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Potential Subgraphs of the Missing Moore Graph

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Abstract: The possible existence of a Moore graph of diameter 2 and degree 57 has been an open question for more than six decades. In this paper, certain subgraphs of this graph, referred to as t-subgraphs, are considered. Exploiting symmetry by assuming a cyclic group of permutations representing edges joining leaf nodes of branches of a tree, a tractable constraint model for t-subgraphs is created. This can be solved using the Google OR Tools CP-SAT solver. Larger potential t-subgraphs than those currently known are constructed. This further extends a construction of certain sets of mutually orthogonal Latin rectangles. The implications for non-existence proofs of the Moore graph are considered.

Keywords: graph theory; Moore graphs; diameter; girth; Google OR-Tools

MSC: 05C12; 05C38; 05B15

1. Introduction

In 1960, Hoffman and Singleton [1] published a study on regular graphs of degree k and diameter d meeting the Moore upper bound on the number of vertices. Here, degree k indicates that there are exactly k edges incident with every vertex, and diameter d indicates that the length of any shortest path between two distinct vertices is at most d. This application of linear algebra to graphs was pioneering work in what became known as algebraic graph theory. For a regular graph of degree k and diameter d the number of vertices $|V|$ satisfies the following:

$$|V| \leq 1 + k\sum_{i=1}^{d}(k-1)^{i-1}.$$

This inequality is obtained by counting the maximum possible number of vertices at distance i from a chosen vertex u [1].The right-hand side of the inequality is known as the Moore bound, and graphs meeting this bound with equality are known as Moore graphs. Extensive surveys on Moore graphs can be found in [2,3]. The case of interest in the current paper is diameter $d = 2$, for which the number of vertices of the Moore graph is $1 + k^2$. Specifically, in this case, Hoffman and Singleton [1] proved that such a Moore graph exists in at most four cases. These are $k = 2$ (the pentagon), $k = 3$ (for which the Petersen graph with 10 vertices is the unique Moore graph—see Figure 1), $k = 7$ (for which the Hoffman–Singleton graph with 50 vertices is the unique Moore graph—see Appendix C) or possibly $k = 57$ (the open case which would have 3250 vertices). This last case is often referred to as the "missing Moore graph". All attempts to construct the missing Moore graph have failed, and as early as 1974, it was noted that the many attempts to prove non-existence had not proved publishable [4]. This situation has continued to the present day. Specific information on the missing Moore graph can be found in [3,5,6]. Even if this

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graph exists, it cannot be highly symmetric. The terminology associated with symmetry in graphs can be found in Appendix A. It is known that the missing Moore graph cannot be distance-transitive [7] or even vertex-transitive [8] and the order of the automorphism group cannot exceed 375 (see [3,6,9] for further information). Here, use will be made of a cyclic group of permutions, with elements describing the edges connecting leaf vertices of pairs of branches of a tree. Further use will also be made of symmetry in this paper in application of Google OR-Tools CP-SAT constraint solver [10], which exploits symmetry breaking. Satisfiability problems often have symmetry properties that can be exploited so that solutions can be found more quickly. When there are many symmetric configurations of variables, a symmetry preprocessor adds constraints that force the solver to consider only a few of these configurations. This avoids unnecessary searching of isomorphic parts of the search space.

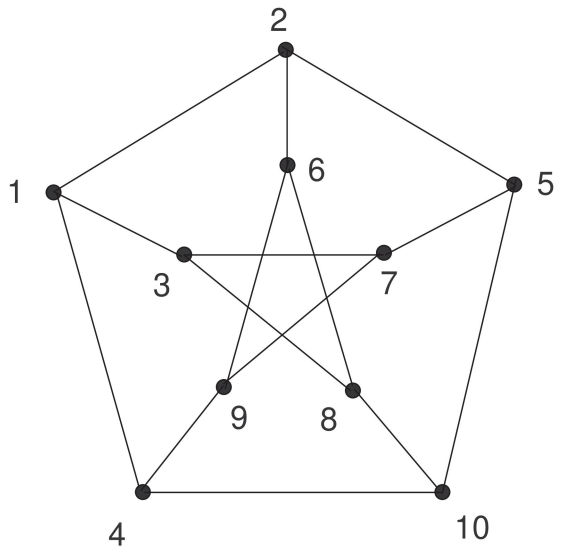

Figure 1. The Petersen graph. Details of the symmetry properties of the Petersen graph can be found in Appendix B.

In this paper, the existence of a sequence of potential subgraphs of the Moore graph of degree 57 will be considered. The existence of a full set of 57 such subgraphs would imply the existence of the Moore graph, but previously only a sequence of 15 such subgraphs was known. Use of the Google OR-Tools CP-SAT solver with the cyclic group mentioned above extends this to 20 subgraphs. These subgraphs can be used to construct certain sets of mutually orthogonal Latin rectangles, so the new results allow larger such sets of Latin rectangles to be constructed.

2. Structure of the Moore Graph

For any vertex u, Let $\Gamma(u)$ be the set of neighbours of u, i.e., the set of vertices adjacent to u. Also, let $\Gamma_2(u)$ be the set of vertices at distance two from u. Construction of a Moore graph will start from a tree with $|\Gamma(u)| = k$ and $|\Gamma_2(u)| = k(k-1)$ as shown in Figure 2. The edges to be added must all be incident with two vertices of $\Gamma_2(u)$.

The girth of a graph is the minimum number of edges in any cycle. Notice that as a Moore graph has degree k, diameter $d = 2$, and $1 + k^2$ vertices, the same tree will apply whichever vertex is chosen as u, and the graph must have girth 5 (i.e., with no cycles of length 3 or 4). Now, consider the remaining edges of the Moore graph, which must connect leaf vertices of pairs of distinct branches to avoid creating cycles of length 3. A leaf vertex of one branch cannot be adjacent to more than one leaf vertex of another branch or a cycle of length 4 would be created. As the degree of a vertex is k, the vertex must be adjacent to one vertex of every other branch. It follows that, for a pair of branches, the edges connecting the leaf vertices form a perfect matching, which will be represented in this paper by a permutation.

In [5], an optimization algorithm was used to create a graph with 3250 vertices, maximum degree 57, and girth 5 with the maximum number of edges that could be found. Define the *deficit* as the required number of edges minus the number of edges found. Even after a run of some 18 months, the deficit found was 41,482 (later improved slightly to 41,431). As evidence suggested the algorithm was effective, this strongly suggested non-existence of the Moore graph of degree 57, but did not, of course, constitute a proof. A further alternative approach will be used here considering certain subgraphs of this Moore graph.

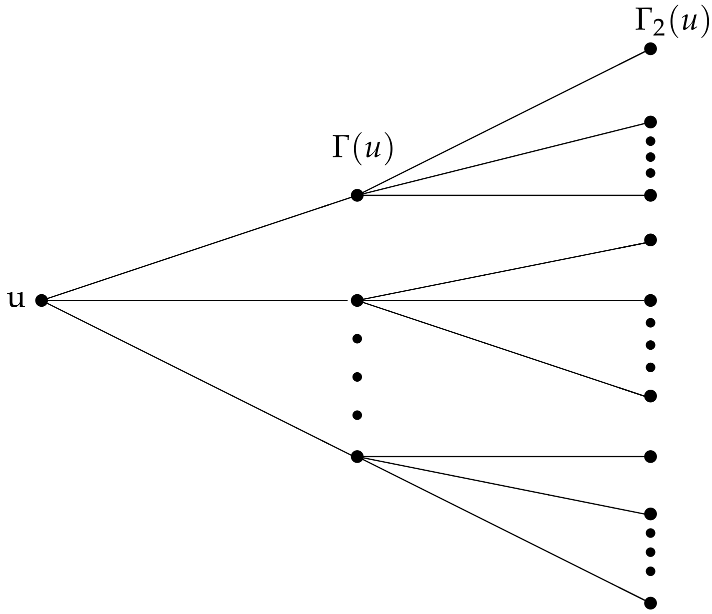

Figure 2. Starting tree with $1 + k^2$ vertices. It is possible to redraw the Petersen graph shown in Figure 1, for example, showing such a starting tree, independently of which vertex is chosen as root.

3. *t*-Subgraphs

Let G be a Moore graph of diameter 2 and degree k. Label the vertices of $\Gamma(u)$ as $w_1, w_2, \ldots, w_k$, and label the leaf vertices v_{ij} with $i \in \{1, 2, \ldots, k\}$ and $j \in \{0, 1, \ldots, k-2\}$. Vertex w_i is adjacent to all $k-1$ vertices $v_{ij'}$ with $j' \in \{0, 1, \ldots, k-2\}$. In some examples presented later, it will be more convenient to simply use $0, 1, \ldots, k-1$ for $w_0, w_1, \ldots, w_{k-1}$ and to simply denote leaf vertices with (i, j).

Definition 1. *A t-subgraph G_t of G $(1 \le t \le k)$ is the subgraph induced by the vertices $w_1, w_2, \ldots, w_t$ and by v_{ij} with $i \in \{1, 2, \ldots, t\}$ and $j \in \{0, 1, \ldots, k-2\}$. As G has girth 5, t-subgraphs cannot contain cycles of length 3 or 4.*

If a t-subgraph with $t = k$ exists, a root vertex u can be added adjacent to $w_1, w_2, \ldots, w_k$ without creating a cycle of length 3 or 4 to create the Moore graph.

Definition 2. *Any such graph with t vertices of degree $k-1$, each adjacent to a disjoint set of $k-1$ vertices of degree t and with girth 5 will be referred to as a potential t-subgraph.*

A leaf node v_{i_1j} is adjacent to precisely one leaf node of a branch i_2. If $i_1 < i_2$, let $q(i_2, v_{i_1j})$ be the vertex $v_{i_2j''}$ adjacent to v_{i_1j}. Then, these edges describe a permutation $\pi_{i_1i_2}$ of $(0, 1, \ldots, k-2)$.

$$\pi_{i_1i_2} = \begin{pmatrix} v_{i_10} & v_{i_11} & \ldots & v_{i_1k-2} \\ q(i_2, v_{i_10}) & q(i_2, v_{i_11}) & \ldots & q(i_2, v_{i_1k-2}) \end{pmatrix}$$

Clearly, $\pi_{i_2i_1} = \pi^{-1}_{i_1i_2}$.

As in [5], it can be assumed without loss of generality that π_{1i} is the identity permutation $1 < i \leq t$. Then, to avoid cycles of length 3 or 4, the following conditions hold, with $i_1 \leq t, i_2 \leq t, i_3 \leq t, i_4 \leq t$ in all cases:

$$\pi_{i_1 i_2}(2 \leq i_1 < i_2) \text{ has no fixed points} \tag{1}$$

(or if p' was a fixed point then vertices $(1, p'), (i_1, p'), (i_2, p')$ would be on a cycle of length 3).

$$\pi_{i_1 i_2}\pi_{i_2 i_3}(i_1 > 1, i_2 > 1, i_3 > 1, i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3) \text{ has no fixed points} \tag{2}$$

(or if p' was a fixed point then vertices $(1, p'), (i_1, p'), (i_2, p''), (i_3, p')$ would be on a cycle of length 4).

$$\pi_{i_1 i_2}\pi_{i_2 i_3}\pi_{i_3 i_1}(i_1 > 1, i_2 > 1, i_3 > 1, i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3) \text{ has no fixed points} \tag{3}$$

(or if p' was a fixed point then vertices $(i_1, p'), (i_2, p''), (i_3, p''')$ would be on a cycle of length 3).

$$\pi_{i_1 i_2}\pi_{i_2 i_3}\pi_{i_3 i_4}\pi_{i_4 i_1}(i_1 > 1, i_2 > 1, i_3 > 1, i_4 > 1, i_1 \neq i_2, i_1 \neq i_3, i_1 \neq i_4, i_2 \neq i_3, i_2 \neq i_3, i_3 \neq i_4) \tag{4}$$

has no fixed points (or if p' was a fixed point then vertices $(i_1, p'), (i_2, p''), (i_3, p'''), (i_4, p'''')$ would be on a cycle of length 4).

If $t = k$, and these conditions are all satisfied, a Moore graph is obtained.

Using a minor modification of the algorithm in [5] (to account for the different vertex degrees) it is possible to find potential t-subgraphs in the degree 57 case with $t \leq 15$, but, despite runs of several weeks, it has not proved possible to find t-subgraphs with $t \geq 16$ on any reasonable timescale. This again suggests the non-existence of the missing Moore graph. The uncertainty about whether $t = 15$ is the best possible motivates a new construction of t-subgraphs. The exact approach presented here will extend the existence of potential t-subgraphs further.

4. Cyclic Permutations and a Constraint Model

Permutations without a fixed point are referred to in the literature as *derangements*. The conditions on products of permutations presented in Section 3 suggest the use of a group of derangements (a permutation group in which all elements except the identity are derangements). The simplest such group of derangements to use is a cyclic group. It is only necessary to show that the relevant products do not equal the identity of the group.

The model will be presented for a t-subgraph, but applies to the full Moore graph if $t = k$. Introduce variables x_{ij} where $x_{ij} = a$ ($a \in \{0, 1, \ldots, (k-2)\}$) if π_{ij} ($i < j$) maps $\{0, 1, \ldots, k-2\}$ to $\{a, 1+a, \ldots, k-2+a\}$ mod $(k-1)$ (a cyclic shift by $a \leq k-2$ positions). As π_{1i} is the identity permutation, we have

$$x_{1i} = x_{i1} = 0 \quad \forall i \in \{2, 3, \ldots, t\} \tag{5}$$

Also, as $\pi_{ji} = \pi_{ij}^{-1}$ it follows that

$$x_{ji} = -x_{ij} \tag{6}$$

For $3 \leq p \leq t$ the condition that there are no cycles of length 3 implies the following:

$$x_{pi} + x_{ij} + x_{jp} \notin \{0, \pm(k-1)\} \tag{7}$$

where $p, i, j \in \{1, 2, \ldots, t\}$, $p > i, p > j, i < j$. The last inequality arises because it is only necessary to consider one orientation of each 3-cycle. As at least one term will have an opposite sign, it is not possible for the sum to be $\pm 2(k-1) \equiv 0$ mod $(k-1)$. There are $\sum_{p=3}^{t}(p-1)(p-2)/2$ such constraints.

Similarly, for $4 \leq p \leq t$, the condition that there are no cycles of length 4 implies that:

$$x_{pi} + x_{ij} + x_{jl} + x_{ml} \notin \{0, \pm(k-1), \pm 2(k-1)\} \quad (8)$$

where $p, i, j, l \in \{1, 2, \ldots, t\}$, $p > i, p > j, p > l, i \neq j, j \neq l, i < l$. The last inequality arises because it is only necessary to consider one orientation of each 4-cycle. As at least one term will have an opposite sign, it is not possible for the sum to be $\pm 3(k-1) \equiv 0 \bmod (k-1)$. There are $\sum_{p=4}^{t}(p-1)(p-2)(p-3)/2$ such constraints. Note that in both Equations (7) and (8) it is possible for the sums of variables to be negative.

5. Results

The powerful Google OR Tools CP-SAT solver has been used to solve these constraints for several values of t. This is a constraint programming solver that uses SAT (satisfiability) methods. Full details can be found at [10]. The computer used was an Intel(R) Core(TM) i3-9100 CPU @ 3.60GHz, 3600 MHz, 4 Cores, and 4Gb RAM running Windows 10. It is only necessary to present the constraints given in Section 4 to the solver. Python code for achieving this can be found in the Supplementary Material. Runs lasted up to about 4 days for the hardest cases. In all cases, the t-subgraph found was checked for validity (correct vertex degrees and girth 5) using independent checking software.

The constraint solver was able to find potential t-subgraphs for $t \leq 20$. In the case $t = 20$, the solver was unable to complete, so the values of the variables relevant to $t = 19$ were fixed in the solver, and the solution was completed. Clearly, the solver was unable to complete $t = 21$ without a similar fixing of variables. However, when fixing the variables from the solution for $t = 19$, the solver reported infeasibility. After attempting to remove the infeasibility by just fixing the variables from the solution for $t = 18$, the solver failed to complete in 5 days. It will be clear in the discussion that follows that after fixing the variables from the solution for $t \leq 18$, the solver will not be able to complete. It appears that $t = 21$ cannot be resolved using the CP-SAT solver, at least using the computers available to the authors. Details of the results with timings can be found in Table 1.

As the number of variables is $\sum_{p=3}^{t}(p-2)$, and it has already been noted in Section 4 that the number of constraints is $\sum_{p=3}^{t}(p-1)(p-2)/2 + \sum_{p=4}^{t}(p-1)(p-2)(p-3)/2$, the size of the model grows rapidly. This is illustrated in Table 2. It is clear why the solver applied to the basic model fails to complete the case where $t \geq 20$. It is also clear that the existence of a solution for a value t does not necessarily imply that the solution can be extended to $t + 1$.

Table 1. CP-SAT results for the model represented by Equations (5) to (8).

Case	Outcome	Time
$t = 16$ basic model	valid solution found	2 min
$t = 17$ basic model	valid solution found	1 min
$t = 18$ basic model	valid solution found	10 min
$t = 19$ basic model	valid solution found	5 h 34 min
$t = 20$ basic model	solver failed to complete	4 days
$t = 20$ variables from $t = 19$ solution fixed	valid solution found	37 min
$t = 21$ basic model	solver failed to complete	4 days
$t = 21$ variables from $t = 19$ solution fixed	infeasibility reported	9 h
$t = 21$ variables from $t = 18$ solution fixed	solver failed to complete	5 days

Table 2. Increase in the numbers of constraints and variables as t increases.

t	Constraints	Variables
16	6020	105
17	7820	120
18	9996	136
19	12,597	153
20	15,675	171
21	19,285	190
22	23,485	210

6. t-Subgraphs, Cliques and Latin Rectangles

In this section, an application of the results on t-subgraphs to the construction of certain sets of mutually orthogonal Latin rectangles will be described in detail. A study of these rectangles may provide a new route to prove the existence or non-existence of the Moore graph of degree 57 and diameter 2.

Definition 3. *A* Latin rectangle *is an $r \times n$ $(r \leq n)$ matrix with elements chosen from $\{1, 2, \dots n\}$ such that each row contains each symbol exactly once and each column contains each symbol at most once. Two $r \times n$ Latin rectangles $A = [a_{ij}]$ and $B = [b_{ij}]$ are* orthogonal *if each ordered pair (a_{ij}, b_{ij}) appears at most once. A set of pairwise orthogonal Latin rectangles is called a set of* mutually orthogonal Latin rectangles *(denoted as MOLRs).*

In this work, we will also call $r_1 \times n$ and $r_2 \times n$ $(r_1 \neq r_2)$ Latin rectangles orthogonal if each ordered pair of symbols (a_{ij}, b_{ij}) has an entry (i, j) that exists in both rectangles and appears at most once.

Definition 4. *A set of $n_1 \times n, (n_1 - 1) \times n, \dots, n_2 \times n$ (with $n_2 < n_1$) mutually orthogonal Latin rectangles is called a decreasing set of mutually orthogonal Latin rectangles. If $n_1 = n$ and $n_2 = 1$, it is called a complete decreasing set of mutually orthogonal Latin rectangles.*

If $r = n$, a Latin rectangle is a Latin square of order n, and in the definition of orthogonality, each ordered pair must appear exactly once.

Here, the extension of a t-subgraph to a $(t+1)$-subgraph will be considered. Note that in the following, the leaf vertices v_{ij} of the starting tree will still be referred to as leaf vertices, even when they cease to be leaf vertices of a tree by the addition of edges. Observe that a t-subgraph has a diameter 3, with paths of the form $v_{ab}, v_{cd}, w_c, v_{ce}$ and w_p, v_{pq}, v_{rs}, w_r always existing. Now, define $G_t^{(3)}$ as the graph with the same vertices as a t-subgraph G_t, but with two vertices adjacent if they are at distance 3 in G_t. When extending from a t-subgraph to a $(t+1)$-subgraph, each leaf vertex of branch $t+1$ must be adjacent to vertices of a clique (set of vertices inducing a complete subgraph) in the leaf vertices of $G_t^{(3)}$ or diameter 3 implies that a cycle of length 3 or 4 will be created. Further, these cliques must be disjoint, or again a cycle of length 4 will be created. Thus, a partition of the leaf vertices of $G_t^{(3)}$ into $k-1$ cliques with t vertices is obtained. Denote these cliques by $C_{t\ell}$ $(\ell \in \{0, 1, \dots, k-2\})$.

It can be observed that cliques $C_{t'\ell_1}, C_{t''\ell_2}$ $(t' \neq t'')$ arising from different t-subgraphs can have at most one vertex in common. Assume that these cliques have vertices $v_{t_1 a}$ and $v_{t_2 b}$ in common. Let $v_{(t'+1)\ell_1}$ be the vertex of branch $t'+1$ adjacent to the vertices of $C_{t'\ell_1}$ and $v_{(t''+1)\ell_2}$ be the vertex of branch $t''+1$ adjacent to the vertices of $C_{t''\ell_2}$. Then, $v_{(t'+1)\ell_1}$, $v_{t_1 a}, v_{(t''+1)\ell_2}, v_{t_2 b}, v_{(t'+1)\ell_1}$ would be a cycle of length 4, contradicting girth 5.

The above set of partitions into cliques of the various t-subgraphs will now be illustrated for the Hoffman–Singleton graph with degree 7 and 50 vertices. The abbreviated notation mentioned in Section 3 will be used. The Hoffman–Singleton graph is unique [1], but many vertex labellings are possible. The particular vertex labelling used here arises from a construction of the Hoffman–Singleton graph using the algorithm presented in [5]. The partitions into cliques are shown in Figure 3. Note that in this figure the labels for branches in the branch/leaf notation are taken from the set $\{0, 1, \ldots, t-1\}$. The cliques for each value of t can be represented in $k-1$ matrices with columns indexed by leaf from 0 to $k-1$ and rows indexed by branch from 0 to $t-1$. The matrices will be taken in reverse order, so the matrix with the largest number of rows comes first. Then, if clique $C_{t\ell}$ contains vertex v_{ij}, an entry $\ell+1$ is placed in position (i,j) of the matrix with t rows. The matrices are presented in Figure 4 and will now be referred to as rectangles.

$$
\begin{array}{lcl}
C_{10}, C_{11}, \ldots, C_{15} & : & \{(0,0)\}, \{(0,1)\}, \{(0,2)\}, \{(0,3)\}, \{(0,4)\}, \{(0,5)\} \\
C_{20}, C_{21}, \ldots, C_{25} & : & \{(0,0),(1,1)\}, \{(0,1),(1,0)\}, \{(0,2),(1,3)\}, \{(0,3),(1,2)\}, \\
 & & \{(0,4),(1,5)\}, \{(0,5),(1,4)\} \\
C_{30}, C_{31}, \ldots, C_{35} & : & \{(0,0),(1,4),(2,2)\}, \{(0,1),(1,3),(2,5)\}, \{(0,2),(1,5),(2,0)\}, \\
 & & \{(0,3),(1,1),(2,4)\}, \{(0,4),(1,0),(2,3)\}, \{(0,5),(1,2),(2,1)\} \\
C_{40}, C_{41}, \ldots, C_{45} & : & \{(0,0),(1,2),(2,5),(3,3)\}, \{(0,1),(1,4),(2,3),(3,2)\}, \\
 & & \{(0,2),(1,0),(2,4),(3,1)\}, \{(0,3),(1,5),(2,1),(3,0)\}, \\
 & & \{(0,4),(1,1),(2,2),(3,5)\}, \{(0,5),(1,3),(2,0),(3,4)\} \\
C_{50}, C_{51}, \ldots, C_{55} & : & \{(0,0),(1,3),(2,4),(3,5),(4,1)\}, \{(0,1),(1,5),(2,2),(3,4),(4,0)\}, \\
 & & \{(0,2),(1,4),(2,1),(3,3),(4,5)\}, \{(0,3),(1,0),(2,5),(3,2),(4,4)\}, \\
 & & \{(0,4),(1,2),(2,0),(3,1),(4,3)\}, \{(0,5),(1,1),(2,3),(3,0),(4,2)\} \\
C_{60}, C_{61}, \ldots, C_{65} & : & \{(0,0),(1,5),(2,3),(3,1),(4,4),(5,2)\}, \{(0,1),(1,2),(2,4),(3,0),(4,5),(5,3)\}, \\
 & & \{(0,2),(1,1),(2,5),(3,4),(4,3),(5,0)\}, \{(0,3),(1,4),(2,0),(3,5),(4,2),(5,1)\}, \\
 & & \{(0,4),(1,3),(2,1),(3,2),(4,0),(5,5)\}, \{(0,5),(1,0),(2,2),(3,3),(4,1),(5,4)\}
\end{array}
$$

Figure 3. Cliques from the Hoffman–Singleton graph.

$$
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 2 & 1 & 5 & 6 & 3 & 4 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 6 & 4 & 5 & 2 & 3 & 1 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 3 & 6 & 1 & 5 & 4 & 2 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{bmatrix}
\quad
\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}
$$

Figure 4. A complete decreasing set of mutually orthogonal latin rectangles from the Hoffman–Singleton graph.

Three observations are necessary here. Firstly, each row of one of the rectangles cannot contain repeated cliques. This is because each clique can only contain one leaf vertex of a single branch or else a circuit of length 4 would be created. Secondly, each column of one of the rectangles cannot contain repeated cliques. This arises because, as mentioned above, the particular labelling of the vertices of the Moore graph is taken from construction by the algorithm presented in [5]. In the algorithm, it was assumed without loss of generality that the leaf vertices were labelled in such a way that v_{0j} was adjacent to leaf vertices $v_{ij}, (i \in \{1, \ldots, k-1\})$. If the same clique appeared twice in the same column, a vertex $v_{(t+1)j'''}$ in branch $t+1$ would be adjacent to vertices $v_{i'j}, v_{i''j}$ with the same leaf label. Then, $v_{(t+1)j'''}, v_{i'j}, v_{0j}, v_{i''j}, v_{(t+1)j'''}$ would be a cycle of length 4. Thus, the rectangles are Latin rectangles. Thirdly, it was noted previously that cliques $C_{t'\ell_1}, C_{t''\ell_2}$ $(t' \neq t'')$ arising from different t-subgraphs can have at most one vertex in common. Thus, any pair of rectangles are orthogonal. It follows that the rectangles presented for $n=6$ are a complete decreasing set of Latin rectangles.

The result of the above construction can be summarized as follows:

Theorem 1. *There exists a complete decreasing set of mutually orthogonal Latin rectangles for* $n = 6$*.*

It can be noted that such a set exists for $n = 6$, although there does not even exist two orthogonal Latin squares for $n = 6$ [11].

The same construction will work for the Moore graph with $k = 57$ if it exists. Even if the Moore graph does not exist, an incomplete decreasing set of mutually orthogonal Latin rectangles can be obtained from any potential t-subgraph that does exist. Thus, the construction in this section gives us:

Theorem 2. *The clique partitions of a potential* t*-subgraph give rise to a decreasing set of mutually orthogonal Latin rectangles, with the number of rows decreasing from* $t - 1$ *to* 1*. If* $t = k$ *(so the Moore graph exists) the clique partition gives rise to a complete decreasing set of mutually orthogonal Latin rectangles, with the number of rows decreasing from* $k - 1$ *to* 1*.*

7. Existence of Decreasing Sets of of Mutually Orthogonal Latin Rectangles

It is known that a set of $n - 1$ mutually orthogonal Latin squares of order n (known as a complete set) exists when n is a prime power [12]. Taking the first $n, n - 1, \ldots, 2$ rows of these squares respectively and adding a single row of distinct elements (which cannot affect orthogonality) gives a complete decreasing set of mutually orthogonal Latin rectangles in the prime power case. This result, together with the result of Hoffman and Singleton [1], shows that although the existence of a Moore graph of diameter 2 implies the existence of a complete decreasing set of mutually orthogonal Latin rectangles, the existence of such a set of rectangles does not imply the existence of the corresponding Moore graph. For non-prime powers, only much smaller sets of mutually orthogonal Latin squares are known. For example, if $n = 56$ the number $N(56)$ of mutually orthogonal Latin squares is only known to satisfy $N(56) \geq 7$ [12]. Thus, only a decreasing set of eight mutually orthogonal Latin rectangles can currently be obtained from Latin squares.

Using the t-subgraph with $t = 15$ mentioned at the end of Section 3 gives a decreasing set of 14 mutually orthogonal Latin rectangles with $n = 56$. The clique partitions and rectangles can be found at [13]. The extension to a t-subgraph with $t = 20$ in this paper gives a decreasing set of 19 mutually orthogonal Latin rectangles with $n = 56$. This set of rectangles can be found in the Supplementary Material, or at [14].

As well as for $n = 56$, the method presented in Section 4 could also be extended to other non-prime power values of n. The method could be used to construct potential t-subgraphs (with $t < n$) which may exist, even where a Moore graph cannot exist. In view of the results of [1], a complete decreasing set of mutually orthogonal rectangles could not be found using this method.

Very recently, Wanless [15] has constructed a decreasing set of 32 mutually orthogonal Latin rectangles with $n = 56$, $n_1 = 32$, and $n_2 = 1$. However, the rectangles do not correspond to a valid potential t-subgraph. Reversing the construction presented in Section 6, the corresponding graph has many cycles of length 3 and 4. It follows that the construction cannot be used as an approach to resolve the question of the existence of the missing Moore graph. The question of the possible existence of a complete decreasing set of mutually orthogonal Latin rectangles with $n = 56$ remains open.

8. Discussion

The assumption of a cyclic permutation means that it would never be possible to prove the non-existence of the missing Moore graph using the approach in Section 4. However, the increasing struggles of CP-SAT to find a solution does suggest that existence of potential t-subgraphs becomes much less likely as t increases. It can be conjectured that no potential t-subgraph exists for $t = 21$. This relatively small subgraph might provide a route to a non-existence proof.

9. Conclusions

The assumption of a cyclic permutation has simplifed the constraint model sufficiently to construct larger potential t-subgraphs. This extends a known construction of mutually orthogonal Latin rectangles and provides some new insight into possible non-existence proofs of the missing Moore graph. At the same time, the results here for $t \leq 20$ illustrate the power and utility of CP-SAT when applied to problems in graph theory.

Future work will continue the 64-year quest to resolve the question of the existence of the missing Moore graph. This may build on the work presented in this paper.

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Appendix A

An *automorphism* of a simple graph G is a permutation π of the vertex set of G such that $\{u, v\}$ is an edge of G if and only if $\{\pi(u), \pi(v)\}$ is an edge of G. The automorphisms of G under composition form a group known as the *automorphism group* of G. A graph G is a *vertex-transitive graph* if, given any two vertices u and v of G, there is an automorphism π such that $\pi(u) = v$. A graph G is a *distance-transitive graph* if, given any four vertices u, v, x, y of G such that $d(u, v) = d(x, y)$, there is an automorphism π such that $\pi(u) = x$ and $\pi(v) = y$.

Appendix B

The automorphism group of the Petersen graph is the symmetric group S_5. In fact the Petersen graph is one of only 12 distance-transitive graphs of degree 3 [16].

Appendix C

The original construction of the Hoffman–Singleton graph can be found in [1]. Several other constructions are known. The construction here is due to Biggs [4]. Biggs first defined a graph G_1 of degree 5 with 36 vertices as follows. Let $L = \{a, b, c, d, e, f\}$, and $N = \{1, 2, 3, 4, 5, 6\}$. The vertex set of G_1 is $L \times N$ and the vertex (l_1, n_1) is adjacent to (l_2, n_2) if an only if the transposition (n_1, n_2) occurs in the position (l_1, l_2) of Table A1. This table is originally due to Coxeter [17].

Table A1. The 'syntheme-duad' table due to Coxeter.

	b	*c*	*d*	*e*	*f*
a	(15)(23)(46)	(14)(26)(35)	(13)(24)(56)	(12)(36)(45)	(16)(25)(34)
b		(12)(34)(56)	(14)(25)(36)	(16)(24)(35)	(13)(26)(45)
c			(16)(23)(45)	(13)(25)(46)	(15)(24)(36)
d				(15)(26)(34)	(12)(35)(46)
e					(14)(23)(56)

This graph G_1 has degree 5 and diameter 3. Now add 14 new vertices denoted $L, N, a, b, c, d, e, f, 1, 2, 3, 4, 5, 6$. Vertex L is adjacent to all vertices in the set $\{a, b, c, d, e, f, N\}$ and vertex N is adjacent to all vertices in the set $\{1, 2, 3, 4, 5, 6, L\}$. In addition, each vertex (l, n) of G_1 is adjacent to l and n. The graph G_2 obtained is the Hoffman–Singleton graph. The automorphism group of this graph is of order 252000 and is obtained from $PSU(3, 5^2)$ by adjoining the field automorphism of $GF(5^2)$. The Hoffman–Singleton graph is distance-transitive.

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