tilde{B}$, which generally is onerous. Method 1, on the other hand, avoids any need of matrix inversion, as B is split alongside A.

Finally, we remark that a "simplified method" was also considered in [4]. The particular setting analyzed in that situation is outside the scope of this paper for two reasons. First, the "simplified method" is actually a simplified problem, as it prescribes the very particular choice $\gamma = \frac{\pi}{4}e$. We, instead, aim at analyzing algorithms that can be applied with generality to any choice γ within some convergence conditions. Second, we are going to discuss the special choice $\gamma = \frac{\pi}{4}e$ as a remark to the convergence theorems, noticing the simplifications that occur in this case. Nonetheless, as this applies to specific problems and not to simplifications inherent in the algorithms, it appears more suitable not to treat these situations within a simplified method in the present analysis.

2.2.2. Modulus-based matrix splitting algorithms for the CHLCP

The proposed real-valued forms allow to write the CHLCP in a way that is formally equivalent to the modulus-based formulation of real-valued HLCPs introduced in [12]. In this setting, we first define

$$z = \Gamma(|x| + x) \qquad w = \Omega(|x| - x),$$

where $\Gamma, \Omega \in \mathbb{R}^{2n\times 2n}$ are two positive diagonal matrices and $x \in \mathbb{R}^{2n}$ is an unknown term. Then, the problem (2.3) can equivalently be written as finding x that solves the modulus-based system

$$A\Gamma(|x| + x) - B\Omega(|x| - x) = q,$$

or, equivalently,

$$(A\Gamma + B\Omega)x - (B\Omega - A\Gamma)|x| = q. \tag{2.13}$$

The equivalence between (2.4) and (2.13) can be demonstrated as in [12, Theorem 2]. Proceeding as in [12] and setting, for simplicity, $\Gamma = \frac{1}{r}I$ with $r \in \mathbb{R}^+$, the following algorithms can be formulated.

Method 2 (*Modulus-Based Matrix Splitting Methods for CHLCPs*). *Given the CHLCP (1.1), let $A, B \in \mathbb{R}^{2n\times 2n}$ and $q \in \mathbb{R}^{2n}$ be defined as in (2.5) or in (2.8). Starting from an initial guess $x^{(0)} \in \mathbb{R}^{2n}$, let the $(k+1)$-th iterate $x^{(k+1)}$ be the solution of the linear system*

$$(M_A + M_B\Omega)x = (N_A + N_B\Omega)x^{(k)} + (B\Omega - A)|x^{(k)}| + rq, \tag{2.14}$$

with r positive constant, Ω positive diagonal matrix of order $2n$, $A = M_A - N_A$ and $B = M_B - N_B$ splittings of A and B, respectively. At each iteration, the complementarity vectors of the methods have the form

$$z^{(k+1)} = \frac{1}{r}\left(|x^{(k+1)}| + x^{(k+1)}\right); \qquad w^{(k+1)} = \frac{1}{r}\Omega\left(|x^{(k+1)}| - x^{(k+1)}\right) \tag{2.15}$$

and the corresponding vectors that, at convergence, satisfy the CHLCP (1.1) can be computed by

$$\begin{pmatrix} z_R^{(k+1)} \\ z_I^{(k+1)} \end{pmatrix} = T_1 z^{(k+1)} \qquad \begin{pmatrix} w_R^{(k+1)} \\ w_I^{(k+1)} \end{pmatrix} = T_2 w^{(k+1)} \tag{2.16}$$

Method 3 (*Accelerated Modulus-Based Matrix Splitting Methods For CHLCPs*). *Let $A, B \in \mathbb{R}^{2n\times 2n}$ and $q \in \mathbb{R}^{2n}$ be the matrices and the vector as defined in (2.5) or in (2.8). Starting from an initial guess $x^{(0)} \in \mathbb{R}^n$, let the $(k+1)$-th iterate $x^{(k+1)}$ be the solution of the system*

$$(M_{A_1} + M_{B_1}\Omega)x + (N_{B_2}\Omega - N_{A_2})|x| = (N_{A_1} + N_{B_1}\Omega)x^{(k)} + (M_{B_2}\Omega - M_{A_2})|x^{(k)}| + rq, \tag{2.17}$$

with r positive constant, Ω positive diagonal matrix of order $2n$ and $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ and $B = M_{B_1} - N_{B_1} = M_{B_2} - N_{B_2}$ (that is, $M_{A_1} - N_{A_1}$, $M_{A_2} - N_{A_2}$ are splittings of A and $M_{B_1} - N_{B_1}$, $M_{B_2} - N_{B_2}$ are splittings of B). At each iteration, the complementarity vectors have the form (2.15) and the corresponding real and imaginary parts of the vectors of the CHLCP (1.1) can be computed by (2.16).

Clearly, for the algorithms to be effective, the sub-problems (2.14) and (2.17) must be easy to solve at each iteration. This can be ensured by a suitable choice of the splitting matrices. In particular, accelerated over-relaxation (AOR) splittings are popular and effective. We are therefore going to focus on these for the convergence analysis. In this regard, let us consider the splittings $A = D_A - L_A - U_A$ and $B = D_B - L_B - U_B$, where D_A, D_B are the diagonal parts of A and B respectively, $-L_A, -L_B$ are the lower-triangular parts, and $-U_A, -U_B$ are the upper-triangular parts. The modulus-based AOR method is defined by Method 2 with the splittings

$$\begin{aligned} M_A &= \frac{D_A - \beta L_A}{\alpha}; \quad N_A = \frac{(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A}{\alpha} \\ M_B &= \frac{D_B - \beta L_B}{\alpha}; \quad N_B = \frac{(1-\alpha)D_B + (\alpha-\beta)L_B + \alpha U_B}{\alpha} \end{aligned} \tag{2.18}$$

while the accelerated modulus-based AOR is defined by Method 3 with the splittings

$$\begin{aligned} M_{A_1} &= \frac{D_A - \beta L_A}{\alpha}; \quad N_{A_1} = \frac{(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A}{\alpha} \\ M_{B_1} &= \frac{D_B - \beta L_B}{\alpha}; \quad N_{B_1} = \frac{(1-\alpha)D_B + (\alpha-\beta)L_B + \alpha U_B}{\alpha} \\ M_{A_2} &= D_A - U_A; \quad N_{A_2} = L_A; \quad M_{B_2} = D_B - U_B; \quad N_{B_2} = L_B. \end{aligned} \tag{2.19}$$

With the special choice $\alpha = 1, \beta = 0$ we obtain the corresponding Jacobi methods, with $\alpha = \beta = 1$ we obtain the Gauss–Seidel methods, and with $\alpha = \beta$ we obtain the successive over-relaxation (SOR) methods.

Finally, it is worth noticing that the proposed modulus-based reformulations afford for the application of other algorithms for HLCPs, such as two-step splittings [26], multisplittings [27,28], two-sweep splittings [29], and preconditioned general splittings [30]. Similarly, it is possible to use modulus-based methods that do not employ splittings, such as the sign-based method [31] or the nonsmooth Newton's method [32]. Alternatively, it is even possible to directly solve the real-valued HLCPs (2.4) and (2.7) by non-modulus-based algorithms. The tailored analysis of all these strategies for the CHLCP is outside of the scope of this paper, but the standard convergence conditions known for these algorithms can safely be applied to the equivalent real-valued problems.

3. Convergence analysis

In this section, we analyze the convergence of the modulus-based matrix splitting algorithms discussed in the previous section. In this regard, in the following we always assume that the CHLCP has a unique solution. In practice, the uniqueness of the solution can be studied by applying the existence results for standard HCLPs [33,34] to the equivalent real-valued problems (2.4) or (2.7).

3.1. Convergence of Method 1

As regards the convergence of Method 1, the following theorem generalizes [4, Theorem 4.1] which, along with its corollaries, is the only convergence result proposed in [4] for the general form of the block-diagonal splitting method for CLCPs. We remark that the [4, Theorem 4.1] is quite general, but difficult to use in practice, as the convergence is characterized by conditions that are generally hard to check. The same applies to the following theorem. Convergence conditions that are easier to check will be studied in the following subsections, with special regard to AOR splittings for the Methods 2 and 3.

Theorem 3.1. *Let $A_{11} = M_{A_{11}} - N_{A_{11}}$, $A_{22} = M_{A_{22}} - N_{A_{22}}$, $B_{11} = M_{B_{11}} - N_{B_{11}}$, and $B_{22} = M_{B_{22}} - N_{B_{22}}$ and let Ω_1, Ω_2 be positive diagonal matrices such that $M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2$ and $M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2$ are H_+-matrices. Furthermore, define*

$$\begin{aligned} W_1 &:= \langle M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2\rangle^{-1}\Big[|N_{B_{11}}\Omega_1 + N_{A_{11}}\Omega_2| + |B_{11}\Omega_1 - A_{11}\Omega_2|\Big] \\ W_2 &:= \langle M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2\rangle^{-1}\Big[|B_{12}\Omega_1 - A_{12}\Omega_2| + |B_{12}\Omega_1 + A_{12}\Omega_2|\Big] \\ V_1 &:= \langle M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2\rangle^{-1}\Big[|N_{B_{22}}\Omega_1 + N_{A_{22}}\Omega_2| + |B_{22}\Omega_1 - A_{22}\Omega_2|\Big] \\ V_2 &:= \langle M_{B_{22}}\Omega_1 + M_{A_2}\Omega_2\rangle^{-1}\Big[|B_{21}\Omega_1 - A_{21}\Omega_2| + |B_{21}\Omega_1 + A_{21}\Omega_2|\Big]. \end{aligned} \tag{3.1}$$

If

$$V_1 \le W_1, \quad V_2 \le W_2, \quad \rho(W_1 + W_2) < 1,$$

then the sequence of iterates generated by Method 1 converges to the solution of the CHLCPs.

Proof. Naming $\boldsymbol{x}^*, \boldsymbol{y}^*$ the unique solution of (2.11), the error at the generic $(k+1)$-th iteration can be evaluated as

$$\begin{cases} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \le |(M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2)^{-1}| \Big[(|N_{B_{11}}\Omega_1 + N_{A_{11}}\Omega_2| + |B_{11}\Omega_1 - A_{11}\Omega_2|)|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ \quad +(|B_{12}\Omega_1 - A_{12}\Omega_2| + |B_{12}\Omega_1 + A_{12}\Omega_2|)|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \Big] \\ |\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^*| \le |(M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2)^{-1}| \Big[(|N_{B_{22}}\Omega_1 + N_{A_{22}}\Omega_2| + |B_{22}\Omega_1 - A_{22}\Omega_2|)|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \\ \quad +(|B_{21}\Omega_1 - A_{21}\Omega_2| + |B_{21}\Omega_1 + A_{21}\Omega_2|)|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \Big], \end{cases}$$

where we have used the triangle inequality and the fact that $||a| - |b|| \le |a - b|$ for any $a, b \in \mathbb{R}$. If $M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2$ and $M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2$ are H_+-matrices, by [35] we can further write

$$\begin{cases} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \le \langle M_{B_{11}}\Omega_1 + M_{A_{11}}\Omega_2\rangle^{-1} \Big[(|N_{B_{11}}\Omega_1 + N_{A_{11}}\Omega_2| + |B_{11}\Omega_1 - A_{11}\Omega_2|)|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ \quad +(|B_{12}\Omega_1 - A_{12}\Omega_2| + |B_{12}\Omega_1 + A_{12}\Omega_2|)|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \Big] \\ |\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^*| \le \langle M_{B_{22}}\Omega_1 + M_{A_{22}}\Omega_2\rangle^{-1} \Big[(|N_{B_{22}}\Omega_1 + N_{A_{22}}\Omega_2| + |B_{22}\Omega_1 - A_{22}\Omega_2|)|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \\ \quad +(|B_{21}\Omega_1 - A_{21}\Omega_2| + |B_{21}\Omega_1 + A_{21}\Omega_2|)|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \Big] \end{cases}$$

Hence, defining V_1, V_2, W_1, W_2 as in (3.1), we have

$$\begin{cases} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \le W_1|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| + W_2|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \\ |\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^*| \le V_1|\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| + V_2|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \end{cases}$$

which is

$$\begin{pmatrix} I & 0 \\ -V_2 & I \end{pmatrix} \begin{pmatrix} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \\ |\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^*| \end{pmatrix} \le \begin{pmatrix} W_1 & W_2 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} |\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ |\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \end{pmatrix}. \tag{3.2}$$

Notice that V_2 is a non-negative matrix, which, by assumption, is smaller than W_2. Furthermore, $\rho(W_2) < 1$ by the assumption $\rho(W_1 + W_2) < 1$ with W_1, W_2 non-negative matrices [23, p. 27]. Hence, $\rho(V_2) < 1$ and the matrix

$$\begin{pmatrix} I & 0 \\ -V_2 & I \end{pmatrix}$$

is a nonsingular M-matrix. Hence, by multiplying both sides of (3.2) by the inverse of such matrix, we obtain

$$\begin{pmatrix} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \\ |\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^*| \end{pmatrix} \le \begin{pmatrix} I & 0 \\ V_2 & I \end{pmatrix} \begin{pmatrix} W_1 & W_2 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} |\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ |\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ V_2W_1 & V_2W_1 + V_1 \end{pmatrix} \begin{pmatrix} |\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ |\boldsymbol{y}^{(k)} - \boldsymbol{y}^*| \end{pmatrix}$$

This last equation is formally equivalent to [4, Eq. (4.5)]. The theorem is then proved by proceeding exactly as in the remainder of [4, Theorem 4.1]. □

Remark 3.1. In the special case of a (non-horizontal) CLCP (i.e., $B_R = I$, $B_I = 0$), Theorem 3.1 reduces to [4, Theorem 4.1] if A, B are defined as in (2.8) and if, for prescribed $r \in \mathbb{R}^+$ and positive diagonal matrix $\Omega \in \mathbb{R}^n$, $\Omega_1 = \frac{1}{2r}\Omega$, $\Omega_2 = \frac{1}{r}I$, and $M_{B_{11}} = M_{B_{22}} = I$.

3.2. *Convergence of the Methods 2 and 3*

The convergence of the Methods 2 and 3 could be studied as in [12] and as in more recent convergence analyses (e.g, see [13]). However, A and B are here block matrices made of the real and imaginary parts of the CHLCP (1.1). Therefore,

- It is desirable to have convergence conditions that are based directly on conditions on A_R, A_I, B_R, B_I, which are smaller than A, B and immediately available in the problem (1.1);
- There may be situations where A and B do not satisfy known convergence conditions, but where we can exploit the block structure to prove that the algorithms are converging.

Thus, we here develop tailored convergence conditions for modulus-based AOR algorithms for CHLCPs.

3.2.1. *Auxiliary lemmas for the general formulation (2.4)–(2.5)*

In the literature, the convergence of modulus-based matrix splitting methods is generally analyzed assuming that the matrices of the problem have positive diagonal (indeed, they are often required to be H_+-matrices or positive definite matrices). Thus, we put ourselves in this condition. In the following, $a_{R_{ij}}, a_{I_{ij}}, b_{R_{ij}}, b_{I_{ij}}$ denote the component of index i, j of the matrices A_R, A_I, B_R, B_I, respectively, for $i, j = 1, \ldots, n$,. Setting A, B as in (2.5), the positivity of the diagonal of A, B implies, for $i = 1, \ldots, n$,

$$\begin{cases} a_{R_{ii}} \tan(\gamma_i)^{-1} - a_{I_{ii}} > 0 \\ -a_{I_{ii}} \tan(\gamma_i)^{-1} + a_{R_{ii}} > 0 \end{cases} \quad \text{and} \quad \begin{cases} b_{R_{ii}} \tan(\gamma_i) - b_{I_{ii}} > 0 \\ -b_{I_{ii}} \tan(\gamma_i) + b_{R_{ii}} > 0 \end{cases}$$

By $0 < \gamma_i < \frac{\pi}{2}$, we have $\tan(\gamma_i) > 0$ for $i = 1, \ldots, n$. Hence, the above relations are satisfied if

$$\begin{cases} a_{R_{ii}} > a_{I_{ii}} \tan(\gamma_i) \\ a_{R_{ii}} > a_{I_{ii}} \tan(\gamma_i)^{-1} \end{cases} \quad \text{and} \quad \begin{cases} b_{R_{ii}} > b_{I_{ii}} \tan(\gamma_i)^{-1} \\ b_{R_{ii}} > b_{I_{ii}} \tan(\gamma_i), \end{cases}$$

which is for

$$a_{R_{ii}} > a_{I_{ii}} \max\{\tan(\gamma_i), \tan(\gamma_i)^{-1}\}; \quad b_{R_{ii}} > b_{I_{ii}} \max\{\tan(\gamma_i), \tan(\gamma_i)^{-1}\}. \tag{3.3}$$

If both these conditions are satisfied for $i = 1, \ldots, n$, then A and B have positive diagonal.

Furthermore, we prove the following lemma, which will be useful for the convergence analysis. For simplicity of notation, as Ω is a diagonal matrix, we denote its diagonal elements ω_{ii} simply by ω_i, $i = 1, \ldots, 2n$.

Lemma 3.1. *Assume that A, B as in (2.5) have positive diagonal, i.e. that (3.3) is satisfied. If*

$$\begin{aligned} &\frac{a_{R_{ii}}\tan(\gamma_i)^{-1}-a_{I_{ii}}}{b_{R_{ii}}\tan(\gamma_i)-b_{I_{ii}}} \le \omega_i \le \frac{|a_{R_{ii}}+a_{I_{ii}}\tan(\gamma_j)^{-1}|}{|(b_{R_{ii}}+b_{I_{ii}}\tan(\gamma_i))|};\\ &\frac{a_{R_{ii}}-a_{I_{ii}}\tan(\gamma_i)^{-1}}{b_{R_{ii}}-b_{I_{ii}}\tan(\gamma_i)} \le \omega_{i+n} \le \frac{|a_{R_{ii}}\tan(\gamma_i)^{-1}+a_{I_{ij}}|}{|b_{R_{ii}}\tan(\gamma_i)+b_{I_{ii}}|} \end{aligned} \tag{3.4}$$

for $i = 1, \dots, n$ and

$$\begin{aligned} &\omega_j \le \min\left\{\frac{|a_{R_{ij}}\tan(\gamma_j)^{-1}-a_{I_{ij}}|}{|(b_{R_{ij}}\tan(\gamma_j)-b_{I_{ij}})|}, \frac{|a_{R_{ij}}+a_{I_{ij}}\tan(\gamma_j)^{-1}|}{|(b_{R_{ij}}+b_{I_{ij}}\tan(\gamma_j))|}\right\},\\ &\omega_{j+n} \le \min\left\{\frac{|a_{R_{ij}}-a_{I_{ij}}\tan(\gamma_j)^{-1}|}{|(b_{R_{ij}}-b_{I_{ij}}\tan(\gamma_j))|}, \frac{|a_{R_{ij}}\tan(\gamma_j)^{-1}+a_{I_{ij}}|}{|(b_{R_{ij}}\tan(\gamma_j)+b_{I_{ij}})|}\right\} \end{aligned} \tag{3.5}$$

$i, j = 1, \dots, n$ with $i \ne j$ for all elements where at least one between $b_{R_{ij}}$ and $b_{I_{ij}}$ is nonzero (and setting to infinity possible bounds with 0 denominator), then

$$D_B\Omega \ge D_A; \quad |L_B|\Omega \le |L_A|; \quad |U_B|\Omega \le |U_A|.$$

Proof. In order to have $D_B\Omega \ge D_A$, the form of A, B in (2.5) requires

$$\begin{cases} (b_{R_{ii}}\tan(\gamma_i)-b_{I_{ii}})\omega_i \ge a_{R_{ii}}\tan(\gamma_i)^{-1}-a_{I_{ii}}\\ (b_{R_{ii}}-b_{I_{ii}}\tan(\gamma_i))\omega_{i+n} \ge a_{R_{ii}}-a_{I_{ii}}\tan(\gamma_i)^{-1} \end{cases}$$

By (3.3), the terms on both sides are positive. Hence, the above conditions are satisfied if

$$\omega_i \ge \frac{a_{R_{ii}}\tan(\gamma_i)^{-1}-a_{I_{ii}}}{b_{R_{ii}}\tan(\gamma_i)-b_{I_{ii}}}; \quad \omega_{i+n} \ge \frac{a_{R_{ii}}-a_{I_{ii}}\tan(\gamma_i)^{-1}}{b_{R_{ii}}-b_{I_{ii}}\tan(\gamma_i)} \tag{3.6}$$

As regards the off-diagonal terms, first notice that, in the special case $b_{R_{ij}} = b_{I_{ij}} = 0$, $|L_B|\Omega \le |L_A|$ and $|U_B|\Omega \le |U_A|$ at the index i, j. Instead, when at least one between $b_{R_{ij}}$ and $b_{I_{ij}}$ is nonzero, analyzing the blocks that constitute A and B we notice that the following relations must be satisfied.

$$\begin{cases} |a_{R_{ij}}\tan(\gamma_j)^{-1}-a_{I_{ij}}| \ge |(b_{R_{ij}}\tan(\gamma_j)-b_{I_{ij}})\omega_j|\\ |a_{R_{ij}}-a_{I_{ij}}\tan(\gamma_j)^{-1}| \ge |(b_{R_{ij}}-b_{I_{ij}}\tan(\gamma_j))\omega_{j+n}| \end{cases} \quad \text{for } i = 1, \dots, n;\ j = 1, \dots, n;\ i \ne j$$

$$\begin{cases} |-a_{R_{ij}}-a_{I_{ij}}\tan(\gamma_j)^{-1}| \ge |(-b_{R_{ij}}-b_{I_{ij}}\tan(\gamma_j))\omega_j|\\ |a_{R_{ij}}\tan(\gamma_j)^{-1}+a_{I_{ij}}| \ge |(b_{R_{ij}}\tan(\gamma_j)+b_{I_{ij}})\omega_{j+n}| \end{cases} \quad \text{for } i = 1, \dots, n;\ j = 1, \dots, n.$$

Each of these equations is immediately satisfied if the right-hand side is zero. In all other cases, we have some requirement on the diagonal entries of Ω. In particular, for $i \ne j$, it is easy to verify that these conditions are satisfied when (3.5) is satisfied, provided that fractions with zero denominator are considered to be infinity. Finally, for the top-right and for the bottom-left blocks, we must consider even the case $i = j$. By the analysis of these blocks, we must have

$$\omega_i \le \frac{|a_{R_{ii}}+a_{I_{ii}}\tan(\gamma_j)^{-1}|}{|(b_{R_{ii}}+b_{I_{ii}}\tan(\gamma_i))|}; \quad \omega_{i+n} \le \frac{|a_{R_{ii}}\tan(\gamma_i)^{-1}+a_{I_{ij}}|}{|(b_{R_{ii}}\tan(\gamma_j)+b_{I_{ii}})|}.$$

These conditions, together with (3.6), finally provide the condition (3.4). □

Remark 3.2. Conditions that require particular relations among the entries of the matrices of the problem are common in the literature of splitting algorithms for HLCPs (see, e.g., [12,13]). Furthermore, there is no contradiction among the conditions of Lemma 3.1. Thus, while it may be impossible to satisfy these conditions for some matrices and some choices of $\boldsymbol{\gamma}$, there also exist many matrices that can satisfy them. For instance, consider the following matrices that could come from classical finite-difference stencils:

$$A_R = \begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2 \end{pmatrix} \quad A_I = \begin{pmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1 \end{pmatrix} \quad B_R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad B_I = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1 \end{pmatrix}.$$

The positivity of the diagonals of A, B is satisfied for $0.5 < \tan(\gamma_i) < 2$. Considering, for instance, $\tan(\gamma_i) = 1.5$, the conditions (3.4) are satisfied for

$$\frac{4}{21} \le \omega_i \le \frac{8}{3}; \quad \frac{2}{3} \le \omega_{i+n} \le \frac{28}{3};$$

for $i = 1, \dots, n$. In the lower triangular part, we have that both B_R and B_I are zero, which immediately implies $|L_B|\Omega \le |L_A|$ for any choice of Ω. Finally, for the upper triangular part, $\omega_i \le 8/9$ is required, for $i = 1, \dots, 2n$. Hence, all the conditions are satisfied for

$$\frac{4}{21} \le \omega_i \le \frac{8}{9}; \quad \frac{2}{3} \le \omega_{i+n} \le \frac{8}{9}.$$

Remark 3.3. The conditions of Lemma 3.1 can be satisfied in a much larger range of parameters than in the modulus-based methods for CLCPs in [4]. Indeed, we here have a range of values of $\tan(\gamma_i)$ that we can consider, while the simplified method of [4] required $\tan(\gamma_i) = 1$ for all i. Furthermore, the choice of B is general, while CLCPs include just the case $B_R = I,\ B_I = 0$.

Remark 3.4. Finally, it is worth remarking that, based on the analysis of Lemma 3.1, it is possible to formulate analogous conditions for matrices that satisfy different assumptions. In particular, by proceeding as in Lemma 3.1, it is straightforward to find corresponding conditions for problems where

$$D_B\Omega \le D_A; \quad |L_B|\Omega \ge |L_A|; \quad |U_B|\Omega \ge |U_A|.$$

In this case, it would also be possible to simply rename the vectors and matrices of the original problem (1.1) to obtain real-valued formulations that directly satisfy Lemma 3.1.

3.2.2. Auxiliary lemmas for commuting matrices

Let us now perform an analysis similar to the previous subsection with regard to the matrices A and B as in (2.8), obtained under the assumption that B_{RI} and T_2 commute. Furthermore, we have earlier remarked that this case implies $B_I = 0$.

First, we notice that A, B have positive diagonal when A_R and B_R have positive diagonal. This is apparent for B. As regards A, this comes from the fact that, carrying out the matrix products of (2.8), we can write

$$A = \frac{1}{2}\begin{pmatrix} (\tan(\boldsymbol{\gamma})^{-1}A_R + A_I)\tan(\boldsymbol{\gamma})^{-1} - \tan(\boldsymbol{\gamma})^{-1}A_I + A_R & (\tan(\boldsymbol{\gamma})^{-1}A_R + A_I)\tan(\boldsymbol{\gamma})^{-1} + \tan(\boldsymbol{\gamma})^{-1}A_I - A_R \\ (\tan(\boldsymbol{\gamma})^{-1}A_R - A_I)\tan(\boldsymbol{\gamma})^{-1} - \tan(\boldsymbol{\gamma})^{-1}A_I - A_R & (\tan(\boldsymbol{\gamma})^{-1}A_R - A_I)\tan(\boldsymbol{\gamma})^{-1} + \tan(\boldsymbol{\gamma})^{-1}A_I + A_R, \end{pmatrix}$$

which, for diagonal elements, reduces to

$$\frac{1}{2}\left(\tan(\gamma_i)^{-2}a_{R_{ii}} + a_{I_{ii}}\tan(\gamma_i)^{-1} - a_{I_{ii}}\tan(\gamma_i)^{-1} + a_{R_{ii}}\right) = \frac{1}{2}\left(\tan(\gamma_i)^{-2}a_{R_{ii}} + a_{R_{ii}}\right)$$

for $i = 1, \ldots, n$. Such terms are evidently positive for $a_{R_{ii}} > 0$. Under this assumption, let us then analyze A and B similarly to the previous subsection.

Lemma 3.2. *Consider A, B as in (2.8) and assume that A_R, B_R have positive diagonal. If*

$$\omega_i \ge \frac{a_{R_{ii}}(\tan(\gamma_i)^{-2} + 1)}{2b_{R_{ii}}}; \quad \omega_{i+n} \ge \frac{a_{R_{ii}}(\tan(\gamma_i)^{-2} + 1)}{2b_{R_{ii}}} \tag{3.7}$$

for $i = 1, \ldots, n$ and

$$\begin{aligned} \omega_j &\le \frac{|a_{R_{ij}}(\tan(\gamma_i)^{-1}\tan(\gamma_j)^{-1} + 1) + a_{I_{ij}}(\tan(\gamma_j)^{-1} - \tan(\gamma_i)^{-1})|}{2|b_{R_{ij}}|} \\ \omega_{j+n} &\le \frac{|a_{R_{ij}}(\tan(\gamma_i)^{-1}\tan(\gamma_j)^{-1} + 1) - a_{I_{ij}}(\tan(\gamma_j)^{-1} - \tan(\gamma_i)^{-1})|}{2|b_{R_{ij}}|} \end{aligned} \tag{3.8}$$

for any index $i, j = 1, \ldots, n$ with $i \ne j$ where $b_{R_{ij}} \ne 0$, then

$$D_B\Omega \ge D_A; \quad |L_B|\Omega \le |L_A|; \quad |U_B|\Omega \le |U_A|.$$

Proof. By the definitions of A and B, the condition $D_B\Omega \ge D_A$ is satisfied if

$$b_{R_{ii}}\omega_i \ge \frac{1}{2}\left(\tan(\gamma_i)^{-2}a_{R_{ii}} + a_{R_{ii}}\right),$$

which is if (3.7) holds.

As regards the off-diagonal terms, the conditions $|L_B|\Omega \le |L_A|$ and $|U_B|\Omega \le |U_A|$ imply, for the diagonal blocks,

$$\begin{cases} \frac{1}{2}|\tan(\gamma_i)^{-1}a_{R_{ij}}\tan(\gamma_j)^{-1} + a_{I_{ij}}\tan(\gamma_j)^{-1} - \tan(\gamma_i)^{-1}a_{I_{ij}} + a_{R_{ij}}| \ge |b_{R_{ij}}|\omega_j \\ \frac{1}{2}|\tan(\gamma_i)^{-1}a_{R_{ij}}\tan(\gamma_j)^{-1} - a_{I_{ij}}\tan(\gamma_j)^{-1} + \tan(\gamma_i)^{-1}a_{I_{ij}} + a_{R_{ij}}| \ge |b_{R_{ij}}|\omega_{j+n} \end{cases}$$

for every entry of indices $i, j = 1, \ldots, n$ with $i \ne j$ where $b_{R_{ij}} \ne 0$. These inequalities are evidently satisfied under the condition (3.8) of the lemma. Instead, the conditions $|L_B|\Omega \le |L_A|$ and $|U_B|\Omega \le |U_A|$ are immediately satisfied on the upper-right and lower-left blocks by the assumption $B_I = 0$. □

Remark 3.5. The proposed conditions do not present contradictions. For instance, avoiding the simple case of a CLCP, let us consider a symmetric B_R (which commutes with $\tan(\boldsymbol{\gamma})^{-1}$ for any constant vector $\boldsymbol{\gamma}$) and $B_I = 0$:

$$A_R = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad A_I = \begin{pmatrix} 1 & 0 & 4 \\ -1 & -1 & 2 \\ 3 & -1 & 2 \end{pmatrix} \quad B_R = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix} \quad B_I = 0$$

It can be verified that $T_2 B_{RI} = B_{RI} T_2$ if $\gamma_1 = \gamma_2 = \cdots = \gamma_n$. Under this assumption, we can apply the conditions of Lemma 3.2. In particular, the positivity of the diagonals of A, B is satisfied by the positivity of the diagonal of A_R and B_R. Considering, for instance, $\tan(\gamma_i) = \frac{2}{3}$ for $i = 1, \dots, n$, the conditions of Lemma 3.2 are satisfied if each diagonal component of Ω is selected in the interval $[0.40625, 1.625]$.

Remark 3.6. We have remarked that the commutativity is possible only when $B_I = 0$. Nonetheless, it is easy to generalize the above results even to the case $A_I = 0$. Indeed, it is sufficient to rename vectors and matrices of the problem (1.1) (in particular, $\tilde{A}$ should be renamed to $\tilde{B}$ and vice-versa), possibly changing sign to both sides of the equation. Then, it is possible to proceed exactly as before, provided that the renamed B_R and T_2 commute.

Remark 3.7. The conditions of Lemma 3.1 are significantly simplified if we consider a CLCP. Indeed, for a CLCP, $b_{R_{ij}} = b_{I_{ij}} = 0$ for all off-diagonal elements. Thus, all the conditions are immediately satisfied off-diagonal. The remaining conditions hold with $b_{R_{ii}} = 1$, providing the following corollary.

Corollary 3.1. *Consider A, B as in (2.8) and assume that A_R has positive diagonal, $B_R = I$, and $B_I = 0$ (hence, $B = I$). If*

$$\frac{\tan(\gamma_i)^{-2} a_{R_{ii}} + a_{R_{ii}}}{2} \le \omega_i; \quad \frac{\tan(\gamma_i)^{-2} a_{R_{ii}} + a_{R_{ii}}}{2} \le \omega_{i+n}$$

for $i = 1, \dots, n$, then $D_B \Omega \ge D_A$, $|L_B|\Omega \le |L_A|$, and $|U_B|\Omega \le |U_A|$.

Remark 3.8. In the simplified setting of [4], $\tan(\gamma_i) = 1$ is taken for $i = 1, \dots, n$, which further reduces the conditions to

$$\omega_i \ge a_{R_{ii}}; \quad \omega_{i+n} \ge a_{R_{ii}}.$$

It is worth noticing that such conditions coincide with the ones that were required on Ω in [5] for the convergence of the modulus-based algorithms for LCPs. The only difference is that the dimension of Ω is here doubled as an effect of the real-valued rewriting of the complex problem. Therefore, the conditions of Lemma 3.2 in the simplified setting of [4] reduce to the convergence conditions of modulus-based matrix splitting methods for real LCPs.

3.2.3. Convergence theorems for modulus-based methods

We now study the convergence of modulus-based matrix splitting methods for CHLCPs, with special regard to AOR splittings. With respect to the direct application of known results, we replace the conditions on the entries of A, B by conditions on A_R, A_I, B_R, B_I via the lemmas of the previous subsections. Furthermore, notice that [12] requires that the signs of off-diagonal elements are the same in A and B. An analogous consideration applies to the condition in [26, Corollary 1]. The more general framework proposed by [13] would instead require $\Omega = D_A D_B^{-1}$ for several of the AOR splittings. Indeed, whenever the diagonal terms appear in both M_A, M_B and in N_B, N_B, the requirements $\langle M_B \rangle \Omega \ge \langle M_A \rangle$ and $|N_A| \ge |N_B|\Omega$ of [13, Theorem 2.4] are conflicting on the diagonal and can be satisfied only by $\Omega = D_A D_B^{-1}$. This situation occurs even for SOR with general α, while Jacobi and Gauss–Seidel are immune, since the diagonal terms appear only in M_A, M_B. We relax these requirements in the following theorem, which can thus be useful for real-valued HLCPs as well, whenever $D_B \Omega \ge D_A$, $|L_A| \ge |L_B|\Omega$, and $|U_A| \ge |U_B|\Omega$.

Theorem 3.2. *Let $A, B \in \mathbb{R}^{2n \times 2n}$ be defined as in (2.5) and let the assumptions of Lemma 3.1 be satisfied. Alternatively, let $A, B \in \mathbb{R}^{2n \times 2n}$ be defined as in (2.8) and let the assumptions of Lemma 3.2 be satisfied. If A and $M_A + M_B \Omega$ are H_+-matrices, then the sequence of iterates generated by the modulus-based AOR methods for the CHLCP (1.1) converges to the solution of the problem for $\alpha \in (0, 1]$ and $0 < \beta \le \alpha$. Given $\alpha \in (1, 2)$, the modulus-based AOR methods are still convergent if the assumptions of Lemmas 3.1 or 3.2 are satisfied with $\Omega \le \frac{\alpha}{2\kappa(\alpha-1)} D_A D_B^{-1}$ for some $\kappa > \frac{\alpha}{2}$ for which $A - \left(1 - \frac{2\kappa-\alpha}{2\alpha\kappa}\right) D_A$ is still an H_+-matrix.*

Proof. Let us start from the first claim and assume $\alpha \in (0, 1]$. Under the assumptions of Lemma 3.1, we have $D_B \Omega \ge D_A$, $|L_A| \ge |L_B|\Omega$, and $|U_A| \ge |U_B|\Omega$. Furthermore, D_A and D_B are positive diagonal matrices. Denoting by $\boldsymbol{x}^*$ the solution of the problem (2.13), we can evaluate the error at the generic iteration $(k+1)$ as

$$\begin{aligned} |\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| &= \left| (M_A + M_B \Omega)^{-1} \left[(N_A + N_B \Omega)(\boldsymbol{x}^{(k)} - \boldsymbol{x}^*) + (B\Omega - A)(|\boldsymbol{x}^{(k)}| - |\boldsymbol{x}^*|) \right] \right| \\ &\le \langle M_A + M_B \Omega \rangle^{-1} (|N_A + N_B \Omega| + |B\Omega - A|) |\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| \\ &:= J |\boldsymbol{x}^{(k)} - \boldsymbol{x}^*|, \end{aligned}$$

with $J = \mathcal{M}^{-1}\mathcal{N}$, where $\mathcal{M} := \langle M_A + M_B \Omega \rangle$ and $\mathcal{N} := |N_A + N_B \Omega| + |B\Omega - A|$. In the previous evaluations, we have used the triangle inequality and the relation $|(M_A + M_B \Omega)^{-1}| \le \langle M_A + M_B \Omega \rangle^{-1}$ when $M_A + M_B \Omega$ is an H_+-matrix [35]. Let us then analyze the matrix $V := \mathcal{M} - \mathcal{N}$ using the AOR splittings (2.18). As regards the diagonal terms, with the assumptions $D_B > 0$, $D_A > 0$, $D_B\Omega \geq D_A$, and $\alpha \in (0, 1]$, we have

$$D_V = \frac{D_A}{\alpha} + \frac{D_B\Omega}{\alpha} - \left|\frac{1-\alpha}{\alpha}(D_A + D_B\Omega)\right| - |D_B\Omega - D_A|$$
$$= \frac{D_A}{\alpha} + \frac{D_B\Omega}{\alpha} - \frac{1-\alpha}{\alpha}(D_A + D_B\Omega) - D_B\Omega + D_A = 2D_A.$$

As regards the lower triangular term $-L_V$, we have

$$-L_V = -\left|\frac{\beta}{\alpha}L_A + \frac{\beta}{\alpha}L_B\Omega\right| - \left|\frac{\alpha-\beta}{\alpha}L_A + \frac{\alpha-\beta}{\alpha}L_B\Omega\right| - |L_A - L_B\Omega|$$

Under the assumptions $0 < \beta \leq \alpha$ and $|L_A| \geq |L_B\Omega|$,

$$-L_V = -\frac{\beta}{\alpha}|L_A + L_B\Omega| - \frac{\alpha-\beta}{\alpha}|L_A + L_B\Omega| - |L_A - L_B\Omega|$$
$$= -|L_A + L_B\Omega| - |L_A - L_B\Omega| = -2|L_A|,$$

where, in the last passage, we have used the fact that $|L_A| \geq |L_B\Omega|$ implies that the sign of L_A prevails in every entry of $|L_A + L_B\Omega|$ and of $|L_A - L_B\Omega|$, so that the two absolute values must have the same sign component-wise. Finally, as regards the upper triangular term $-U_V$,

$$-U_V = -|U_A + U_B\Omega| - |U_A - U_B\Omega|,$$

the assumption $|U_A| \geq |U_B\Omega|$ implies that the sign of U_A prevails in every component of the two absolute values of the previous equation, so that we obtain

$$U_V = -2|U_A|.$$

It follows that $V = 2\langle A\rangle$, and therefore it is a nonsingular M-matrix. Furthermore, $\mathcal{M} - \mathcal{N}$ is an M-splitting because $\mathcal{M}$ is an M-matrix and $\mathcal{N}$ is non-negative. Therefore, $\rho(J) < 1$ by [35, Theorem 3.4], proving the convergence of the algorithms.

As regards the second claim, we just need to repeat the analysis for D_V, which, for $\alpha \in (1, 2)$, becomes

$$D_V = \frac{D_A}{\alpha} + \frac{D_B\Omega}{\alpha} + \frac{1-\alpha}{\alpha}(D_A + D_B\Omega) - D_B\Omega + D_A = 2\frac{D_A}{\alpha} - 2D_B\Omega\frac{\alpha-1}{\alpha}$$

If $\Omega \leq \frac{\alpha}{2\kappa(\alpha-1)}D_A D_B^{-1}$,

$$D_V \geq 2\frac{D_A}{\alpha} - \frac{D_A}{\kappa} = \frac{2\kappa-\alpha}{\alpha\kappa}D_A$$

Notice that $\frac{\alpha}{2\kappa(\alpha-1)} > 0$ for $\alpha \in (1, 2)$, so Ω is a positive diagonal matrix as required by the algorithms. Furthermore, $\kappa > \alpha/2$ ensures the positivity of the diagonal matrix D_V by the previous relation. Finally, we notice that the analysis of the triangular parts of V remains unchanged for $0 < \beta \leq \alpha$. Therefore, the algorithm still converges if

$$-2|L_A| - 2|U_A| + \frac{2\kappa-\alpha}{\alpha\kappa}D_A = -2|L_A| - 2|U_A| + 2D_A - 2D_A + \frac{2\kappa-\alpha}{\alpha\kappa}D_A = 2\langle A\rangle - 2D_A + \frac{2\kappa-\alpha}{\alpha\kappa}D_A$$

is an M-matrix, which is if the matrix

$$A - \left(1 - \frac{2\kappa-\alpha}{2\alpha\kappa}\right)D_A$$

is an H_+-matrix. It is worth noticing that the requirement tends to reduce to A being an H_+ matrix as κ increases, which, however, implies that a smaller Ω must be chosen. □

The previous theorem can easily be adapted to the common Jacobi, Gauss–Seidel, and SOR splittings by the following corollary.

Corollary 3.2. *Let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.5) and let the assumptions of Lemma 3.1 be satisfied. Alternatively, let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.8) and let the assumptions of Lemma 3.2 be satisfied. If A is an H_+-matrix, then the modulus-based Jacobi method for the CHLCP (1.1) converges.*

If, in addition, $D_A - L_A + (D_B - L_B)\Omega$ is an H_+-matrix, the modulus-based Gauss–Seidel methods for the CHLCP (1.1) converges.

Finally, if $\frac{D_A-\alpha L_A}{\alpha} + \frac{D_B-\alpha L_B}{\alpha}\Omega$ is an H_+-matrix, the modulus-based SOR method for the CHLCP (1.1) converges for $\alpha \in (0, 1]$. Convergence for $\alpha \in (1, 2)$ is ensured if, moreover, $\Omega \leq \frac{\alpha}{2\kappa(\alpha-1)}D_A D_B^{-1}$ for some $\kappa > \frac{\alpha}{2}$ for which $A - \left(1 - \frac{2\kappa-\alpha}{2\alpha\kappa}\right)D_A$ is still an H_+-matrix.

Proof. It follows directly from Theorem 3.2 considering $\alpha = 1, \beta = 0$ for Jacobi, $\alpha = \beta = 1$ for Gauss–Seidel, and $\alpha = \beta$ for SOR. □

Remark 3.9. Notice that Theorem 3.2 can be straightforwardly adapted to the case where $D_B\Omega \leq D_A, |L_B|\Omega \geq |L_A|, |U_B|\Omega \geq |U_A|$ under the assumption that B is an H_+-matrix. Indeed, in this case, following Remark 3.4, we can find conditions analogous to those of Lemma 3.1. Under these conditions and repeating the passages of Theorem 3.2, we obtain that the iteration matrix V is an M-matrix.

Corollary 3.3. *If $\tilde{B} = I$ (i.e., in the special case of a CLCP), the conditions on Ω of Theorem 3.2 reduce to those required by Remark 3.7 for general γ and by Remark 3.8 for $\gamma = \frac{\pi}{4}e$.*

Proof. The proof follows directly by the fact that, under the assumptions of the corollary, $D_B\Omega \geq D_A$, $|L_A| \geq |L_B|\Omega$, and $|U_A| \geq |U_B|\Omega$ by the Remarks 3.7 or 3.8, depending on the considered case. □

Interestingly, for a CLCP with $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, Theorem 3.2 with Corollary 3.3 reduces to something similar to the convergence theorem of modulus-based algorithms for real-valued LCPs [36]. Indeed, while [36] requires $\Omega \geq D_A$, our analysis requires an analogous condition of just the real part of A, i.e., $\omega_i \geq a_{R_{ii}}, \omega_{i+n} \geq a_{R_{ii}}$.

3.2.4. Convergence theorems for accelerated modulus-based methods

Theorem 3.3. *Let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.5) and let the assumptions of Lemma 3.1 be satisfied. Alternatively, let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.8) and let the assumptions of Lemma 3.2 be satisfied. Furthermore, consider the AOR splittings in (2.18) and assume that A and $M_A + M_B\Omega$ are H_+-matrices and that $\langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{A_2} + N_{B_2}\Omega|$ is an M-matrix. Then, the sequence of iterates generated by the accelerated modulus-based AOR methods for the CHLCP (1.1) converges to the solution of the problem if $\alpha \in (0, 1]$ and $0 < \beta \leq \alpha$.*

If $\alpha \in (1, 2)$, the modulus-based AOR methods for the CHLCP (1.1) are still convergent if the assumptions of Lemma 3.1 are satisfied with $\Omega \leq \frac{\alpha}{2\kappa(\alpha-1)} D_A D_B^{-1}$ for some $\kappa > \frac{\alpha}{2}$ for which $A - \left(1 - \frac{2\kappa-\alpha}{2\alpha\kappa}\right) D_A$ is still an H_+-matrix.

Proof. Starting as in the proof to Theorem 3.2, we can evaluate the error as

$$|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \leq \langle M_{A_1} + M_{B_1}\Omega\rangle^{-1} \Big[|N_{A_1} + N_{B_1}\Omega||\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| + + |M_{B_2}\Omega - M_{A_2}||\boldsymbol{x}^{(k)} - \boldsymbol{x}^*| + |N_{B_2}\Omega - N_{A_2}||\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \Big]$$

which is

$$(I - \langle M_{A_1} + M_{B_1}\Omega\rangle^{-1}|N_{B_2}\Omega - N_{A_2}|)|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \leq \leq \langle M_{A_1} + M_{B_1}\Omega\rangle^{-1}\Big[|N_{A_1} + N_{B_1}\Omega| + |M_{B_2}\Omega - M_{A_2}|\Big]|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*|.$$

The term $\langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{A_2} + N_{B_2}\Omega|$ is an M-matrix by hypothesis and, evidently, is smaller than or equal to $\langle M_{A_1} + M_{B_1}\Omega\rangle$. Furthermore, both $\langle M_{A_1} + M_{B_1}\Omega\rangle$ and $\langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{A_2} + N_{B_2}\Omega|$ are obviously Z-matrices. Therefore,

$$\langle M_{A_1} + M_{B_1}\Omega\rangle^{-1}(\langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{A_2} + N_{B_2}\Omega|) = I - \langle M_{A_1} + M_{B_1}\Omega\rangle^{-1}|N_{A_2} + N_{B_2}\Omega|$$

is an M-matrix (e.g., see [37, p. 127]) and its inverse is non-negative. Hence, we can write the error as

$$|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^*| \leq J|\boldsymbol{x}^{(k)} - \boldsymbol{x}^*|$$

with

$$J := (\langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{B_2}\Omega - N_{A_2}|)^{-1}\Big[|N_{A_1} + N_{B_1}\Omega| + |M_{B_2}\Omega - M_{A_2}|\Big]$$

Let us now analyze the term

$$V = \langle M_{A_1} + M_{B_1}\Omega\rangle - |N_{B_2}\Omega - N_{A_2}| - |N_{A_1} + N_{B_1}\Omega| - |M_{B_2}\Omega - M_{A_2}|$$

With the accelerated AOR splittings (2.19), $-|N_{B_2}\Omega - N_{A_2}| - |M_{B_2}\Omega - M_{A_2}| = -|B\Omega - A|$. Furthermore, the splitting matrices $M_{A_1}, M_{B_1}, N_{A_1}, N_{B_1}$ are the same as M_A, M_B, N_A, N_B of the non-accelerated AOR (2.18). Therefore, the matrix V is identical to the same matrix V of the proof of Theorem 3.2. The analysis of its convergence can then be carried out as in Theorem 3.2. □

Analogously to Theorem 3.2, it is easy to write corollaries to Theorem 3.3 to address widely used splittings and the commuting case.

Corollary 3.4. *Let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.5) and let the assumptions of Lemma 3.1 be satisfied. Alternatively, let $A, B \in \mathbb{R}^{2n\times 2n}$ be defined as in (2.8) and let the assumptions of Lemma 3.2 be satisfied. If A is an H_+-matrix and $\langle D_A + D_B\Omega\rangle - |L_A + L_B\Omega|$ be an M-matrix, then the accelerated modulus-based Jacobi method for the CHLCP (1.1) converges.*

If, in addition, $D_A - L_A + (D_B - L_B)\Omega$ is an H_+-matrix and $\langle D_A + D_B\Omega\rangle - 2|L_A + L_B\Omega|$ is an M-matrix, the accelerated modulus-based Gauss–Seidel method for the CHLCP (1.1) converges.

Finally, if $\frac{D_A - \alpha L_A}{\alpha} + \frac{D_B - \alpha L_B}{\alpha}\Omega$ is an H_+-matrix and $\frac{D_A}{\alpha} + \frac{D_B}{\alpha}\Omega - 2|L_A + L_B\Omega|$ is an M-matrix, the accelerated modulus-based SOR method for the CHLCP (1.1) converges for $\alpha \in (0, 1]$. Convergence of SOR for $\alpha \in (1, 2)$ is ensured if, moreover, $\Omega \leq \frac{\alpha}{2\kappa(\alpha-1)} D_A D_B^{-1}$ for some $\kappa > \frac{\alpha}{2}$ for which $A - \left(1 - \frac{2\kappa-\alpha}{2\alpha\kappa}\right) D_A$ is still an H_+-matrix.

Notice that, similarly to standard modulus-based algorithms, the above convergence analysis can be adapted to the case $D_B\Omega \leq D_A, |L_B|\Omega \geq |L_A|, |U_B|\Omega \geq |U_A|$ (in this regard, see Remark 3.9). Finally, we also remark that Theorem 3.3 is different from currently known convergence conditions for accelerated modulus-based AOR methods for real HLCPs. Indeed, the current convergence results in [12] consider general splittings, but require that the sign of the entries of several splitting matrices are the same. This is not required by Theorem 3.3.

4. Numerical experiments

We evaluate the analyzed algorithms by the following numerical experiments, which cover all the situations considered in the previous sections. Through the definition of the problems, I denotes an identity matrix of suitable dimension. When A_R, A_I, B_R, B_I are block-matrices, they are intended to be of order $n = m^2$, where m denotes the order of the blocks.

- **Problem 4.1** coincides with the first numerical example of [4], which we use to revisit (non-horizontal) CLCPs in light of our analysis on the CHLCP. In particular, Problem 4.1 consists of the CLCP with matrix $\tilde{A} = A_R + iA_I$ where

  - $A_R = \bar{A} + \mu I$, with $\bar{A} \in \mathbb{R}^{n\times n}$ block-tridiagonal matrix of blocks $\{-I, W, -I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{-1, 4, -1\}$;
  - $A_I \in \mathbb{R}^{n\times n}$ is the block-diagonal matrix of blocks $\{\bar{W}\}$, where $\bar{W} \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $\bar{W} = triadiag\{-1, 0, -1\}$.

  For a better reproducibility, we choose the know term $\boldsymbol{q}$ as in [4], where it was computed as $\boldsymbol{q} = \tilde{A}\tilde{\boldsymbol{z}} - \tilde{\boldsymbol{w}}$ where $\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}}$ were taken as the vectors of components $\tilde{z}_i = \tilde{w}_i = 1 + i$, for $i = i, \dots, n$. However, it should be noticed that these vectors are unfeasible as solutions of the CLCP, as they violate the condition $Re(\tilde{\boldsymbol{z}}^* \tilde{\boldsymbol{w}}) = 0$. Therefore, the solution will be different from $\tilde{z}_i = \tilde{w}_i = 1 + i$, for $i = i, \dots, n$, and we will evaluate the convergence of the algorithms by residual evaluations.
- **Problem 4.2** refers to the case of a CHLCP where B_{RI} and T_2 commute and where $B_{RI} \neq I$, so that we do not fall into a CLCP as in Problem 4.1. Hence, this problem (and the following ones) is novel in the literature. For this problem, we define the matrices $\tilde{A} = A_R + iA_I$ and $\tilde{B} = B_R + iB_I$ of the problem (1.1) as follows.

  - $A_R = \bar{A} + \mu I$, with $\bar{A} \in \mathbb{R}^{n\times n}$ non-symmetric block-tridiagonal matrix of blocks $\{-1.5I, W, -0.5I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{-1.5, 4, -0.5\}$;
  - $A_I \in \mathbb{R}^{n\times n}$ is the block-diagonal matrix of blocks $\{\bar{W}\}$, where $\bar{W} \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $\bar{W} = triadiag\{-1, 0, -1\}$;
  - $B_R \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{-0.25I, W, -0.25I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{-0.25, 4, -0.25\}$;
  - $B_I \in \mathbb{R}^{n\times n}$ is zero.

- **Problem 4.3** is a general CHLCP where no commutativity properties can be exploited. As such, it is more general than the Problems 4.1 and 4.2. In particular, we define $\tilde{A} = A_R + iA_I$ and $\tilde{B} = B_R + iB_I$ of the problem (1.1) as follows.

  - $A_R \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{-1.5, W, -0.5I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{-1.5, 4, -0.5\}$;
  - $A_I \in \mathbb{R}^{n\times n}$ is the block-diagonal matrix of blocks $4 * triadiag\{-1, -1, 0\}$;
  - $B_R \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{0.5I, W, 0.5I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{0.5, 4, 0.5\}$;
  - $B_I = -4I \in \mathbb{R}^{n\times n}$.

- **Problem 4** is another general CHLCP. We define $\tilde{A} = A_R + iA_I$ and $\tilde{B} = B_R + iB_I$ of the problem (1.1) as follows.

  - $A_R \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{0, W, -I\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{-1, 4, 0\}$;
  - $A_I \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{-I, W, 0\}$, where $W \in \mathbb{R}^{m\times m}$ is the tridiagonal matrix $W = triadiag\{0, -4, -1\}$;
  - $B_R \in \mathbb{R}^{n\times n}$ is the block-tridiagonal matrix of blocks $\{-I, 4I, -0.5I\}$;
  - $B_I \in \mathbb{R}^{n\times n}$ is the block-diagonal matrix made of tridiagonal blocks $triadiag\{-1, -4, -0.5\}$.

With the exception of Problem 4.1, the term $\boldsymbol{q}$ is computed as $\boldsymbol{q} = \tilde{A}\tilde{\boldsymbol{z}} - \tilde{B}\tilde{\boldsymbol{w}}$ where $\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{w}}$ are taken as vectors that satisfy the conditions of the CHCLP (1.1). In particular, when $\boldsymbol{\gamma} = \pi/4\boldsymbol{e}$, we take $\tilde{z}_i = 1 + i$, $\tilde{w}_i = 1 - i$, for $i = i, \dots, n$. For other choices of γ, we take $\tilde{z}_i = 1 + \tan(\gamma)i$, $\tilde{w}_i = \tan(\gamma) - i$, which satisfy all the conditions of the problem. The values of μ, r, Ω, γ, and the starting iterate are changed in the numerical experiments and are reported in the following subsections. To use comparable parameters in all algorithms, in Method 1 we set $\Omega_2 = \frac{1}{r}I$ and scale with respect to r. We then set $\Omega_1 = \Omega$.

The solution algorithms are implemented in Fortran and run on an M1 MacBook Pro (16", 2021) with 32 GBs of RAM. The methods are stopped when the ℓ_2-norm of the residual

$$\mathbf{res}_k := A_{RI}\begin{pmatrix} \boldsymbol{z}_R^{(k)} \\ \boldsymbol{z}_I^{(k)} \end{pmatrix} - B_{RI}\begin{pmatrix} \boldsymbol{w}_R^{(k)} \\ \boldsymbol{w}_I^{(k)} \end{pmatrix} - \begin{pmatrix} \boldsymbol{q}_R \\ \boldsymbol{q}_I \end{pmatrix}$$

is smaller than a tolerance $tol = 10^{-6}$. Indeed, notice that the non-negativity and complementarity conditions are satisfied at each iteration by the modulus-based formulations (see Eqs. (2.12) and (2.15)). In turn, this ensures that the conditions of the CHLCP are satisfied via Eq. (2.2). Therefore, checking that the residual of the linear system (2.1) is sufficiently small ensures that we are close to the solution of the problem.

Table 1
Solution of Problem 4.1 by the considered modulus-based Jacobi, Gauss–Seidel and SOR methods, using $\boldsymbol{x}^{(0)} = \mathbf{0}$, $r = 2$, $\mu = 4$, $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, and $\Omega = diag(A_R) = 8I$. In SOR, we set $\alpha = 1.1$.

m	Method	it	res	t	m	Method	it	res	t
	MJ	35	9.6E−7	8.0E−2		MJ	37	7.9E−7	0.54
	AMJ	28	4.8E−7	5.9E−2		AMJ	29	6.6E−7	0.44
	BDMJ	30	6.1E−7	5.5E−2		BDMJ	31	6.8E−7	0.45
	MGS	28	4.8E−7	5.1E−2		MGS	29	6.6E−7	0.41
100	AMGS	19	4.5E−7	3.7E−2	250	AMGS	20	3.2E−7	0.30
	BDMGS	24	9.7E−7	4.4E−2		BDMGS	25	8.9E−7	0.35
	MSOR	26	6.4E−7	4.8E−2		MSOR	27	8.1E−7	0.40
	AMSOR	17	3.5E−7	3.4E−2		AMSOR	17	8.5E−7	0.26
	BDMSOR	22	9.9E−7	4.1E−2		BDMSOR	23	8.1E−7	0.33

m	Method	it	res	t	m	Method	it	res	t
	MJ	38	8.9E−7	2.4		MJ	40	5.6E−7	11.4
	AMJ	30	6.7E−7	2.0		AMJ	31	6.3E−7	9.0
	BDMJ	32	6.1E−7	2.1		BDMJ	33	6.2E−7	9.4
	MGS	30	6.7E−7	1.9		MGS	31	6.3E−7	8.6
500	AMGS	20	6.4E−7	1.3	1000	AMGS	21	6.0E−7	5.8
	BDMGS	26	6.6E−7	1.6		BDMGS	27	6.1E−7	7.5
	MSOR	28	7.3E−7	1.8		MSOR	29	6.0E−7	8.2
	AMSOR	18	5.6E−7	1.2		AMSOR	19	4.9E−7	5.3
	BDMSOR	24	5.6E−7	1.5		BDMSOR	25	5.4E−7	7.0

Table 2
Sensitivity to the starting iterate. Mean of the results of 10 consecutive runs with different random starting iterates, $m = 250$.

Method	it	res	t	Method	it	res	t	Method	it	res	t
MJ	36	1.0E−6	0.54	MGS	28	9.5E−7	0.41	MSOR	27	5.6E−7	0.40
AMJ	28	9.4E−7	0.45	AMGS	19	7.3E−7	0.29	AMSOR	20	3.9E−7	0.31
BDMJ	52	9.2E−7	0.77	BDMGS	26	6.9E−7	0.38	BDMSOR	32	7.6E−7	0.48

Table 3
Solution to Problem 4.1 with $\boldsymbol{\gamma} = \pi/5\boldsymbol{e}$, $\Omega = 12I$ (left) and $\boldsymbol{\gamma} = \frac{4}{15}\pi\boldsymbol{e}$, $\Omega = 8I$ (right). Dimension $m = 500$ and SOR parameter set at $\alpha = 1.05$.

Method	it	res	t	Method	it	res	t
MJ	32	8.3E−7	2.1	MJ	40	7.5E−7	2.6
AMJ	24	7.1E−7	1.6	AMJ	31	9.7E−7	2.1
BDMJ	32	8.0E−7	2.1	BDMJ	34	9.2E−7	2.2
MGS	24	6.1E−7	1.5	MGS	31	9.7E−7	2.0
AMGS	21	5.1E−7	1.4	AMGS	23	3.8E−7	1.5
BDMGS	27	4.8E−7	1.7	BDMGS	29	4.3E−7	1.8
MSOR	24	4.4E−7	1.5	MSOR	30	9.4E−7	1.9
AMSOR	20	4.4E−7	1.3	AMSOR	22	2.9E−7	1.4
BDMSOR	26	4.4E−7	1.7	BDMSOR	28	3.6E−7	1.8

4.1. Revisiting some results on complex-valued LCPs

Let us consider Problem 4.1. As in [4], we set the starting iterate $\boldsymbol{x}^{(0)} = \mathbf{0}$, $r = 2$, $\mu = 4$, $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, and $\Omega = diag(A_R)$. It is easy to verify that, with these choices, the conditions of Lemma 3.2 – Corollary 3.1 are satisfied. Furthermore, A is an H_+-matrix. Hence, by Theorem 3.2 – Corollary 3.2, the modulus-based Jacobi (MJ), modulus-based Gauss–Seidel (MGS), and modulus-based SOR (MSOR) with $\alpha \in (0, 1)$ converge to the solution of this problem. As regards $\alpha > 1$, A remains an H_+-matrix for several combinations of κ and α. For instance, choosing $\alpha = 1.1$, Theorem 3.2 – Corollary 3.2 ensure the convergence of the algorithms for Ω roughly smaller than $31I$ with $\kappa = 1.4$. The choice $\Omega = diag(A_R) = 8I$ is well within the admissible range. Analogous considerations apply to the accelerated methods AMJ, AMGS, AMSOR 3.3 via Corollary 3.4.

reports the results obtained from solving the problem with MJ, MGS, MSOR, AMJ, AMGS, AMSOR and the corresponding algorithms in [4], here denoted by block-diagonal modulus-based Jacobi (BDMJ), Gauss–Seidel (BDMGS), and SOR (BDMSOR). In the table, *it* denotes the number of iterations at convergence, *res* the residual in Euclidean norm, and *t* the computational time (in seconds).

The results of substantially confirm what was observed in [4], but they also afford for some new insights. In particular:

- In [4], SOR splittings were not considered. , instead, shows that SOR can provide a further acceleration of the convergence.
- In [4], no comparison with accelerated algorithms was performed. The results of , instead, demonstrate that accelerated modulus-based methods for LCPs can outperform the block-diagonal splittings proposed in [4]. This is particularly evident using the Gauss–Seidel and the SOR splittings, and for problems of large dimension;

Table 4
Solution to Problem 4.2 by the considered modulus-based Jacobi, Gauss–Seidel and SOR methods, using $m = 250$, $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, $r = 2$, $\mu = 0$, $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, and different choices of Ω. In SOR, we set $\alpha = 1.1$.

Ω	Method	it	res	t	Ω	Method	it	res	t
	MJ	1278	9.6E−7	19.0		MJ	1606	9.8E−7	24.1
	AMJ	920	9.5E−7	14.5		AMJ	1283	9.5E−7	20.4
	BDMJ	1275	9.9E−7	18.7		BDMJ	1604	9.9E−7	24.2
	MGS	770	9.9E−7	11.3		MGS	1063	9.8E−7	15.8
I	AMGS	394	8.8E−7	6.1	$1.5I$	AMGS	729	9.7E−7	11.4
	BDMGS	770	9.9E−7	11.3		BDMGS	1063	9.9E−7	15.9
	MSOR	650	9.7E−7	9.8		MSOR	913	9.9E−7	14.1
	AMSOR	269	8.4E−7	4.3		AMSOR	577	9.9E−7	9.3
	BDMSOR	651	9.2E−7	9.9		BDMSOR	914	9.5E−7	14.1

Ω	Method	it	res	t
	MJ	1935	9.6E−7	29.1
	AMJ	1647	9.7E−7	26.3
	BDMJ	1933	9.9E−7	29.1
	MGS	1356	9.8E−7	20.2
$2I$	AMGS	1061	9.9E−7	16.3
	BDMGS	1356	9.9E−7	20.2
	MSOR	1176	9.9E−7	18.1
	AMSOR	881	9.4E−7	14.1
	BDMSOR	1177	9.6E−7	17.9

- The algorithms [4] appear to be more sensitive than standard modulus-based methods to the choice of the starting iterate. Indeed, consider, for instance, $m = 250$. With respect to , we generate the starting iterate by the Fortran *random_number* function intrinsic subroutine. Taking the mean of 10 consecutive runs with different random numbers, we get the results in .
  It seems that this issue does not appear with constant starting iterates: for instance, we did not observe significant changes in the convergence speed of all methods when we set $\boldsymbol{x}^{(0)} = \boldsymbol{e}$ or $\boldsymbol{x}^{(0)} = -\boldsymbol{e}$. Notice that this issue may be problem specific. Indeed, Problem 4.1 is made by structured matrices and the solution to Problem 4.1 is mostly constant. On the other hand, the algorithms in [4] tend to "mix" old and new iterates by using the block-structure of the equivalent real problem in addition to the chosen splitting. It is possible that this leads to conflicting contributions in the iterations of the algorithms in [4] applied to Problem 4.1 when $\boldsymbol{x}^{(0)}$ is a random vector.
- The proposed convergence conditions for all the MAOR and AMAOR methods make it easy to evaluate the convergence when $\boldsymbol{\gamma}$ is changed. Indeed, Lemma 3.1 with Corollary 3.1 provides very simple conditions for any choice of $\boldsymbol{\gamma}$. For instance, with the above choice $\boldsymbol{\gamma} = \pi/4\boldsymbol{e}$, the requirement was $\Omega \geq 8I$ (and Ω smaller than approximately $31I$ for SOR with $\alpha = 1.1$). In Table 5, we report the results for different choices of $\boldsymbol{\gamma}$. This is a major difference with respect to [4], where only $\gamma = \pi/4\boldsymbol{e}$ was considered within a simplified setting. In particular, in Table 5 we consider $\boldsymbol{\gamma} = \pi/5\boldsymbol{e}$ and $\boldsymbol{\gamma} = \frac{4}{15}\pi\boldsymbol{e}$. The convergence requirements for the standard and accelerated modulus-based algorithms are roughly $\Omega \geq 11.58I$ for $\boldsymbol{\gamma} = \pi/5\boldsymbol{e}$ and $\Omega \geq 7.24I$ for $\boldsymbol{\gamma} = \frac{4}{15}\pi\boldsymbol{e}$. Hence, we choose $\Omega = 12I$ in the first case and $\Omega = 8I$, which also ensure the convergence of SOR methods for $\alpha = 1.05$ (evaluated considering, for instance, $\kappa = 8$). As regards the convergence of the methods in [4], in general their convergence cannot be easily evaluated: indeed [4, Theorem 4.1] should be used, which implies computing spectral radii and inverse matrices. Hence, we evaluate the convergence of these latter algorithms only numerically.
  shows that all the algorithms could solve all the problems. However, the block-diagonal methods appear to be generally less competitive. For instance, for $\boldsymbol{\gamma} = \pi/5\boldsymbol{e}$, $\Omega = 12I$, they were roughly on par with standard modulus-based methods and significantly slower than accelerated algorithms.

4.2. CHLCPs with commuting matrices

Consider Problem 4.2 with $\mu = 0$ (hence, A_R is not strictly diagonally dominant) and $\boldsymbol{x}^{(0)} = \boldsymbol{0}$. The choice $B_I = 0$ and symmetric B_R ensures that B_{RI} and T_2 commute when $\boldsymbol{\gamma}$ is constant. Considering, for instance, $\boldsymbol{\gamma} = \pi/4\boldsymbol{e}$, it is easy to apply Lemma 3.2 and verify that the assumptions of the lemma are satisfied for $\omega_i \in [1, 2]$. Furthermore, A is an H_+ matrix, as can be easily verified using a small dimension and exploiting the structure of the matrices to generalize to larger dimensions. Hence, the (standard and accelerated) modulus-based AOR methods for CHLCPs (with $\alpha \in (0, 1)$) converge by Theorem 3.2 and its corollaries. In this context, it is interesting to notice that it would be more complicated to evaluate the convergence of these algorithms by known convergence theorems for real-valued problems. Indeed, the matrix A of the equivalent real-valued HLCP has a non-banal structure and is not even diagonally dominant. Furthermore, the analysis of convergence would not be possible by standard results on the accelerated methods, as current convergence conditions [12] would require that the elements of A and B be linked by relations that are not satisfied here.

reports the results for solving Problem 4.2 with $m = 250$ for various choices of Ω. We notice that all the algorithms converge, including SOR which, for this problem, would be outside of the theoretical convergence domain for $\alpha = 1.1$. Once again, the accelerated algorithms are faster than the others, while block-diagonal splittings are on par with the Jacobi methods. In all cases,

Table 5
Solution to Problem 4.2 with $m = 500$, $\boldsymbol{x}^{(0)} = \mathbf{0}$, $r = 2$, $\mu = 3$, $\boldsymbol{\gamma} = \frac{\pi}{5}\boldsymbol{e}$, and $\Omega = 2.75\boldsymbol{I}$. In SOR, we set $\alpha = 1.1$.

Method	it	res	t
MJ	43	6.2E−7	2.9
AMJ	35	7.5E−7	2.3
BDMJ	42	7.5E−7	2.8
MGS	27	7.1E−7	1.7
AMGS	19	4.7E−7	1.2
BDMGS	27	6.4E−7	1.7
MSOR	22	5.0E−7	1.4
AMSOR	13	4.8E−7	0.86
BDMSOR	22	4.7E−7	1.4

Table 6
Solution to Problem 4.3 (left, $\boldsymbol{\gamma} = \frac{\pi}{4}\boldsymbol{e}$, and $\Omega = \boldsymbol{I}$) and of Problem 4.4 (right, $\boldsymbol{\gamma} = \frac{\pi}{3}\boldsymbol{e}$, and $\Omega = \tan(\frac{\pi}{3})^{-1}\boldsymbol{I}$) with $m = 500$, $\boldsymbol{x}^{(0)} = \mathbf{0}$, and $r = 2$. In SOR, we set $\alpha = 1.05$.

Method	it	res	t	Method	it	res	t
MJ	49	7.8E−7	3.2	MJ	39	7.8E−7	2.5
AMJ	19	9.2E−7	1.3	AMJ	25	9.1E−7	1.7
BDMJ	26	6.2E−7	1.7	BDMJ	22	7.4E−7	1.4
MGS	25	6.0E−7	1.6	MGS	26	9.2E−7	1.7
AMGS	16	8.8E−7	1.0	AMGS	20	9.3E−7	1.3
BDMGS	17	9.7E−7	1.1	BDMGS	22	7.4E−7	1.4
MSOR	23	9.3E−7	1.5	MSOR	29	8.8E−7	1.9
AMSOR	14	3.8E−7	0.92	AMSOR	28	7.6E−7	1.8
BDMSOR	22	5.6E−7	1.4	BDMSOR	28	8.6E−7	1.8

the choice of Ω affects the convergence speed of the algorithms. The convergence, however, is slower than in Problem 4.1. This can be explained by the fact that, although A_R, B_R are diagonally dominant, the same cannot be said for the matrix A of the equivalent real-valued HLCP. The algorithms remain convergent by Theorem 3.2 and its corollaries, but the lack of diagonal dominance can arguably affect the convergence, especially using AOR splittings (which use the diagonals in the splitting matrices M_A and M_B). The fact itself of having convergence without A being diagonally dominant is interesting (for instance, common convergence conditions of many Jacobi and Gauss–Seidel methods require either strictly or irreducibly diagonally dominant matrices) and can be explained by its being an H_+-matrix. Furthermore, although many iterations are required, the computational times remain small. This makes the algorithms effective even for large problems, at least when the matrices are sparse.

Let us now consider different choices of $\boldsymbol{\gamma}$. In this regard, Lemma 3.2 and Theorem 3.2 are general. In particular, it can be noticed that, setting, for instance, $\omega_i = 1.5$, $i = 1, \ldots, n$, the assumptions of Lemma 3.2 are satisfied when $\gamma_i \in [\arctan\sqrt{0.5}, \arctan\sqrt{2}] \approx [0.6155, 0.9553]$ radians. However, for the above matrices, the condition that A is an H_+ matrix is no longer satisfied and we would then be outside of the theoretical convergence domain described by Theorem 3.2 (although the conditions are just sufficient and, in practice, it can be verified that all the analyzed methods are still convergent). Thus, in order to remain within the theoretical convergence, we strengthen the diagonal dominance of A_R by setting $\mu = 3$.

For $\gamma = \pi/5\boldsymbol{e}$, Theorem 3.2 is satisfied for $\omega_i \in [2.5326, 2.8944]$ and A_R is still an H_+-matrix. Choosing, for instance, $\omega_i = 2.75$ for $i = 1, \ldots, 2n$ and $m = 500$, we obtain the results in . The theoretical convergence conditions of SOR with $\alpha = 1.1$ (evaluated with $\kappa = 14$) are satisfied as well.

demonstrates that the proposed algorithms can solve CHLCPs with a general choice of the angle $\boldsymbol{\gamma}$ and the effectiveness of the proposed convergence conditions. As regards the efficiency, considerations similar to those made on can be applied.

4.3. CHLCPs with general matrices

Finally, we consider Problems 4.3 and 4.4, for which we must resort to the non-commutative formulation. For Problem 4.3, we consider $\gamma = \pi/4\boldsymbol{e}$ and $\Omega = \boldsymbol{I}$. The convergence conditions are satisfied, even for the SOR method (where we have set $\alpha = 1.05$ and checked the condition setting, for instance, $\kappa = 5$). In Problem 4.4, we set $\gamma = \pi/3$. With this choice, based on Lemma 3.1 – Theorems 3.2 and 3.3, the convergence of (possibly accelerated) modulus-based AOR methods is ensured when $\Omega = \tan(\frac{\pi}{3})^{-1}\boldsymbol{I}$ for $\alpha \in (0, 1]$. The results of the solution of these problems with dimension $m = 500$ are reported in .

shows that the analyzed modulus-based method can efficiently solve CHLCPs even when no commutativity properties can be exploited, and with different choices of the angle $\boldsymbol{\gamma}$. All the algorithms can exploit the sparsity of the matrices and can reach a good approximation of the solution in just a few seconds. In this context, accelerated modulus-based SOR methods appear to be particularly effective to solve the considered problems.

5. Conclusions

We have introduced the complex-valued horizontal linear complementarity problem as a generalization of the complex linear complementarity problem. We have provided two equivalent real-valued reformulations of the CHLCP, depending on whether the problem satisfies some commutative properties. For both reformulations, we have then analyzed standard and accelerated modulus-based matrix splitting methods applied to the equivalent real-valued HLCP, and we have generalized to the CHLCP the modulus-based matrix splitting algorithm for CLCPs recently proposed in [4]. In our analysis, we have particularly analyzed (standard and accelerated) AOR methods for the equivalent real-valued HLCP, providing tailored convergence conditions that exploit the structure of the matrices of the equivalent problem. Some of these results also extend the convergence of accelerated modulus-based AOR methods for general HLCPs. Finally, we have solved several numerical experiments. Here, we have first revisited the CLCP based on our analysis on the CHLCP, adding new results on accelerated algorithms, SOR splittings, and general choices of the angle γ of the problem. Then, we have solved CHLCPs in the commutative and non-commutative settings. The results have shown that the analyzed algorithms can successfully solve all these kinds of problems, that the proposed convergence conditions are effective, and that accelerated AOR methods are often the most effective of the considered algorithms.

Acronyms

AMAOR	Accelerated MAOR
AOR	Accelerated over-relaxation
BDM	Block-diagonal modulus-based
BDMGS	BDM Gauss–Seidel
BDMJ	BDM Jacobi
BDMSOR	BDM SOR
CHLCP	Complex-valued horizontal linear complementarity problem
CLCP	Complex-valued linear complementarity problem
LCP	Linear complementarity problem
MAOR	Modulus-based AOR
MGS	Modulus-based Gauss–Seidel
MJ	Modulus-based Jacobi
MSOR	Modulus-based SOR
SOR	Successive over relaxation

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Data availability

Data will be made available on request.

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