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Theory of elastic wave propagation in a fluid-saturated multi-porous medium with multi-permeability

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Abstract: This paper establishes the concept of elastic wave propagation in a multi-porous medium with different permeabilities by assuming there are n distinct pore fluid phases. The dynamic equation of motion of elastic wave propagation through this multi-porous medium is derived based on Lagrangian mechanics. In this regard, the generalized form of mass coefficients and then the energy loss due to the fluid phases in terms of dissipation coefficients are presented for low-frequency limits with the help of Darcy's law of multi-phases system. The elastic coefficients of the constitutive equation in terms of compliance matrix are identified using a series of Gedanken experiments. Some significant results regarding the compressional and rotational waves in a multi-porosity medium are derived. The validation of the theory has been shown by comparing it with the existing theory of single and double porosity. It is observed that there are $(n + 1)$ compressional waves corresponding to solid and fluid phases, whereas only one rotational wave is associated with the solid phase. The concept of multi-porosity theory can contribute to a deeper understanding of wave behaviour in a porous medium.

Key words: Lagrangian mechanics, Elastic wave, Multi-porosity, Multi-permeability, Darcy's law, Gedanken experiment
MSC codes: 70S05, 74F10, 74A20, 35L05, 74J10

1 Introduction

Porous media have emerged as fascinating materials due to their intricate structure and diverse applications in numerous scientific and engineering fields. Their ability to efficiently transport and store fluids and their unique mechanical properties have paved the way for significant advancements in various industries. One notable area of study involving porous media is the propagation of elastic waves. Elastic wave propagation in a fluid-saturated and unsaturated porous medium has gained considerable interest due to its relevance in various fields, including geophysics, acoustics, and materials science. Understanding how elastic waves propagate through porous media can provide valuable insights into seismic activities, sound transmission, and the behaviour of composite materials.

The theory of elastic wave propagation in a fluid-saturated porous medium was initially introduced and developed by Biot [1, 2]. In this work, only one type of pore structure (single porosity) saturated by compressible viscous fluid is considered in the low and high-frequency range. Subsequently, numerous researchers, including Power [3], Sorek et al. [4], Edelman [5], Albers [6], Silva et al. [7], Berjamin [8], and others, have contributed to the understanding and advancement of wave propagation in porous media. Additionally, Arora et al. [9] explored body wave propagation in composite solids saturated by two immiscible fluids, while Gubaidullin [10] investigated wave propagation saturated with bubbly liquids. The usefulness of the theory of single porosity has been discussed in many books and research papers, a few of them being Zhou et al. [11], Battiato et al. [12] and Jeng and Cui [13]. Although the theory of single porosity has found successful applications across various research fields, it has

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limitations due to the assumption of homogeneous porosity. When fractures or cracks occur in the medium, this assumption becomes invalid. One approach to deal with this type of non-homogeneity is to construct a locally homogeneous model by considering two types of pore structure: matrix pore (primary pore) and fractured pore (secondary pore). This topic has been explored by Connel and Budiansky [14], Budiansky and Connel [15], Elsworth and Bai [16], and Berryman and Wang [17]. Then, different authors discussed the theory and derivation of elastic wave propagation using various approaches. Arbogast [18], Daly and Roose [19] derived the model using homogenization theory, Tuncay and Corapcioglu [20] employed the volume averaging approach, and Berryman and Wang [21] utilized Lagrangian mechanics. Furthermore, Pride and Berryman [22, 23] investigated the equation of motion, attenuation, and fluid flow in a double porosity and permeability medium. Also, the discussion of fluid flow in a porous medium can be facilitated through Darcy's law, and the stability of Darcy's equations of flow in porous media has been explored by Payne and Straughan [24]. Khalili [25], Zhao and Chen [26] examined the coupling effects between solid-fluid and fluid-fluid interactions in fractured porous media. After that, Ba et al. [27] extended the Biot-Rayleigh theory for a single porosity medium to a double porosity medium. Gentile and Straughan [28], and Rohan et al. [29] extended the linear theory to non-linear elasticity and heterogeneous double poroelastic media, respectively. Zhang et al. [30] employed the Hamiltonian principle to model double porosity with penny-shaped cracks as secondary pore structures. Whereas, Novikov et al. [31] discussed double porosity theory considering different fracture concentrations or percolation lengths within the system. Guo et al. [32] and Guo and Gurevich [33] delved into the frequency-dependent seismic wave anisotropy in porous media, specifically with intersecting fractures at different angles. Recently, Corapcioglu and Tuncay [34] summarized Biot's single porosity theory and discussed double porosity systems saturated by two immiscible fluids in a book chapter. Since the equations of motion governing elastic wave propagation in single and double-porous media are represented by a system of partial differential equations, questions regarding stability, existence, and uniqueness naturally arise. Researchers such as Svanadze [35], Straughan [36], Ciarletta et al. [37], and Xiong et al. [38] have explored and provided insights into these questions.

Mathematical theories of porous media are widely employed to analyze various natural phenomena, including seismic activities, heat transfer, fluid flow, and mechanical behaviour. Seismic waves play a pivotal role in impacting and causing destruction to man-made structures during earthquakes. Considering the Earth's layers as porous media, the study of seismic wave propagation within such media holds immense significance. Two primary categories of seismic waves exist: body waves and surface waves. Tuncay and Corapcioglu [39, 40] derived and discussed body waves in single and double porous media saturated with two immiscible fluids. In the literature, particular attention has been given to two prominent types of surface waves: Rayleigh and Love waves. Numerous research studies, such as those conducted by Dai et al. [41, 42], Sharma [43, 44], Pal and Ghorai [45], Manna and Anjali [46], Gupta et al. [47, 48] and Rajak et al. [49], have investigated Rayleigh and Love waves in single and double porous media. Manna et al. [50] and Pramanik et al. [51] explored Love wave propagation in a coated anisotropic porous layer with the influence of a point source as an earthquake epicenter. Vashishth and Bareja [52], as well as Bhat and Manna [53], analyzed Love wave propagation in composite porous media with piezoelectric and fiber-reinforced material, respectively. Kumar et al. [54] also discussed the propagation of Love-type waves in thermoelastic solids, considering porous rock as a double porous medium. Additionally, two other types of surface waves, namely Torsional waves and SH waves, have been extensively explored within porous media. These waves have been investigated by Sing et al. [55], Pramanik and Manna [56], Gupta et al. [57], and Kumari et al. [58], among others. Notably, Vashishth and Gupta [59] delved into the decoupling of plane waves in a piezo-poroelastic medium, while Sharma [60] examined the impact of the piezoelectric effect on the harmonic plane waves within the same medium.

The theory of single and double porosity was initially developed by considering one and two types of pore structures, respectively. However, it is important to note that the micro-structure of the solid skeleton may give rise to more than two types of pore structures. The concept of porosity can be further extended to triple and quadruple porosity in porous rock, where three and four types of pore structures are considered, respectively. In the case of triple porosity, these pore structures consist of macro porosity, which is the largest visible pore structure, meso porosity as the intermediate structure, and micro porosity as the smallest. Similarly, for quadruple porosity, the four types of pore structures are macro, meso, micro, and sub-micro, each progressively smaller in scale. Experimental work on triple and quadruple porous media, focusing on gas and petrophysical analysis, has been conducted by researchers such as Bai and Roegiers [61], Aguilera [62], Olusola et al. [63], and He et al. [64]. Also, the modelling of methane recovery in the triple porosity medium was discussed by Zou et al. [65]. The theories of double and triple porosity, as well as the concepts of uniqueness and stability, have been extensively explored and discussed by Straughan [66, 67]. Arusoaie and Chirita [68] have investigated compressional wave behaviour in triple porosity media, while Zampoli and Chirita [69] have examined Rayleigh waves in a triple thermo-poroelastic medium. In continuation to the previous development in single, double, and triple porosity, one can extend the concept of multi-porosity for a better understanding of porous media by considering a greater number of distinct pore structures. In this context, the pore structures can be categorized from largest to smallest as macro, meso, sub-meso, micro, sub-micro, nano, and so on. This comprehensive approach will contribute to a deeper understanding of wave behaviour in porous rock. Consequently, there is a strong need for the mathematical, physical, and numerical analysis of multi-porosity theory. The present work focuses on the complete generalization of the mathematical theory of multi-porous media.

In this paper, the theory of generalized multi-porosity medium with multi-permeability has been discussed and analyzed. The equation of motion for elastic wave propagation is presented using Lagrangian mechanics, taking into account the coupling effect between solid-solid, solid-fluid, and fluid-fluid. The mass and dissipation coefficients that appear in the equation of motion are evaluated and shown how to calculate them in general in terms of measurable parameters. For this, the general multi-porosity system reduces to a single porosity system in various limiting cases, and Darcy's law of a multi-phase system is employed. A generalized method for determining the elastic parameters in the constitutive equation is also demonstrated using a series of Gedanken tests. The Biot-Willis parameter (Biot and Willis [70]) for single porosity medium is also extended for multi-porosity medium. The equations of motion for compressional and rotational waves are developed, and some significant results in the multi-porosity medium are reported. The validity of the model is shown in terms of the particular cases by comparing it with single and double porosity theories. Finally, the dispersion equations for compressional and rotational waves, taking a triple porosity medium as an example, have been comprehensively examined numerically and graphically using MATLAB software.

2 Problem formulation

There are many types of models that exist to represent porous medium. Namely, single porosity, double porosity or fractured porosity models with fluid-saturated and unsaturated porous medium. In these models, at most, two types of pore structures are considered: only one pore structure for single porosity and two pore structures (primary and secondary) for double porosity medium. However, in nature, not all pore structures are of the same type, and these pores may not be homogeneously distributed throughout the medium. Single porosity models are successful in representing porous medium, but they fail to adequately describe the fractured porous medium. In fractured porous medium, two types of pore structures are considered: simple pores and fractured (or fissured) pores. Extending the idea of single and double (fractured) porous medium into multi-porous medium by considering n-different types of pore structure with different permeability. The permeability of different pore structures is different, leading to n distinct pressure fields for n different types of pore structures. This multi-porous medium has $n+1$ phases, such as a solid phase and n fluid phases in the n different types of pore structure, with the possibility of fluid mass exchange between different pore structures. Figure 1 illustrates the concept of multi-porosity in two ways: (a) zooming in from larger to smaller porosity structures into the micro-structure level and (b) showcasing diverse porosity types based on size and shape variations.

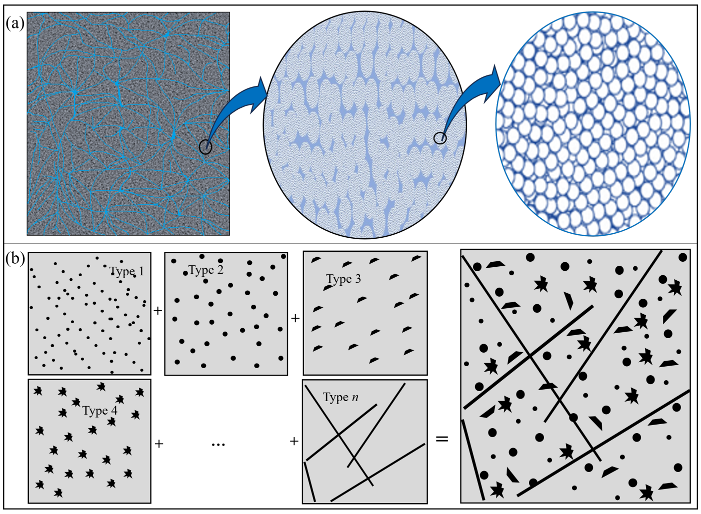

Figure 1: The schematic diagram of a multi-porous medium

3 Equations of motion

To derive the equation of motion in a multi-porous medium, we will use the Lagrangian formulation, which allows us to generate the equation of motion in any set of coordinates. Let $\mathbf{U}^{(0)}$ be the particle displacement in the solid phase, and $\mathbf{U}^{(k)}$ be the particle displacement in the type-k pore fluid phase. Also, let σ_{ij} be the components of stress in the solid phase and $\bar{p}^{(k)}$ are the macroscopic fluid pressure in the type-k pore fluid phase. If the kinetic energy is defined by E and potential energy by D, then Lagrange's equation of motion can be written in the Cartesian coordinate system (x_1, x_2, x_3) as

$$\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial \dot{U}_i^{(0)}}\right) + \frac{\partial D}{\partial \dot{U}_i^{(0)}} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3}, \quad i = 1, 2, 3, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial \dot{U}_i^{(k)}}\right) + \frac{\partial D}{\partial \dot{U}_i^{(k)}} = -\frac{\partial \bar{p}^{(k)}}{\partial x_i}, \quad k = 1, 2, 3, ..., n; \; i = 1, 2, 3, \tag{2}$$

where, the macroscopic fluid pressures $\bar{p}^{(k)}$ are related to the microscopic or internal pore fluid pressure $p^{(k)}$ by the relation $\bar{p}^{(k)} = \nu^{(k)}\phi^{(k)}p^{(k)}$, with $\nu^{(k)}$ are the volume fraction occupied by type-k pore fluid among all pores fluid and $\phi^{(k)}$ are the porosities.

For a system with n fluid phases, the kinetic energy E is given by the generalization of Biot's approach (Biot [1]) as

$$2E = \sum_{i=0}^{n} \rho_{ii} \dot{\mathbf{U}}^{(i)} \cdot \dot{\mathbf{U}}^{(i)} + 2 \sum_{\substack{i,j=0 \\ i<j}}^{n} \rho_{ij} \dot{\mathbf{U}}^{(i)} \cdot \dot{\mathbf{U}}^{(j)}, \tag{3}$$

where, the mass coefficients ρ_{ij} account for the non-uniformity of fluid flow through the pores and are related to the solid density ρ_s and fluid density ρ_f. The expression for ρ_{ij} in relation to other measurable quantities of the multi-porous medium is defined in Section 4.

Energy loss is commonly described using the term dissipation and is essential for fluid motion, hence cannot be avoided in this circumstance. For a system with n fluid phases, the appropriate dissipation function can be written as

$$2D = \sum_{\substack{i,j=0 \\ i<j}}^{n} b_{ij} \left(\dot{\mathbf{U}}^{(i)} - \dot{\mathbf{U}}^{(j)}\right) \cdot \left(\dot{\mathbf{U}}^{(i)} - \dot{\mathbf{U}}^{(j)}\right), \tag{4}$$

where, b_{ij} are the dissipation coefficients, represent the coupling between solid-fluid and fluid-fluid phases. The other causes of attenuation, especially for partially saturated rock (Miksis [72]) are neglected. The fluid-fluid coupling coefficients $b_{ij}(i < j; i, j = 1, 2, ..., n)$ are generally expected as very small and sometimes these are neglected. The expression for b_{ij} in terms of permeabilities, porosities and viscosity of the fluid are defined in section 5.

From the Lagrangian's equations (1) and (2), using (3) and (4), the equations of motion can be formulated as

$$\sigma_{ij,j} = \sum_{r=0}^{n} \rho_{0r} \ddot{U}_i^{(r)} + \sum_{r=1}^{n} b_{0r} \left(\dot{U}_i^{(0)} - \dot{U}_i^{(r)} \right), \quad i = 1, 2, 3, \tag{5}$$

$$-\bar{p}_{,i}^{(k)} = \sum_{r=0}^{n} \rho_{kr} \ddot{U}_i^{(r)} - \sum_{r=0}^{k-1} b_{rk} \left(\dot{U}_i^{(r)} - \dot{U}_i^{(k)} \right) + \sum_{m=k+1}^{n} b_{km} \left(\dot{U}_i^{(k)} - \dot{U}_i^{(m)} \right), \quad i = 1, 2, 3, \tag{6}$$

which represent coupling between the solid and n types $(k = 1, 2, 3, ..., n)$ of fluid phases. The values of the mass and dissipation coefficients in terms of physically measurable quantities are presented in the next two sections.

4 Mass coefficients

It is obvious that the volume fractions $\nu^{(k)}$ and porosities $\phi^{(k)}$ of type-k pores, as well as the solid and fluid densities ρ_s and ρ_l, must all influence the mass coefficients ρ_{ij} that arise in the kinetic energy E. The total porosity of the multi-porous medium is defined by $\phi = \sum_{k=1}^{n} \nu^{(k)} \phi^{(k)}$.

The kinetic energy of the single porosity medium with three mass coefficients, and solid and only one fluid displacement $\mathbf{U}^{(0)}$, $\mathbf{U}^{(1)}$, respectively, can be written as

$$2E = \begin{pmatrix} \dot{\mathbf{U}}^{(0)} & \dot{\mathbf{U}}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{\rho}_{00} & \tilde{\rho}_{01} \\ \tilde{\rho}_{01} & \tilde{\rho}_{11} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}}^{(0)} \\ \dot{\mathbf{U}}^{(1)} \end{pmatrix}. \tag{7}$$

For this system of single porosity medium, Biot [1, 2] has shown that the total mass density ρ in terms of mass coefficient is $\rho = \tilde{\rho}_{00} + 2\tilde{\rho}_{01} + \tilde{\rho}_{11}$ and in terms of solid and fluid density can be written as $\rho = (1 - \phi)\rho_s + \phi\rho_f$. Furthermore, it has been shown that $\tilde{\rho}_{00} + \tilde{\rho}_{01} = (1 - \phi)\rho_s$ and $\tilde{\rho}_{01} + \tilde{\rho}_{11} = \phi\rho_f$. The individual expression of these mass coefficients can be obtained by introducing the electrical tortuosity parameter $\tilde{\tau}$ as

$$\tilde{\rho}_{00} = (1 - \phi)\rho_s + (\tilde{\tau} - 1)\phi\rho_f, \quad \tilde{\rho}_{01} = -(\tilde{\tau} - 1)\phi\rho_f, \quad \tilde{\rho}_{11} = \tilde{\tau}\phi\rho_f. \tag{8}$$

By introducing the factor $\tilde{r}\,(0 \le \tilde{r} \le 1)$, dependent on the micro-geometry of the medium, Berryman [71] has established that

$$\tilde{\tau} = 1 + \tilde{r}\frac{1 - \phi}{\phi} \tag{9}$$

With the help of above single porosity system we will find the mass coefficients for multi-porous medium.

For a multi-porous medium, the kinetic energy E from equation (3) can be represented in matrix format as

$$2E = \begin{pmatrix} \dot{\mathbf{U}}^{(0)} & \dot{\mathbf{U}}^{(1)} & \cdots & \dot{\mathbf{U}}^{(n)} \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} & \cdots & \rho_{0n} \\ \rho_{10} & \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n0} & \rho_{n1} & \cdots & \rho_{nn} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}}^{(0)} \\ \dot{\mathbf{U}}^{(1)} \\ \vdots \\ \dot{\mathbf{U}}^{(n)} \end{pmatrix}. \tag{10}$$

As a next step, we will consider some limiting cases to convert this multi-porosity system into a single porosity system to get the relations among mass coefficients.

Case 1: Let us consider that the solid and all the fluid phases move in unison, i.e., $\dot{\mathbf{U}}^{(0)} = \dot{\mathbf{U}}^{(1)} = \cdots = \dot{\mathbf{U}}^{(n)} = \dot{\mathbf{U}}$ (say). The total kinetic energy (10) is then simplified and it may be written as

$$2E = \left(\sum_{i=0}^{n} \rho_{ii} + 2 \sum_{\substack{i,j=0 \\ i<j}}^{n} \rho_{ij} \right) \dot{\mathbf{U}} \cdot \dot{\mathbf{U}} \tag{11}$$

and, so by the analogy with the single porosity medium, we have

$$\sum_{i=0}^{n} \rho_{ii} + 2 \sum_{\substack{i,j=0 \\ i<j}}^{n} \rho_{ij} = (1-\phi)\rho_s + \phi\rho_f. \tag{12}$$

Case 2: Now, let us consider that all the fluid phases move at a uniform velocity distinct from that of the solid, i.e., $\dot{\mathbf{U}}^{(1)} = \dot{\mathbf{U}}^{(2)} = \cdots = \dot{\mathbf{U}}^{(n)} = \dot{\mathbf{U}}$ (say), then the kinetic energy (10) can be simplified and expressed as

$$2E = \begin{pmatrix} \dot{\mathbf{U}}^{(0)} & \dot{\mathbf{U}} \end{pmatrix} \begin{pmatrix} \rho_{00} & \sum_{k=1}^{n} \rho_{0k} \\ \sum_{k=1}^{n} \rho_{0k} & \sum_{k=1}^{n} \rho_{kk} + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}}^{(0)} \\ \dot{\mathbf{U}} \end{pmatrix}. \tag{13}$$

Now by the direct relationship between the matrix elements in (13) and (7), we have three equations as

$$\rho_{00} = (1-\phi)\rho_s + (\tau-1)\phi\rho_f, \tag{14}$$

$$\sum_{k=1}^{n} \rho_{0k} = -(\tau-1)\phi\rho_f, \tag{15}$$

$$\sum_{k=1}^{n} \rho_{kk} + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} = \tau\phi\rho_f. \tag{16}$$

The equations (12), (14), (15) and (16) are not all independent. There are $\frac{(n+1)(n+2)}{2}$ unknown mass coefficients, so we need more equations to find out the individual values of mass coefficients.

Case 3: Again, let us consider that the fluid phase in type-k pore oscillate independently and all the other fluid phases in type-i pore ($i \neq k$) with solid move in unison, i.e., $\dot{\mathbf{U}}^{(0)} = \dot{\mathbf{U}}^{(1)} = \cdots = \dot{\mathbf{U}}^{(k-1)} = \dot{\mathbf{U}}^{(k+1)} = \cdots = \dot{\mathbf{U}}^{(n)} = \dot{\mathbf{U}}$ (say), then the kinetic energy (10) can be simplified as

$$2E = \begin{pmatrix} \dot{\mathbf{U}} & \dot{\mathbf{U}}^{(k)} \end{pmatrix} \begin{pmatrix} \sum_{\substack{i,j=0 \\ i,j\neq k}}^{n} \rho_{ij} & \sum_{\substack{j=0 \\ j\neq k}}^{n} \rho_{kj} \\ \sum_{\substack{j=0 \\ j\neq k}}^{n} \rho_{kj} & \rho_{kk} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}} \\ \dot{\mathbf{U}}^{(k)} \end{pmatrix}, \quad k = 1, 2, 3, ..., n. \tag{17}$$

Similarly, we can find three equations for each values of k by the analogy with the single porosity medium. However, since we regard solid and pore fluid phases as a single unit in this instance, we must carefully account for the components of the system that are present in the medium. Let $\tau^{(k)}$ be the tortuosity of the type-k pore alone, then for each $k\,(k = 1, 2, 3, ..., n)$ we have

$$\sum_{\substack{i,j=0 \\ i,j\neq k}}^{n} \rho_{ij} = (1-\phi)\rho_s + \sum_{\substack{i=1 \\ i\neq k}}^{n} \nu^{(i)}\phi^{(i)}\rho_f + \left(\tau^{(k)} - 1\right)\nu^{(k)}\phi^{(k)}\rho_f, \tag{18}$$

$$\sum_{\substack{j=0 \\ j\neq k}}^{n} \rho_{kj} = -\left(\tau^{(k)} - 1\right)\nu^{(k)}\phi^{(k)}\rho_f, \tag{19}$$

$$\rho_{kk} = \tau^{(k)}\nu^{(k)}\phi^{(k)}\rho_f. \tag{20}$$

Case 4: Let us consider that the solid and type-k pore fluid phase move in unison, and all the other fluid phases other than the fluid in type-k pore move in unison, i.e.,

$$\dot{\mathbf{U}}^{(0)} = \dot{\mathbf{U}}^{(k)} = \dot{\mathbf{U}}_1 \text{ (say) and } \dot{\mathbf{U}}^{(1)} = \dot{\mathbf{U}}^{(2)} = \cdots = \dot{\mathbf{U}}^{(k-1)} = \dot{\mathbf{U}}^{(k+1)} = \cdots = \dot{\mathbf{U}}^{(n)} = \dot{\mathbf{U}}_2 \text{ (say)},$$

then the kinetic energy (10) can be written as

$$2E = \begin{pmatrix} \dot{\mathbf{U}}_1 & \dot{\mathbf{U}}_2 \end{pmatrix} \begin{pmatrix} \rho_{00} + 2\rho_{0k} + \rho_{kk} & \sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{0j} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{kj} \\ \sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{0j} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{kj} & \sum_{\substack{i,j=1 \\ i,j\neq k}}^{n} \rho_{ij} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}}_1 \\ \dot{\mathbf{U}}_2 \end{pmatrix}, \quad k = 1, 2, ..., n. \tag{21}$$

As in the previous cases, by the analogy of single porosity, for each $k\,(k = 1, 2, 3, ..., n)$ we can get

$$\rho_{00} + 2\rho_{0k} + \rho_{kk} = (1-\phi)\rho_s + \nu^{(k)}\phi^{(k)}\rho_f + \left(\tau^{(1,2,\ldots,k-1,k+1,\ldots,n)} - 1\right)\left(\phi - \nu^{(k)}\phi^{(k)}\right)\rho_f, \tag{22}$$

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{0j} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \rho_{kj} = -\left(\tau^{(1,2,\ldots,k-1,k+1,\ldots,n)} - 1\right)\left(\phi - \nu^{(k)}\phi^{(k)}\right)\rho_f, \tag{23}$$

$$\sum_{\substack{i,j=1 \\ i,j\neq k}}^{n} \rho_{ij} = \tau^{(1,2,\ldots,k-1,k+1,\ldots,n)}\left(\phi - \nu^{(k)}\phi^{(k)}\right)\rho_f, \tag{24}$$

where, $\tau^{(1,2,\ldots,k-1,k+1,\ldots,n)}$ is the combined tortuosity of all type-$i\,(i \neq k)$ pore structures.

Case 5: Let us consider that the type-l and type-m pore fluid phases move in unison, and all other fluid phases in type-$k\,(k \neq l, m)$ with solid phase move in unison for $0 < l < m \leq n$, i.e.,

$$\dot{\mathbf{U}}^{(l)} = \dot{\mathbf{U}}^{(m)} = \dot{\mathbf{U}}_1 \text{ (say) and}$$
$$\dot{\mathbf{U}}^{(0)} = \dot{\mathbf{U}}^{(1)} = \cdots = \dot{\mathbf{U}}^{(l-1)} = \dot{\mathbf{U}}^{(l+1)} = \cdots = \dot{\mathbf{U}}^{(m-1)} = \dot{\mathbf{U}}^{(m+1)} = \cdots = \dot{\mathbf{U}}^{(n)} = \dot{\mathbf{U}}_2 \text{ (say)},$$

then the kinetic energy (10) can be written as

$$2E = \begin{pmatrix} \dot{\mathbf{U}}_1 & \dot{\mathbf{U}}_2 \end{pmatrix} \begin{pmatrix} \rho_{ll} + 2\rho_{lm} + \rho_{mm} & \sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{lj} + \sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{mj} \\ \sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{lj} + \sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{mj} & \sum_{\substack{i,j=0 \\ i,j\neq l,m}}^{n} \rho_{ij} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{U}}_1 \\ \dot{\mathbf{U}}_2 \end{pmatrix}. \tag{25}$$

Similarly, for each $l, m\,(l, m = 1, 2, 3, ..., n)$, we have the three equations as

$$\rho_{ll} + 2\rho_{lm} + \rho_{mm} = \tau^{(l,m)}\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right)\rho_f, \tag{26}$$

$$\sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{lj} + \sum_{\substack{j=0 \\ j\neq l,m}}^{n} \rho_{mj} = -\left(\tau^{(l,m)} - 1\right)\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right)\rho_f, \tag{27}$$

$$\sum_{\substack{i,j=0 \\ i,j\neq l,m}}^{n} \rho_{ij} = (1-\phi)\rho_s + \left(\phi - \nu^{(l)}\phi^{(l)} - \nu^{(m)}\phi^{(m)}\right)\rho_f + \left(\tau^{(l,m)} - 1\right)\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right)\rho_f. \tag{28}$$

Next, we can consider triple equal velocity and others equally independently move in unison. More equations with ρ_{ij} can be obtained in this manner, although not all equations are independent. This approach should be repeated until the number of independent equations matches the number of unknown mass coefficients based on the number of different types of pore structures.

After a tedious calculations of the above equations (12-28), the explicit expression for mass coefficients in general for n-types of pore structures can be written as

$$\rho_{00} = (1-\phi)\rho_s + (\tau - 1)\phi\rho_f, \quad \rho_{kk} = \tau^{(k)}\nu^{(k)}\phi^{(k)}\rho_f, \tag{29}$$

$$2\rho_{0k}/\rho_f = \left(\tau^{(1,2,\ldots,k-1,k+1,\ldots,n)} - 1\right)\left(\phi - \nu^{(k)}\phi^{(k)}\right) - \left(\tau^{(k)} - 1\right)\nu^{(k)}\phi^{(k)} - (\tau - 1)\phi, \tag{30}$$

$$2\rho_{lm}/\rho_f = \tau^{(l,m)}\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right) - \tau^{(l)}\nu^{(l)}\phi^{(l)} - \tau^{(m)}\nu^{(m)}\phi^{(m)}, \quad 0 < l < m \leq n, \tag{31}$$
$$l, m, k = 1, 2, 3, ..., n,$$

with the relation between the tortuosities as

$$(n-2)\tau\phi = \sum_{k=1}^{n} \tau^{(1,2,\ldots,k-1,k+1,\ldots,n)} \left(\phi - \nu^{(k)}\phi^{(k)}\right) - \sum_{k=1}^{n} \tau^{(k)}\nu^{(k)}\phi^{(k)}, \quad n \geq 2, \tag{32}$$

where,

$$\begin{aligned} \tau^{(1,2,\ldots,k-1,k+1,\ldots,n)} \left(\phi - \nu^{(k)}\phi^{(k)}\right) = -\sum_{\substack{j=1\\ j\neq k}}^{n} \tau^{(k,j)} \left(\nu^{(k)}\phi^{(k)} + \nu^{(j)}\phi^{(j)}\right) + (n-2)\tau^{(k)}\nu^{(k)}\phi^{(k)} \\ + \sum_{\substack{j=1\\ j\neq k}}^{n} \tau^{(j)}\nu^{(j)}\phi^{(j)} + \tau\phi, \quad k = 1,2,...,n. \end{aligned} \tag{33}$$

After simplification, the above equations (29-33), can be rewritten as

$$\rho_{00} = (1-\phi)\rho_s + (\tau - 1)\phi\rho_f, \quad \rho_{kk} = \tau^{(k)}\nu^{(k)}\phi^{(k)}\rho_f, \tag{34}$$

$$\begin{aligned} 2\rho_{0k}/\rho_f = \sum_{\substack{l,m=1\\ l<m\\ l,m\neq k}}^{n} \left(\tau^{(l,m)} - 1\right)\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right) - (n-3)\sum_{\substack{j=1\\ j\neq k}}^{n} \left(\tau^{(j)} - 1\right)\nu^{(j)}\phi^{(j)} \\ - \left(\tau^{(k)} - 1\right)\nu^{(k)}\phi^{(k)} - (\tau - 1)\phi, \quad n \geq 3, \end{aligned} \tag{35}$$

$$2\rho_{lm}/\rho_f = \tau^{(l,m)}\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right) - \tau^{(l)}\nu^{(l)}\phi^{(l)} - \tau^{(m)}\nu^{(m)}\phi^{(m)}, \quad 0 < l < m \leq n, \tag{36}$$

$$l, m, k = 1, 2, 3, ..., n,$$

with the relation between the tortuosities as

$$\tau\phi = \sum_{\substack{l,m=1\\ l<m}}^{n} \tau^{(l,m)}\left(\nu^{(l)}\phi^{(l)} + \nu^{(m)}\phi^{(m)}\right) - (n-2)\sum_{k=1}^{n} \tau^{(k)}\nu^{(k)}\phi^{(k)}, \quad n \geq 2, \tag{37}$$

The tortuosity parameters τ, $\tau^{(k)}$, $\tau^{(l,m)}$ $(l, m, k = 1, 2, ..., n)$ can be estimated using the equation (9), or may be obtained using the electrical methods given by Johnson et al. [73], and also recently Graczyk and Matyka [74] developed deep learning method to find tortuosity from image of a porous medium.

In particular, the mass coefficients for triple-porosity medium can be derived as

$$\left.\begin{aligned} &\rho_{00} = (1-\phi)\rho_s + (\tau - 1)\phi\rho_f, \quad \rho_{ll} = \tau^{(l)}\nu^{(l)}\phi^{(l)}\rho_f, \\ &2\rho_{0l} = \left(\tau^{(i,j)} - 1\right)\left(\nu^{(i)}\phi^{(i)} + \nu^{(j)}\phi^{(j)}\right)\rho_f - \left(\tau^{(l)} - 1\right)\nu^{(l)}\phi^{(l)}\rho_f - (\tau - 1)\phi\rho_f, \quad i, j \neq l, \\ &2\rho_{ij} = \tau^{(i,j)}\left(\nu^{(i)}\phi^{(i)} + \nu^{(j)}\phi^{(j)}\right)\rho_f - \tau^{(i)}\nu^{(i)}\phi^{(i)}\rho_f - \tau^{(j)}\nu^{(j)}\phi^{(j)}\rho_f, \quad i, j, l = 1, 2, 3 \end{aligned}\right\} \tag{38}$$

with the relation between the tortuosities as

$$\tau\phi = \tau^{(1,3)}\left(\nu^{(1)}\phi^{(1)} + \nu^{(3)}\phi^{(3)}\right) + \tau^{(1,2)}\left(\nu^{(1)}\phi^{(1)} + \nu^{(2)}\phi^{(2)}\right) + \tau^{(2,3)}\left(\nu^{(2)}\phi^{(2)} + \nu^{(3)}\phi^{(3)}\right) - \sum_{k=1}^{3} \tau^{(k)}\nu^{(k)}\phi^{(k)}. \tag{39}$$

5 Dissipation coefficients

The dissipation coefficients may be determined by neglecting the mass coefficients in the low-frequency limit as of Berryman and Wang [17, 21]. It is simple to grasp that the dissipation coefficients rely on the fluid flow through the medium, i.e., the medium's permeability and the fluid viscosity. The permeability coefficients can be calculated from Darcy's law by

$$\sum_{j=1}^{n} \frac{L^{(lj)}}{\eta} p_{,ii}^{(j)} = \dot{\zeta}^{(l)}, \quad l = 1, 2, 3, ..., n, \tag{40}$$

where, η represents the fluid's shear viscosity, $L^{(lj)}$ are the permeabilities with cross-coupling coefficients, and the increment of fluid content in type-k pore structures are given by

$$\zeta^{(k)} = -\nu^{(k)}\phi^{(k)}\nabla\cdot\left(\mathbf{U}^{(k)} - \mathbf{U}^{(0)}\right), \quad k = 1, 2, 3, ..., n. \tag{41}$$

The equation (40) can be written in the matrix form as

$$\eta \begin{pmatrix} \dot{\zeta}^{(1)} \\ \dot{\zeta}^{(2)} \\ \dot{\zeta}^{(3)} \\ \vdots \\ \dot{\zeta}^{(n)} \end{pmatrix} = \begin{pmatrix} L^{(11)} & L^{(12)} & L^{(13)} & \cdots & L^{(1n)} \\ L^{(21)} & L^{(22)} & L^{(23)} & \cdots & L^{(2n)} \\ L^{(31)} & L^{(32)} & L^{(33)} & \cdots & L^{(3n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L^{(n1)} & L^{(n2)} & L^{(n3)} & \cdots & L^{(nn)} \end{pmatrix} \begin{pmatrix} p^{(1)}_{,ii} \\ p^{(2)}_{,ii} \\ p^{(3)}_{,ii} \\ \vdots \\ p^{(n)}_{,ii} \end{pmatrix}. \tag{42}$$

From equations (5), (6), extracting the term as per our requirement and then taking the divergence, we get

$$\begin{pmatrix} \sum_{\substack{j=0 \\ j\neq 1}}^{n} b_{1j} & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ -b_{12} & \sum_{\substack{j=0 \\ j\neq 2}}^{n} b_{2j} & -b_{23} & \cdots & -b_{2n} \\ -b_{13} & -b_{23} & \sum_{\substack{j=0 \\ j\neq 3}}^{n} b_{3j} & \cdots & -b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1n} & -b_{2n} & -b_{3n} & \cdots & \sum_{\substack{j=0 \\ j\neq n}}^{n} b_{nj} \end{pmatrix} \begin{pmatrix} \nabla\cdot\left(\dot{\mathbf{U}}^{(1)} - \dot{\mathbf{U}}^{(0)}\right) \\ \nabla\cdot\left(\dot{\mathbf{U}}^{(2)} - \dot{\mathbf{U}}^{(0)}\right) \\ \nabla\cdot\left(\dot{\mathbf{U}}^{(3)} - \dot{\mathbf{U}}^{(0)}\right) \\ \vdots \\ \nabla\cdot\left(\dot{\mathbf{U}}^{(n)} - \dot{\mathbf{U}}^{(0)}\right) \end{pmatrix} = - \begin{pmatrix} \bar{p}^{(1)}_{,ii} \\ \bar{p}^{(2)}_{,ii} \\ \bar{p}^{(3)}_{,ii} \\ \vdots \\ \bar{p}^{(n)}_{,ii} \end{pmatrix}. \tag{43}$$

With the help of equation (41), comparing equations (42) and (43), we have

$$\mathbf{B} = \eta\mathbf{\Phi}\mathbf{L}^{-1}\mathbf{\Phi}, \tag{44}$$

where, $\mathbf{B}$, $\mathbf{L}$ are the symmetric matrices of dissipation coefficients b's and permeability coefficients L's defined in (43) and (42), respectively, and $\mathbf{\Phi}$ is given by

$$\mathbf{\Phi} = \begin{pmatrix} \nu^{(1)}\phi^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \nu^{(2)}\phi^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & \nu^{(3)}\phi^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu^{(n)}\phi^{(n)} \end{pmatrix}. \tag{45}$$

Finally, the dissipation coefficients of the multi-porous medium can be derived from (44) as

$$b_{ij} = \frac{\eta\nu^{(i)}\nu^{(j)}\phi^{(i)}\phi^{(j)}}{\det(\mathbf{L})}\,(-1)^{i+j+1}\,M_{ij}, \quad i \neq j;\, i, j = 1, 2, 3, ..., n, \tag{46}$$

$$b_{0j} = \frac{\eta\nu^{(j)}\phi^{(j)}}{\det(\mathbf{L})}\sum_{k=1}^{n}(-1)^{j+k}\,\nu^{(k)}\phi^{(k)}M_{jk}, \quad j = 1, 2, 3, ..., n, \tag{47}$$

where, M_{ij} is the minor of $L^{(ij)}$ in the matrix $\mathbf{L}$. If we consider that the cross-coupling term between the fluid contents in different type of pore structure are zero i.e., $b_{ij} = 0\,(i \neq j)$, then the simple expression of these dissipation coefficients between solid and fluid phases, in terms of porosities and diagonal permeabilities can be written as

$$b_{0j} = \left(\nu^{(j)}\phi^{(j)}\right)^2 \frac{\eta}{L^{(jj)}}, \quad j = 1, 2, 3, ..., n. \tag{48}$$

In particular, the dissipation coefficients for triple-porosity medium in terms of porosity and permeabilities can be written as (cf. (46) and (47))

$$\left.\begin{aligned}
b_{01} &= \frac{\eta\nu^{(1)}\phi^{(1)}}{\det(\mathbf{L})}\left[\nu^{(1)}\phi^{(1)}\left\{L^{(22)}L^{(33)}-L^{(23)}L^{(32)}\right\}+\nu^{(2)}\phi^{(2)}\left\{L^{(13)}L^{(32)}-L^{(12)}L^{(33)}\right\}\right.\\
&\qquad\left.+\nu^{(3)}\phi^{(3)}\left\{L^{(12)}L^{(23)}-L^{(13)}L^{(22)}\right\}\right],\\
b_{02} &= \frac{\eta\nu^{(2)}\phi^{(2)}}{\det(\mathbf{L})}\left[\nu^{(1)}\phi^{(1)}\left\{L^{(31)}L^{(32)}-L^{(21)}L^{(33)}\right\}+\nu^{(2)}\phi^{(2)}\left\{L^{(11)}L^{(33)}-L^{(13)}L^{(31)}\right\}\right.\\
&\qquad\left.+\nu^{(3)}\phi^{(3)}\left\{L^{(21)}L^{(13)}-L^{(11)}L^{(23)}\right\}\right],\\
b_{03} &= \frac{\eta\nu^{(3)}\phi^{(3)}}{\det(\mathbf{L})}\left[\nu^{(1)}\phi^{(1)}\left\{L^{(21)}L^{(32)}-L^{(31)}L^{(22)}\right\}+\nu^{(2)}\phi^{(2)}\left\{L^{(12)}L^{(31)}-L^{(11)}L^{(32)}\right\}\right.\\
&\qquad\left.+\nu^{(3)}\phi^{(3)}\left\{L^{(11)}L^{(22)}-L^{(21)}L^{(12)}\right\}\right],
\end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned}
b_{12} &= \frac{\eta\nu^{(1)}\nu^{(2)}\phi^{(1)}\phi^{(2)}}{\det(\mathbf{L})}\left\{L^{(12)}L^{(33)}-L^{(13)}L^{(32)}\right\},\\
b_{13} &= \frac{\eta\nu^{(1)}\nu^{(3)}\phi^{(1)}\phi^{(3)}}{\det(\mathbf{L})}\left\{L^{(13)}L^{(22)}-L^{(12)}L^{(13)}\right\},\\
b_{23} &= \frac{\eta\nu^{(2)}\nu^{(3)}\phi^{(2)}\phi^{(3)}}{\det(\mathbf{L})}\left\{L^{(11)}L^{(23)}-L^{(12)}L^{(13)}\right\},
\end{aligned}\right\} \tag{50}$$

where, $\mathbf{L}$ is a 3×3 symmetric matrix of permeabilities given by

$$\mathbf{L}=\begin{pmatrix} L^{(11)} & L^{(12)} & L^{(13)}\\ L^{(21)} & L^{(22)} & L^{(23)}\\ L^{(31)} & L^{(32)} & L^{(33)}\end{pmatrix}. \tag{51}$$

The dissipation coefficients in the multi-porous medium have now been fully identified.

6 Constitutive equations

The theory proposed in this context incorporates an implicit assumption regarding the presence of a macroscopic length scale. At this scale, the rock, fluid mixture, and each distinct pore structure are considered to exhibit a degree of homogeneity and isotropy. These assumptions are made for convenience, as they simplify the analysis process. However, it is important to acknowledge that non-homogeneity and anisotropy are pertinent aspects that can be addressed through a straightforward extension of the current work. Let us consider that the confining (external) pressure p_c and the fluid pressure $p^{(k)}$ in the type-k pore structures are the independent variables, and the increment of fluid content in type-k pore structure $\zeta^{(k)}$ with the volumetric strain e are the dependent variables. Then by the generalization of single and double porosity (Berryman and Wang [21]), the multi-porosity theory with $\frac{(n+1)(n+2)}{2}$ independent coefficients h_{ij} ($h_{ij}=h_{ji}$), the linear relationship can be written as

$$\begin{pmatrix} e\\ -\zeta^{(1)}\\ -\zeta^{(2)}\\ \vdots\\ -\zeta^{(n)}\end{pmatrix}=\begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1(n+1)}\\ h_{21} & h_{22} & h_{23} & \cdots & h_{2(n+1)}\\ h_{31} & h_{32} & h_{33} & \cdots & h_{3(n+1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ h_{(n+1)1} & h_{(n+1)2} & h_{(n+1)3} & \cdots & h_{(n+1)(n+1)}\end{pmatrix}\begin{pmatrix} -p_c\\ -p^{(1)}\\ -p^{(2)}\\ \vdots\\ -p^{(n)}\end{pmatrix}. \tag{52}$$

To determine the unknown coefficients h_{ij} in terms of known quantities, we consider a series of Gedanken experiments with short-term and long-term time limits.

Undrained test, Short time

In the undrained test, it is assumed that the multi-porous material is surrounded by an impermeable jacket, so fluid can not escape from the jacketed sample. Again, for a short time, each fluid phase behaves as undrained following a sudden change in confining pressure (Elsworth and Bai [16]). Then, the individual increment of fluid contents can not change, so

$$\delta\zeta^{(i)} = 0, \quad i = 1, 2, ..., n.$$

Which implies from the equation (52) as

$$\begin{pmatrix} \delta e \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1(n+1)} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2(n+1)} \\ h_{31} & h_{32} & h_{33} & \cdots & h_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{(n+1)1} & h_{(n+1)2} & h_{(n+1)3} & \cdots & h_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} -\delta p_c \\ -\delta p^{(1)} \\ -\delta p^{(2)} \\ \vdots \\ -\delta p^{(n)} \end{pmatrix}. \tag{53}$$

Then the generalized Skempton-like coefficients for the multi-porous medium can be defined as

$$B_{EB}^{(i)} = \left.\frac{\partial p^{(i)}}{\partial p_c}\right|_{\delta\zeta^{(j)}=0}, \quad i = 1, 2, ..., n, \forall j. \tag{54}$$

So the n number of Skempton coefficient for the multi-porous medium can be find in terms of h_{ij} from the matrix equation as

$$\begin{pmatrix} h_{22} & h_{23} & \cdots & h_{2(n+1)} \\ h_{32} & h_{33} & \cdots & h_{3(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{(n+1)2} & h_{(n+1)3} & \cdots & h_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} B_{EB}^{(1)} \\ B_{EB}^{(2)} \\ \vdots \\ B_{EB}^{(n)} \end{pmatrix} = - \begin{pmatrix} h_{12} \\ h_{13} \\ \vdots \\ h_{1(n+1)} \end{pmatrix}. \tag{55}$$

Again the effective undrained modulus can be found from the equation (53) as

$$\begin{aligned} \frac{1}{K_{u^{EB}}} &\equiv -\left.\frac{\partial e}{\partial p_c}\right|_{\delta\zeta^{(j)}=0}, \quad \forall j. \\ &= h_{11} + \sum_{j=1}^{n} h_{1(j+1)} B_{EB}^{(j)}. \end{aligned} \tag{56}$$

Undrained test, Long time

In the long-time undrained test, the multi-porous system reduces to a single-porous system, and all the pore pressure becomes equal. Then we have for the undrained test, long-time behaviour as

$$\left.\begin{aligned} &\delta p^{(1)} = \delta p^{(2)} = ... = \delta p^{(n)} = \delta p, \\ &\delta\zeta \equiv \delta\zeta^{(1)} + \delta\zeta^{(2)} + ... + \delta\zeta^{(n)} = 0. \end{aligned}\right\} \tag{57}$$

Then using equation (57) from equation (52) we have

$$\left.\begin{aligned} &\delta e = -h_{11}\delta p_c - \delta p \textstyle\sum_{j=1}^{n} h_{1(j+1)}, \\ &0 = -\delta p_c \textstyle\sum_{j=1}^{n} h_{1(j+1)} - \delta p \textstyle\sum_{i,j=1}^{n} h_{(i+1)(j+1)}. \end{aligned}\right\} \tag{58}$$

Then the total pore pressure buildup coefficient is given by

$$B \equiv \left.\frac{\partial p}{\partial p_c}\right|_{\delta\zeta=0} = -\frac{\sum_{j=1}^{n} h_{1(j+1)}}{\sum_{i,j=1}^{n} h_{(i+1)(j+1)}}, \tag{59}$$

and the undrained bulk modulus is given by

$$\frac{1}{K_u} \equiv -\left.\frac{\partial e}{\partial p_c}\right|_{\delta\zeta=0} = h_{11} + B\sum_{j=1}^{n} h_{1(j+1)}. \tag{60}$$

Drained test, Long time

In the drained test, a tube is added to the impermeable jacketed sample to allow fluid escape. Also, the multi-porous system reduces to a single-porous system at the long-time limit. Then, in the long-time drained test, the fluid pressure will remain unchanged, equaling the atmospheric pressure, and we have

$$\delta p^{(1)} = \delta p^{(2)} = ... = \delta p^{(n)} = 0, \tag{61}$$

which implies from equation (52) that

$$\delta e = -h_{11}\delta p_c.$$

Therefore, the total drained bulk modulus of the multi-porous system is given by

$$\frac{1}{K} \equiv -\left.\frac{\partial e}{\partial p_c}\right|_{\delta p^{(j)}=0} = h_{11}, \quad \forall j. \tag{62}$$

Undrained k-type pore and drained all other types of pore structures ($k = 1, 2, 3, ..., n$)**, Intermediate time**

In this case, the tubes are inserted into all types of pore structures other than the k-type pore structure of the jacketed sample. Then by the analogy of drained and undrained tests, we have

$$\delta\zeta^{(k)} = 0, \quad \delta p^{(j)} = 0, \quad j = 1, 2, ..., n\,(j \neq k). \tag{63}$$

Using equation (63) the equation (52) can be rewritten as

$$\begin{pmatrix} \delta e \\ -\delta\zeta^{(1)} \\ -\delta\zeta^{(2)} \\ \vdots \\ -\delta\zeta^{(k-1)} \\ 0 \\ -\delta\zeta^{(k+1)} \\ \vdots \\ -\delta\zeta^{(n)} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1(k+1)} & \cdots & h_{1(n+1)} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2(k+1)} & \cdots & h_{2(n+1)} \\ h_{31} & h_{32} & h_{33} & \cdots & h_{3(k+1)} & \cdots & h_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{k1} & h_{k2} & h_{k3} & \cdots & h_{k(k+1)} & \cdots & h_{k(n+1)} \\ h_{(k+1)1} & h_{(k+1)2} & h_{(k+1)3} & \cdots & h_{(k+1)(k+1)} & \cdots & h_{(k+1)(n+1)} \\ h_{(k+2)1} & h_{(k+2)2} & h_{(k+2)3} & \cdots & h_{(k+2)(k+1)} & \cdots & h_{(k+2)(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{(n+1)1} & h_{(n+1)2} & h_{(n+1)3} & \cdots & h_{(n+1)(k+1)} & \cdots & h_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} -\delta p_c \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\delta p^{(k)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{64}$$

From the equation (64) two equations can be extracted as

$$\left.\begin{aligned} \delta e &= -h_{11}\delta p_c - h_{1(k+1)}\delta p^{(k)}, \\ 0 &= -h_{(k+1)1}\delta p_c - h_{(k+1)(k+1)}\delta p^{(k)}, \end{aligned}\right\} \tag{65}$$

thus the pore pressure buildup coefficients in the k-type pore structures are given by

$$B\left[U^{(k)}\right] \equiv \left.\frac{\partial p^{(k)}}{\partial p_c}\right|_{\delta\zeta^{(k)}=0=\delta p^{(j)}} = -\frac{h_{(k+1)1}}{h_{(k+1)(k+1)}}, \quad k = 1, 2, ..., n, \tag{66}$$

and the effective undrained moduli for the k-type pore structure phases are determined by

$$\frac{1}{K\left[U^{(k)}\right]} \equiv -\left.\frac{\partial e}{\partial p_c}\right|_{\delta\zeta^{(k)}=0=\delta p^{(j)}} = h_{11} + h_{1(k+1)}B\left[U^{(k)}\right], \quad k = 1, 2, ..., n. \tag{67}$$

The coefficients $h_{(i+1)(j+1)}$ ($i < j; i, j = 1, 2, ..., n$) are the measure of cross-coupling between the type-i and type-j pore fluid phases. If there is no cross-coupling between the fluid phases, then the pore pressure buildup coefficients and generalized Skempton-like coefficients defined in equations (55) and (66) are the same.

Undrained l, m-types pore structures and drained all other types pore structures ($l, m = 1, 2, 3, ..., n$; $l < m$)**, Intermediate time**

In this case, the tubes are inserted in all types of pore structures other than the l, m-type pores structure of the jacketed sample. Then, by the analogy of drained and undrained tests, we have

$$\delta\zeta^{(l)} = 0 = \delta\zeta^{(m)}, \quad l < m, \text{ and } \quad \delta p^{(j)} = 0, \quad j = 1, 2, ..., n\,(j \neq l, m). \tag{68}$$

Using equation (68) the equation (52) can be rewritten as

$$\begin{pmatrix} \delta e \\ -\delta\zeta^{(1)} \\ -\delta\zeta^{(2)} \\ \vdots \\ -\delta\zeta^{(l-1)} \\ 0 \\ -\delta\zeta^{(l+1)} \\ \vdots \\ -\delta\zeta^{(m-1)} \\ 0 \\ -\delta\zeta^{(m+1)} \\ \vdots \\ -\delta\zeta^{(n)} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1(l+1)} & \cdots & h_{1(m+1)} & \cdots & h_{1(n+1)} \\ h_{21} & h_{22} & \cdots & h_{2(l+1)} & \cdots & h_{2(m+1)} & \cdots & h_{2(n+1)} \\ h_{31} & h_{32} & \cdots & h_{3(l+1)} & \cdots & h_{3(m+1)} & \cdots & h_{3(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{l1} & h_{l2} & \cdots & h_{l(l+1)} & \cdots & h_{l(m+1)} & \cdots & h_{l(n+1)} \\ h_{(l+1)1} & h_{(l+1)2} & \cdots & h_{(l+1)(l+1)} & \cdots & h_{(l+1)(m+1)} & \cdots & h_{(l+1)(n+1)} \\ h_{(l+2)1} & h_{(l+2)2} & \cdots & h_{(l+2)(l+1)} & \cdots & h_{(l+2)(m+1)} & \cdots & h_{(l+2)(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{m(l+1)} & \cdots & h_{m(m+1)} & \cdots & h_{m(n+1)} \\ h_{(m+1)1} & h_{(m+1)2} & \cdots & h_{(m+1)(l+1)} & \cdots & h_{(m+1)(m+1)} & \cdots & h_{(m+1)(n+1)} \\ h_{(m+2)1} & h_{(m+2)2} & \cdots & h_{(m+2)(l+1)} & \cdots & h_{(m+2)(m+1)} & \cdots & h_{(m+2)(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{(n+1)1} & h_{(n+1)2} & \cdots & h_{(n+1)(l+1)} & \cdots & h_{(n+1)(m+1)} & \cdots & h_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} -\delta p_c \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\delta p^{(l)} \\ 0 \\ \vdots \\ 0 \\ -\delta p^{(m)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{69}$$

From the equation (69) three equations can be extracted as

$$\left.\begin{array}{l} \delta e = -h_{11}\delta p_c - h_{1(l+1)}\delta p^{(l)} - h_{1(m+1)}\delta p^{(m)}, \\ 0 = -h_{(l+1)1}\delta p_c - h_{(l+1)(l+1)}\delta p^{(l)} - h_{(l+1)(m+1)}\delta p^{(m)}, \\ 0 = -h_{(m+1)1}\delta p_c - h_{(m+1)(l+1)}\delta p^{(l)} - h_{(m+1)(m+1)}\delta p^{(m)}, \end{array}\right\} \tag{70}$$

thus the two joint pore pressure buildup coefficients in the l, m-types pore structures are defined by

$$B\left[U_{lm}^{(l)}\right] \equiv \left.\frac{\partial p^{(l)}}{\partial p_c}\right|_{\delta\zeta^{(l)}=\delta\zeta^{(m)}=0=\delta p^{(j)}} = \frac{h_{1(m+1)}h_{(l+1)(m+1)} - h_{1(l+1)}h_{(m+1)(m+1)}}{h_{(l+1)(l+1)}h_{(m+1)(m+1)} - a^2_{(l+1)(m+1)}}, \tag{71}$$

$$B\left[U_{lm}^{(m)}\right] \equiv \left.\frac{\partial p^{(m)}}{\partial p_c}\right|_{\delta\zeta^{(l)}=\delta\zeta^{(m)}=0=\delta p^{(j)}} = \frac{h_{1(l+1)}h_{(l+1)(m+1)} - h_{1(m+1)}h_{(l+1)(l+1)}}{h_{(l+1)(l+1)}h_{(m+1)(m+1)} - a^2_{(l+1)(m+1)}}, \tag{72}$$

and the common effective undrained moduli for the l, m-type pore structure phases are determined by

$$\begin{aligned} \frac{1}{K\left[U^{(l,m)}\right]} &\equiv -\left.\frac{\partial e}{\partial p_c}\right|_{\delta\zeta^{(l)}=\delta\zeta^{(m)}=0=\delta p^{(j)}} \\ &= h_{11} + h_{1(l+1)}B\left[U_{lm}^{(l)}\right] + h_{1(m+1)}B\left[U_{lm}^{(m)}\right], \quad l, m = 1, 2, ..., n(l < m). \end{aligned} \tag{73}$$

Here also, if the cross-coupling coefficients $h_{(i+1)(j+1)} = 0$, then the joint pore pressure buildup coefficients defined in equations (71), (72) are the same with generalized Skempton-like coefficients defined in equation (55).

Fluid injection test, Long time

In this test the change of pore pressure in all types of pore structure are same and there is no change in confining pressure, so we have

$$\delta p^{(1)} = \delta p^{(2)} = ... = \delta p^{(n)} = \delta p,$$
$$\delta\zeta = \sum_{j=1}^{n} \delta\zeta^{(j)},$$

and the storage coefficient can be given by

$$S \equiv \left.\frac{\partial\zeta}{\partial p}\right|_{\delta p_c=0} = \sum_{i,j=1}^{n} h_{(i+1)(j+1)}. \tag{74}$$

Generalized set of Biot-Willis parameters

As per the analogy of single and double porosity theory the Biot-Willis parameters are generalized for multi-porosity theory. From equation (60) using the equation (62), we have

$$\sum_{j=1}^{n} h_{1(j+1)} = -\frac{\alpha}{K}, \tag{75}$$

where, $\alpha = \frac{1}{B}\left(1 - \frac{K}{K_u}\right)$. Again, using equation (62) from equation (67), we have

$$h_{1(j+1)} = -\frac{\bar{\alpha}^{(j)}}{K}, \quad j = 1, 2, ..., n, \tag{76}$$

where, $\bar{\alpha}^{(j)} = \frac{1}{B\left[U^{(j)}\right]}\left(1 - \frac{K}{K\left[U^{(j)}\right]}\right)$ is the generalized Biot-Willis parameter for the j-type pore structure. Thus, from equation (75) using equation (76) we have the relation

$$\alpha = \sum_{j=1}^{n} \bar{\alpha}^{(j)}. \tag{77}$$

Using the equation (76) from equation (66), we have the diagonal terms are

$$h_{(j+1)(j+1)} = \frac{\bar{\alpha}^{(j)}}{K\,B\left[U^{(j)}\right]}, \quad j = 1, 2, ..., n. \tag{78}$$

With the help of equations (76-78) from equation (59), we have

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} h_{(i+1)(j+1)} = \frac{1}{2K}\left\{\frac{\alpha}{B} - \sum_{i=1}^{n}\frac{\bar{\alpha}^{(i)}}{B\left[U^{(i)}\right]}\right\}. \tag{79}$$

From equations (71), (72) with the help of equations (76), (78), the remaining off-diagonal terms (for $n > 2$) are given by

$$h_{(i+1)(j+1)} = \frac{\bar{\alpha}^{(i)}\bar{\alpha}^{(j)}}{KB\left[U^{(i)}\right]B\left[U^{(j)}\right]}\,\frac{B\left[U^{(i)}\right]B\left[U^{(j)}_{ij}\right] - B\left[U^{(j)}\right]B\left[U^{(i)}_{ij}\right]}{\bar{\alpha}^{(j)}B\left[U^{(j)}_{ij}\right] - \bar{\alpha}^{(i)}B\left[U^{(i)}_{ij}\right]}, \quad i, j = 1, 2, ..., n(i < j), \tag{80}$$

and the common effective undrained modulus (cf. eq. (73)) can be rewritten in terms of generalized Biot-Willis parameters and undrained modulus as

$$\frac{1}{K\left[U^{(l,m)}\right]} = \frac{1}{K}\left(1 - \bar{\alpha}^{(l)}B\left[U^{(l)}_{lm}\right] - \bar{\alpha}^{(m)}B\left[U^{(m)}_{lm}\right]\right), \quad l, m = 1, 2, ..., n(l < m). \tag{81}$$

Now, the constitutive equation for isotropic linear elastic multi-porous medium in terms of compliance matrix can be written as

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta^{(1)} \\ -\zeta^{(2)} \\ \vdots \\ -\zeta^{(n)} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & -\beta^{(1)} & -\beta^{(2)} & \cdots & -\beta^{(n)} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & -\beta^{(1)} & -\beta^{(2)} & \cdots & -\beta^{(n)} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & -\beta^{(1)} & -\beta^{(2)} & \cdots & -\beta^{(n)} & 0 & 0 & 0 \\ -\beta^{(1)} & -\beta^{(1)} & -\beta^{(1)} & h_{22} & h_{23} & \cdots & h_{2(n+1)} & 0 & 0 & 0 \\ -\beta^{(2)} & -\beta^{(2)} & -\beta^{(2)} & h_{32} & h_{33} & \cdots & h_{3(n+1)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\beta^{(n)} & -\beta^{(n)} & -\beta^{(n)} & h_{(n+1)2} & h_{(n+1)3} & \cdots & h_{(n+1)(n+1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2\mu} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p^{(1)} \\ -p^{(2)} \\ \vdots \\ -p^{(n)} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}, \tag{82}$$

where, the coefficients S_{11}, S_{12} are the usual drained elastic compliances written in terms of Lame's constants λ, μ and the poroelastic expansion coefficients $\beta^{(j)}$'s are written in terms of generalized Biot-Willis parameters as

$$S_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}, \quad S_{12} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}, \quad \text{and} \quad \beta^{(j)} = \frac{\bar{\alpha}^{(j)}}{3K}, \quad j = 1, 2, ..., n. \tag{83}$$

7 Compressional and rotational waves

Let us consider the Cartesian coordinate system (x_1, x_2, x_3) and $U_i^{(0)}$, $U_i^{(k)}$ are the solid, fluid particle displacements in the type-k pore $(k = 1, 2, ..., n)$ along the x_i $(i = 1, 2, 3)$ direction, respectively. Let us define the strain components and volumetric strains in k-th phases, respectively, as

$$\epsilon_{ij}^{(k)} = \frac{1}{2}\left(\frac{\partial U_i^{(k)}}{\partial x_j} + \frac{\partial U_j^{(k)}}{\partial x_i}\right), \quad i, j = 1, 2, 3, \tag{84}$$

$$\bar{\epsilon}^{(k)} = \epsilon_{11}^{(k)} + \epsilon_{22}^{(k)} + \epsilon_{33}^{(k)}, \quad k = 0, 1, 2, ..., n. \tag{85}$$

Then, the constitutive equation in terms of stiffness matrix for the isotropic multi-porous medium can be written from equation (82) in the form (Lu and Shiou [75])

$$\begin{pmatrix}
C_{11} & C_{12} & C_{12} & -\gamma^{(1)} & -\gamma^{(2)} & \cdots & -\gamma^{(n)} & 0 & 0 & 0\\
C_{12} & C_{11} & C_{12} & -\gamma^{(1)} & -\gamma^{(2)} & \cdots & -\gamma^{(n)} & 0 & 0 & 0\\
C_{12} & C_{12} & C_{11} & -\gamma^{(1)} & -\gamma^{(2)} & \cdots & -\gamma^{(n)} & 0 & 0 & 0\\
-\gamma^{(1)} & -\gamma^{(1)} & -\gamma^{(1)} & C_{22} & C_{23} & \cdots & C_{2(n+1)} & 0 & 0 & 0\\
-\gamma^{(2)} & -\gamma^{(2)} & -\gamma^{(2)} & C_{32} & C_{33} & \cdots & C_{3(n+1)} & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
-\gamma^{(n)} & -\gamma^{(n)} & -\gamma^{(n)} & C_{(n+1)2} & C_{(n+1)3} & \cdots & C_{(n+1)(n+1)} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{11}-C_{12} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & C_{11}-C_{12} & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & C_{11}-C_{12}
\end{pmatrix}
\times \begin{pmatrix} \epsilon_{11}^{(0)} \\ \epsilon_{22}^{(0)} \\ \epsilon_{33}^{(0)} \\ -\zeta^{(1)} \\ -\zeta^{(2)} \\ \vdots \\ -\zeta^{(n)} \\ \epsilon_{23}^{(0)} \\ \epsilon_{13}^{(0)} \\ \epsilon_{12}^{(0)} \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p^{(1)} \\ -p^{(2)} \\ \vdots \\ -p^{(n)} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}. \tag{86}$$

Thus, the constitutive equation (86) for isotropic multi-porous medium can be rewritten in simple form as

$$\left.\begin{array}{l}
\sigma_{ii} = (C_{11} - C_{12})\,\epsilon_{ii}^{(0)} + C_{12}\bar{\epsilon}^{(0)} + \sum_{j=1}^{n}\gamma^{(j)}\zeta^{(j)}, \quad i = 1, 2, 3,\\
-p^{(k)} = -\gamma^{(k)}\bar{\epsilon}^{(0)} - \sum_{j=1}^{n} C_{(k+1)(j+1)}\zeta^{(j)}, \quad k = 1, 2, ..., n,\\
\sigma_{ij} = (C_{11} - C_{12})\,\epsilon_{ij}^{(0)}, \quad i \neq j; i, j = 1, 2, 3.
\end{array}\right\} \tag{87}$$

With the help of Einstein's summation of the repeating index, the following two equations can be derived as

$$\sigma_{ij,j} = \frac{C_{11} + C_{12}}{2}\frac{\partial \bar{\epsilon}^{(0)}}{\partial x_i} + \frac{C_{11} - C_{12}}{2}\nabla^2 U_i^{(0)} + \sum_{l=1}^{n}\gamma^{(l)}\frac{\partial \zeta^{(l)}}{\partial x_i}, \quad i = 1, 2, 3, \tag{88}$$

$$-p_{,ii}^{(k)} = -\gamma^{(k)}\nabla^2\bar{\epsilon}^{(0)} + \sum_{j=1}^{n}\nu^{(j)}\phi^{(j)}C_{(k+1)(j+1)}\nabla^2\left(\bar{\epsilon}^{(j)} - \bar{\epsilon}^{(0)}\right), \quad k = 1, 2, ..., n. \tag{89}$$

The above equation can be used to prove the following theorems.

Theorem 1. *Let there be n different types of pore structures with different permeability in a fluid-saturated multi-porous medium. Then there exist $(n+1)$ types of dispersive compressional waves, with one corresponding to the solid phase and the remaining n waves corresponding to fluid phases.*

Proof. Let us consider there are n different types of pore structures with different permeability of a multi-porous medium saturated by a viscous fluid. Thus, differentiating the equations of motion (5),

(6) with respect to x_i $(i = 1, 2, 3)$ and then adding, we get

$$\begin{aligned}
\sigma_{ij,ji} &= \sum_{r=0}^{n} \rho_{0r} \ddot{\bar{\epsilon}}^{(r)} + \sum_{r=1}^{n} b_{0r} \left(\dot{\bar{\epsilon}}^{(0)} - \dot{\bar{\epsilon}}^{(r)} \right), \quad (90) \\
-\bar{p}^{(k)}_{,ii} &= \sum_{r=0}^{n} \rho_{kr} \ddot{\bar{\epsilon}}^{(r)} - \sum_{r=0}^{k-1} b_{rk} \left(\dot{\bar{\epsilon}}^{(r)} - \dot{\bar{\epsilon}}^{(k)} \right) + \sum_{m=k+1}^{n} b_{km} \left(\dot{\bar{\epsilon}}^{(k)} - \dot{\bar{\epsilon}}^{(m)} \right), \quad k = 1, 2, ..., n, \quad (91)
\end{aligned}$$

by considering the Einstein's summation of the repeating index.

Again, differentiating equation (88) w.r.t x_i and then adding for $i = 1, 2, 3$, we have

$$\sigma_{ij,ji} = C_{11} \nabla^2 \bar{\epsilon}^{(0)} - \sum_{l=1}^{n} \gamma^{(l)} \nu^{(l)} \phi^{(l)} \nabla^2 \left(\bar{\epsilon}^{(l)} - \bar{\epsilon}^{(0)} \right). \quad (92)$$

Now, comparing the equations (90) & (92) and (89) & (91), we have

$$C_{11} \nabla^2 \bar{\epsilon}^{(0)} - \sum_{r=1}^{n} \gamma^{(r)} \nu^{(r)} \phi^{(r)} \nabla^2 \left(\bar{\epsilon}^{(r)} - \bar{\epsilon}^{(0)} \right) = \sum_{r=0}^{n} \rho_{0r} \ddot{\bar{\epsilon}}^{(r)} + \sum_{r=1}^{n} b_{0r} \left(\dot{\bar{\epsilon}}^{(0)} - \dot{\bar{\epsilon}}^{(r)} \right), \quad (93)$$

$$\begin{aligned}
&-\gamma^{(k)} \nu^{(k)} \phi^{(k)} \nabla^2 \bar{\epsilon}^{(0)} + \nu^{(k)} \phi^{(k)} \sum_{r=1}^{n} \nu^{(r)} \phi^{(r)} C_{(k+1)(r+1)} \nabla^2 \left(\bar{\epsilon}^{(r)} - \bar{\epsilon}^{(0)} \right) \\
&\quad = \sum_{r=0}^{n} \rho_{kr} \ddot{\bar{\epsilon}}^{(r)} - \sum_{r=0}^{k-1} b_{rk} \left(\dot{\bar{\epsilon}}^{(r)} - \dot{\bar{\epsilon}}^{(k)} \right) + \sum_{m=k+1}^{n} b_{km} \left(\dot{\bar{\epsilon}}^{(k)} - \dot{\bar{\epsilon}}^{(m)} \right), \quad k = 1, 2, ..., n. \quad (94)
\end{aligned}$$

Equations (93) & (94) are the $(n + 1)$ coupled equation of dilatations $\bar{\epsilon}^{(0)}$, $\bar{\epsilon}^{(1)}$, $\bar{\epsilon}^{(2)}$, ..., $\bar{\epsilon}^{(n)}$. These equations represent the equation of motion for $(n + 1)$ compressional waves in multi-porous medium with equation (93) corresponding to solid phase and equation (94) corresponding to fluid phases.

Let us consider the solution of the coupled equations (93), (94) is in the form

$$\bar{\epsilon}^{(l)} = C_l \, e^{i(\beta x_1 - \omega t)}, \quad l = 0, 1, 2, ..., n. \quad (95)$$

Here, the wavenumber β is a complex quantity defined by $\beta = \beta_1 (1 + i\theta)$, θ is the attenuation coefficient and the phase velocity of the compressional wave is given by $V_d = \frac{\omega}{\beta_1}$. Then, substituting equation (95) in equation (93) and (94), we have

$$\left. \begin{aligned}
&F_0^{(0)} C_0 + \sum_{j=1}^{n} F_j^{(0)} C_j = 0, \\
&F_0^{(l)} C_0 + \sum_{j=1}^{n} F_j^{(l)} C_j = 0, \quad l = 1, 2, ..., n,
\end{aligned} \right\} \quad (96)$$

where,

$$\left. \begin{aligned}
F_0^{(0)} &= C_{11} + \textstyle\sum_{j=1}^{n} \gamma^{(j)} \nu^{(j)} \phi^{(j)} - \dfrac{V_d^2}{(1 + i\theta)^2} q_{00}, \\
F_j^{(0)} &= -\gamma^{(j)} \nu^{(j)} \phi^{(j)} - \dfrac{V_d^2}{(1 + i\theta)^2} q_{0j}, \\
F_0^{(l)} &= -\nu^{(l)} \phi^{(l)} \left(\gamma^{(l)} + \textstyle\sum_{j=1}^{n} \nu^{(j)} \phi^{(j)} C_{(l+1)(j+1)} \right) - \dfrac{V_d^2}{(1 + i\theta)^2} q_{0l}, \\
F_j^{(l)} &= \nu^{(l)} \phi^{(l)} \nu^{(j)} \phi^{(j)} C_{(l+1)(j+1)} - \dfrac{V_d^2}{(1 + i\theta)^2} q_{lj}, \quad j, l = 1, 2, ..., n,
\end{aligned} \right\} \quad (97)$$

$$q_{jk} = \begin{cases} \rho_{jk} - \frac{i}{\omega} b_{jk}, & j \neq k, \\ \rho_{jj} + \frac{i}{\omega} \sum_{\substack{l=0 \\ l \neq j}}^{n} b_{jl}, & j = k, \end{cases} \quad j, k = 0, 1, 2, ..., n. \quad (98)$$

To get the non-trivial solution of the system of equation (96) in C_j $(j = 0, 1, 2, ..., n)$, the determinant of the coefficient matrix must be zero, i.e.,

$$\det\left(\mathbf{N}\mathbf{A_d}\mathbf{N} - \frac{V_d^2}{(1+i\theta)^2}\mathbf{Q_d}\right) = 0, \tag{99}$$

where, the matrices $\mathbf{N}$, $\mathbf{Q_d}$ and $\mathbf{A_d}$ of order $(n+1) \times (n+1)$ are defined by

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \nu^{(1)}\phi^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \nu^{(2)}\phi^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu^{(n)}\phi^{(n)} \end{pmatrix}, \quad \mathbf{Q_d} = \begin{pmatrix} q_{00} & q_{01} & q_{02} & \cdots & q_{0n} \\ q_{01} & q_{11} & q_{12} & \cdots & q_{1n} \\ q_{02} & q_{12} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{0n} & q_{1n} & q_{2n} & \cdots & q_{nn} \end{pmatrix}, \tag{100}$$

$$\mathbf{A_d} = \begin{pmatrix} C_{11} + \sum_{j=1}^{n}\gamma^{(j)}\nu^{(j)}\phi^{(j)} & -\gamma^{(1)} & -\gamma^{(2)} & \cdots & -\gamma^{(n)} \\ -\left(\gamma^{(1)} + \sum_{j=1}^{n}\nu^{(j)}\phi^{(j)}C_{2(j+1)}\right) & C_{22} & C_{23} & \cdots & C_{2(n+1)} \\ -\left(\gamma^{(2)} + \sum_{j=1}^{n}\nu^{(j)}\phi^{(j)}C_{3(j+1)}\right) & C_{32} & C_{33} & \cdots & C_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\left(\gamma^{(n)} + \sum_{j=1}^{n}\nu^{(j)}\phi^{(j)}C_{(n+1)(j+1)}\right) & C_{(n+1)2} & C_{(n+1)3} & \cdots & C_{(n+1)(n+1)} \end{pmatrix}. \tag{101}$$

Equation (99) represents the relation between velocity (V_d), frequency (ω) and different parameters in the multi-porous medium, which is nothing but the dispersion relation. This is a $(n+1)$ degree polynomial of V_d^2, and the roots of this equation represent phase velocity corresponding to $(n+1)$ compressional waves for each value of frequency. This proves the theorem. □

Corollary 1. *If the multi-porous medium is dissipationless saturated by non-viscous fluid, then there exist $(n+1)$ types of non-dispersive compressional waves, with one corresponding to the solid phase and the remaining n waves corresponding to fluid phases.*

Proof. When the multi-porous medium is saturated with a non-viscous fluid and is dissipationless, the dissipation coefficients vanish, i.e., $b_{ij} = 0$, $i, j = 0, 1, 2, ..., n (i \neq j)$. Consequently, the matrix $\mathbf{Q_d}$ is transformed into a frequency-independent matrix represented as

$$\mathbf{Q_d'} = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots & \rho_{0n} \\ \rho_{01} & \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{02} & \rho_{12} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{0n} & \rho_{1n} & \rho_{2n} & \cdots & \rho_{nn} \end{pmatrix}. \tag{102}$$

As a result, the dispersion equation (99) transforms into

$$\det\left(\mathbf{N}\mathbf{A_d}\mathbf{N} - \frac{V_d^2}{(1+i\theta)^2}\mathbf{Q_d'}\right) = 0. \tag{103}$$

The equation (103) is polynomial of degree $(n+1)$ of V_d^2 and gives the velocity of $(n+1)$ types of compressional waves. It can be seen that these velocities are independent of the frequency, which implies that all compressional waves are non-dispersive in the dissipationless multi-porous medium. This completes the proof. □

From equations (93), (94), it can be noted that because of the coupling between the waves, the $(n+1)$ types of compressional waves propagate jointly in the multi-porous medium. Also, if there is no relative motion between solid and fluids phases, then the dissipations due to the fluids friction will automatically disappear. This creates a compatibility condition for the existence of an elastic wave, which is given by

$$\frac{C_{11}\rho}{R\sum_{l=0}^{n}\rho_{0l}} = \frac{-\gamma^{(1)}\nu^{(1)}\phi^{(1)}\rho}{R\sum_{l=0}^{n}\rho_{1l}} = \frac{-\gamma^{(2)}\nu^{(2)}\phi^{(2)}\rho}{R\sum_{l=0}^{n}\rho_{2l}} = \cdots = \frac{-\gamma^{(n)}\nu^{(n)}\phi^{(n)}\rho}{R\sum_{l=0}^{n}\rho_{nl}} = 1, \tag{104}$$

and the velocity of this single elastic wave is $V_c = \left(\frac{R}{\rho}\right)^{\frac{1}{2}}$, where

$$R = C_{11} - \sum_{k=1}^{n} \gamma^{(k)} \nu^{(k)} \phi^{(k)}, \quad \rho = \sum_{i,j=0}^{n} \rho_{ij}. \tag{105}$$

Theorem 2. *Let there be n different types of pore structures with different permeability in a multi-porous medium saturated by a viscous fluid. Then there exists only one dispersive rotational wave regardless of the number of pore types n.*

Proof. Let us define the components of rotation vector about the x_1, x_2, x_3 axis for the k-th phase, respectively as

$$\Omega_1^{(k)} = \frac{1}{2}\left(\frac{\partial U_3^{(k)}}{\partial x_2} - \frac{\partial U_2^{(k)}}{\partial x_3}\right), \Omega_2^{(k)} = \frac{1}{2}\left(\frac{\partial U_1^{(k)}}{\partial x_3} - \frac{\partial U_3^{(k)}}{\partial x_1}\right), \Omega_3^{(k)} = \frac{1}{2}\left(\frac{\partial U_2^{(k)}}{\partial x_1} - \frac{\partial U_1^{(k)}}{\partial x_2}\right), k = 0, 1, ..., n. \tag{106}$$

From equation of motion (5) and constitutive equation (88), with the help of equation (41), we have

$$\frac{C_{11}+C_{12}}{2}\frac{\partial \bar{\epsilon}^{(0)}}{\partial x_i} + \frac{C_{11}-C_{12}}{2}\nabla^2 U_i^{(0)} - \sum_{l=1}^{n} \gamma^{(l)} \nu^{(l)} \phi^{(l)} \frac{\partial}{\partial x_i}\left(\bar{\epsilon}^{(l)} - \bar{\epsilon}^{(0)}\right)$$
$$= \sum_{l=0}^{n} \rho_{0l} \ddot{U}_i^{(l)} + \sum_{l=1}^{n} b_{0l}\left(\dot{U}_i^{(0)} - \dot{U}_i^{(l)}\right), \quad i = 1, 2, 3. \tag{107}$$

In equation (107), for $i = 3$ differentiating w.r.t. x_2, for $i = 2$ differentiating w.r.t. x_3 and then subtracting each other, we have

$$\frac{C_{11}-C_{12}}{2}\nabla^2 \Omega_1^{(0)} = \sum_{l=0}^{n} \rho_{0l} \ddot{\Omega}_1^{(l)} + \sum_{l=1}^{n} b_{0l}\left(\dot{\Omega}_1^{(0)} - \dot{\Omega}_1^{(l)}\right). \tag{108}$$

Similarly from (107), following two equations can be derived as

$$\frac{C_{11}-C_{12}}{2}\nabla^2 \Omega_2^{(0)} = \sum_{l=0}^{n} \rho_{0l} \ddot{\Omega}_2^{(l)} + \sum_{l=1}^{n} b_{0l}\left(\dot{\Omega}_2^{(0)} - \dot{\Omega}_2^{(l)}\right), \tag{109}$$

$$\frac{C_{11}-C_{12}}{2}\nabla^2 \Omega_3^{(0)} = \sum_{l=0}^{n} \rho_{0l} \ddot{\Omega}_3^{(l)} + \sum_{l=1}^{n} b_{0l}\left(\dot{\Omega}_3^{(0)} - \dot{\Omega}_3^{(l)}\right). \tag{110}$$

Adding the above three equations (108)-(110), we get

$$\frac{C_{11}-C_{12}}{2}\nabla^2 \mathbf{\Omega}^{(0)} = \sum_{l=0}^{n} \rho_{0l} \ddot{\mathbf{\Omega}}^{(l)} + \sum_{l=1}^{n} b_{0l}\left(\dot{\mathbf{\Omega}}^{(0)} - \dot{\mathbf{\Omega}}^{(l)}\right). \tag{111}$$

By performing the same operations on equations (6), (89), we get

$$\sum_{l=0}^{n} \rho_{kl} \ddot{\mathbf{\Omega}}^{(l)} - \sum_{l=0}^{k-1} b_{lk}\left(\dot{\mathbf{\Omega}}^{(l)} - \dot{\mathbf{\Omega}}^{(k)}\right) + \sum_{l=k+1}^{n} b_{kl}\left(\dot{\mathbf{\Omega}}^{(k)} - \dot{\mathbf{\Omega}}^{(l)}\right) = 0, \quad k = 1, 2, ..., n. \tag{112}$$

Equations (111), (112) govern the propagation of rotational waves in the multi-porous medium.

Now, consider the rotational plane harmonic wave propagating along the x_1 direction, then the solution of the wave propagation (cf. equations (111),(112)) about x_3-axis can be written as

$$\Omega_3^{(l)} = C_l\, e^{i(\beta x_1 - \omega t)}, \quad l = 0, 1, 2, ..., n. \tag{113}$$

From equations (111),(112) using equation (113), we have

$$\left.\begin{aligned} & G_0^{(0)} C_0 + \sum_{j=1}^{n} G_j^{(0)} C_j = 0, \\ & G_0^{(l)} C_0 + \sum_{j=1}^{n} G_j^{(l)} C_j = 0, \quad l = 1, 2, ..., n, \end{aligned}\right\} \tag{114}$$

where,

$$\left.\begin{aligned} &G_0^{(0)} = \tfrac{C_{11}-C_{12}}{2} - \frac{V_r^2}{(1+i\theta)^2} q_{00}, \quad G_j^{(0)} = -\frac{V_r^2}{(1+i\theta)^2} q_{0j}, \\ &G_j^{(l)} = q_{jl}, \quad j,l = 0,1,2,...,n, \end{aligned}\right\} \tag{115}$$

and, $V_r = \frac{\omega}{\beta_1}$ is the phase velocity of the rotational wave propagation. To get the non-trivial solution of the system of equation (114) in C_j $(j = 0,1,2,...,n)$, the determinant of the coefficient matrix must be zero, which gives

$$\frac{C_{11}-C_{12}}{2}\begin{vmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{12} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{vmatrix} = \frac{V_r^2}{(1+i\theta)^2}\begin{vmatrix} q_{00} & q_{01} & q_{02} & \cdots & q_{0n} \\ q_{01} & q_{11} & q_{12} & \cdots & q_{1n} \\ q_{02} & q_{12} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{0n} & q_{1n} & q_{2n} & \cdots & q_{nn} \end{vmatrix} \tag{116}$$

Equation (116) represents the relation between velocity (V_r), frequency (ω) and different parameters in the multi-porous medium, which is nothing but the dispersion relation. This is a one degree polynomial of V_r^2, which implies that there is only one dispersive rotational wave in the multi-porous medium. This proves the theorem. □

Corollary 2. *If the multi-porous medium is dissipationless saturated by non-viscous fluid, then there exist only one rotational wave, which is non-dispersive and corresponding to the solid phase.*

Proof. If the saturated fluid in pore structures is non-viscous (friction-less) and the wave is purely elastic, then the dissipation coefficients vanish, i.e., $b_{ij} = 0$, $i, j = 0,1,2,...,n (i \neq j)$. Similar to the methodology employed in Corollary 1, we can establish the proposed statement. However, in this case, we will adopt a distinct approach to demonstrate the non-dispersive nature of the one rotational wave, which is associated with the solid phase. In this scenario, the rotational equation of motion (112) is transformed into

$$\sum_{l=1}^{n} \rho_{kl} \ddot{\mathbf{\Omega}}^{(l)} = -\rho_{0k} \ddot{\mathbf{\Omega}}^{(0)}, \quad k = 1,2,...,n. \tag{117}$$

From equation (117), we can get the rotation of the fluid phases depending individually on the rotation of solid phase by the relation

$$\mathbf{\Omega}^{(k)} = d_k\, \mathbf{\Omega}^{(0)}, \quad k = 1,2,...,n, \tag{118}$$

where, the constant coefficients d_k's are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = -\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}^{-1} \begin{pmatrix} \rho_{01} \\ \rho_{02} \\ \vdots \\ \rho_{0n} \end{pmatrix}. \tag{119}$$

Using equation (118), eliminating the rotational components corresponding to pore fluid phases from equation (111), we have

$$\frac{C_{11}-C_{12}}{2} \nabla^2 \mathbf{\Omega}^{(0)} = \left(\rho_{00} + \sum_{l=1}^{n} \rho_{0l} d_l\right) \ddot{\mathbf{\Omega}}^{(0)}. \tag{120}$$

Since there is no coupling of rotational waves between solid and fluid phases, the rotational wave in the multi-porous medium is in the solid matrix only. This implies that there is only one rotational wave, which is for solid medium, and the velocity of the corresponding wave is given by

$$v_r = \left[\frac{1}{2} \frac{C_{11}-C_{12}}{\rho_{00} + \sum_{l=1}^{n} \rho_{0l} d_l}\right]^{\frac{1}{2}}. \tag{121}$$

The velocity of this wave is independent of frequency, which implies that the rotational wave is non-dispersive in the multi-porous medium without dissipation. This completes the proof. □

8 Particular cases and validity of the model

In this section, the particular cases and validity of the multi-porous theory is demonstrated by comparing it to existing theories for particular circumstances.

8.1 Multi-porosity theory with no-coupling between fluid-fluid

Equation of motion: If there is no fluid-fluid interaction i.e., $\rho_{ij} = b_{ij} = 0\,(i \neq j;\, i,j = 1,2,...,n)$, then the equation of motion for multi-porosity theory (cf. Eqs. (5), (6)) is reduces to

$$\begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots & \rho_{0n} \\ \rho_{01} & \rho_{11} & 0 & \cdots & 0 \\ \rho_{02} & 0 & \rho_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{0n} & 0 & 0 & \cdots & \rho_{nn} \end{pmatrix} \begin{pmatrix} \ddot{U}_i^{(0)} \\ \ddot{U}_i^{(1)} \\ \ddot{U}_i^{(2)} \\ \vdots \\ \ddot{U}_i^{(n)} \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^n b_{0k} & -b_{01} & -b_{02} & \cdots & -b_{0n} \\ -b_{01} & b_{01} & 0 & \cdots & 0 \\ -b_{02} & 0 & b_{02} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{0n} & 0 & 0 & \cdots & b_{0n} \end{pmatrix} \begin{pmatrix} \dot{U}_i^{(0)} \\ \dot{U}_i^{(1)} \\ \dot{U}_i^{(2)} \\ \vdots \\ \dot{U}_i^{(n)} \end{pmatrix} = \begin{pmatrix} \sigma_{ij,j} \\ -\bar{p}_{,i}^{(1)} \\ -\bar{p}_{,i}^{(2)} \\ \vdots \\ -\bar{p}_{,i}^{(n)} \end{pmatrix}, \quad i = 1,2,3. \tag{122}$$

Mass and dissipation coefficients: The non-zero mass and dissipation coefficients are given as

$$\left.\begin{aligned} &\rho_{00} = (1-\phi)\rho_s + (\tau - 1)\phi\rho_f, \quad \rho_{jj} = \tau^{(j)}\nu^{(j)}\phi^{(j)}\rho_f, \\ &\rho_{0j} = -\left(\tau^{(j)} - 1\right)\nu^{(j)}\phi^{(j)}\rho_f, \\ &b_{0j} = \eta\frac{\left(\nu^{(j)}\phi^{(j)}\right)^2}{L^{(jj)}}, \quad j = 1,2,...,n. \end{aligned}\right\} \tag{123}$$

Elastic coefficients: If there is no coupling between fluid-fluid phases, then $h_{(i+1)(j+1)} = 0$, $(i \neq j;\, i,j = 1,2,...,n)$. Thus, the elastic coefficients of the stiffness matrix defined in (86) are given in the simple form as

$$\left.\begin{aligned} &C_{11} = K_u^* + \tfrac{4}{3}\mu, \quad C_{12} = K_u^* - \tfrac{2}{3}\mu, \quad \gamma^{(j)} = -K_u^* B\left[U^{(j)}\right], \\ &C_{(i+1)(i+1)} = \frac{\left(\sum_{\substack{l=1 \\ l\neq i}}^n B\left[U^{(l)}\right]\bar{\alpha}^{(l)}\right) - 1}{\bar{\alpha}^{(i)}}\gamma^{(i)}, \quad C_{(i+1)(j+1)} = B\left[U^{(i)}\right]B\left[U^{(j)}\right]K_u^*,\ i \neq j, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i,j = 1,2,...,n, \end{aligned}\right\} \tag{124}$$

where,

$$K_u^* = \frac{K}{1 - \sum_{l=1}^n B\left[U^{(l)}\right]\bar{\alpha}^{(l)}}.$$

The equation of motion (122), mass and dissipation coefficients in equation (123) and the elastic coefficients defined in equation (124) are the complete description of multi-porosity theory of elastic wave propagation with negligible fluid-fluid interactions.

Also, in this case the coefficients h_{ij} are given by

$$h_{11} = \frac{1}{K} = \sum_{j=1}^n \frac{\nu^{(j)}}{K\left[U^{(j)}\right]}, \quad h_{1(j+1)} = -\frac{\bar{\alpha}^{(j)}}{K}, \quad h_{(j+1)(j+1)} = \frac{\bar{\alpha}^{(j)}}{KB\left[U^{(j)}\right]}, \quad j = 1,2,...,n, \tag{125}$$

where, all the generalized Biot-Wills parameters are connected by the relation

$$\sum_{j=1}^n \frac{\bar{\alpha}^{(j)}}{B\left[U^{(j)}\right]} = \frac{\alpha}{B}. \tag{126}$$

The obtained results closely align with the findings of Mehrabian and Abousleiman [76], when we consider the total drained bulk modulus is the weighted harmonic mean of constituents' compressibilities and neglecting fluid-fluid interactions, who also discuss the distinctions, particularly detailed on page 2762. However, our work fully aligns for particular cases of double-porosity with the work of Berryman and Wang [17, 21] shown in the subsequent subsection 8.2.

8.2 Double porosity theory

Equation of motion: For $n = 2$, the equation of motion for multi-porosity theory (cf. Eqs. (5), (6)) is reduces to

$$\begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} \ddot{U}_i^{(0)} \\ \ddot{U}_i^{(1)} \\ \ddot{U}_i^{(2)} \end{pmatrix} + \begin{pmatrix} b_{01}+b_{02} & -b_{01} & -b_{02} \\ -b_{01} & b_{01}+b_{12} & -b_{12} \\ -b_{02} & -b_{12} & b_{02}+b_{12} \end{pmatrix} \begin{pmatrix} \dot{U}_i^{(0)} \\ \dot{U}_i^{(1)} \\ \dot{U}_i^{(2)} \end{pmatrix} = \begin{pmatrix} \sigma_{ij,j} \\ -\bar{p}_{,i}^{(1)} \\ -\bar{p}_{,i}^{(2)} \end{pmatrix}, \quad i = 1,2,3 \tag{127}$$

Mass coefficients: The mass coefficients $\rho_{ij}\,(i,j = 0,1,2)$ can be derived from the triple-porosity theory (c.f. Eq. (38) for $\nu^{(3)} = 0$, $\tau^{(1,3)} \equiv \tau^{(1)}$, $\tau^{(2,3)} \equiv \tau^{(2)}$, $\tau^{(1,2)} = \tau$) as

$$\left.\begin{aligned} &\rho_{00} = (1-\phi)\rho_s + (\tau-1)\phi\rho_f, \quad \rho_{kk} = \tau^{(k)}\nu^{(k)}\phi^{(k)}\rho_f, \\ &2\rho_{0k} = \left(\tau^{(l)}-1\right)\nu^{(l)}\phi^{(l)}\rho_f - \left(\tau^{(k)}-1\right)\nu^{(k)}\phi^{(k)}\rho_f - (\tau-1)\phi\rho_f, \quad l \neq k, \\ &2\rho_{12} = \tau\phi\rho_f - \tau^{(1)}\nu^{(1)}\phi^{(1)}\rho_f - \tau^{(2)}\nu^{(2)}\phi^{(2)}\rho_f, \quad l,k = 1,2. \end{aligned}\right\} \tag{128}$$

Dissipation coefficients: The dissipation coefficients $b_{ij}(i,j = 0,1,2; i \neq j)$ can be derived from the multi-porosity theory (cf. Eqs. (46) and (47)) for $n = 2$ as

$$\left.\begin{aligned} b_{01} &= \frac{\eta\nu^{(1)}\phi^{(1)}}{L^{(22)}L^{(11)} - L^{(21)}L^{(12)}}\left[\nu^{(1)}\phi^{(1)}L^{(22)} - \nu^{(2)}\phi^{(2)}L^{(12)}\right], \\ b_{02} &= \frac{\eta\nu^{(2)}\phi^{(2)}}{L^{(22)}L^{(11)} - L^{(21)}L^{(12)}}\left[\nu^{(2)}\phi^{(2)}L^{(11)} - \nu^{(1)}\phi^{(1)}L^{(21)}\right], \\ b_{12} &= \frac{\eta\nu^{(1)}\nu^{(2)}\phi^{(1)}\phi^{(2)}}{L^{(22)}L^{(11)} - L^{(21)}L^{(12)}}L^{(12)}. \end{aligned}\right\} \tag{129}$$

Elastic coefficients: For the double-porosity model the six independent coefficients h_{ij} are given by (cf. Eqs. (62), (76), (78), (79))

$$h_{11} = \frac{1}{K},\ h_{1(j+1)} = -\frac{\bar{\alpha}^{(j)}}{K},\ h_{(j+1)(j+1)} = \frac{\bar{\alpha}^{(j)}}{K\,B\left[U^{(j)}\right]},\ h_{23} = \frac{1}{2K}\left\{\frac{\alpha}{B} - \sum_{i=1}^{n}\frac{\bar{\alpha}^{(i)}}{B\left[U^{(i)}\right]}\right\},\ j = 1,2 \tag{130}$$

The equation of motion (127), mass and dissipation coefficients in equations (128), (129) and the constitutive equation derived using the independent coefficients h_{ij} in equation (130) are in good agreement with the results of double porosity theory described by Berryman and Wang [17, 21].

8.3 Single porosity theory

Equation of motion: For $n = 1$, the equation of motion for multi-porosity theory (cf. Eqs. (5), (6)) is reduces to

$$\left.\begin{aligned} \sigma_{ij,j} &= \rho_{00}\ddot{U}_i^{(0)} + \rho_{01}\ddot{U}_i^{(1)} + b_{01}\left(\dot{U}_i^{(0)} - \dot{U}_i^{(1)}\right), \\ -\bar{p}_{,i}^{(1)} &= \rho_{01}\ddot{U}_i^{(0)} + \rho_{11}\ddot{U}_i^{(1)} - b_{01}\left(\dot{U}_i^{(0)} - \dot{U}_i^{(1)}\right), \end{aligned}\right\} \quad i = 1,2,3. \tag{131}$$

Mass coefficients: The mass coefficients $\rho_{ij}\,(i,j = 0,1)$ can be derived from the triple-porosity theory (c.f. Eq. (38) for $\nu^{(1)} = 1$, $\nu^{(2)} = 0$, $\nu^{(3)} = 0$, $\tau^{(1)} = \tau$, $\phi^{(1)} = \phi$) as

$$\rho_{00} = (1-\phi)\rho_s + (\tau-1)\phi\rho_f, \quad \rho_{01} = -(\tau-1)\phi\rho_f, \quad \rho_{11} = \tau\phi\rho_f. \tag{132}$$

Dissipation coefficient: The dissipation coefficient b_{01} can be derived from the multi-porosity theory (cf. Eq. (47)) for $n = 1$ as

$$b_{01} = \eta\frac{\phi^2}{L^{(11)}}. \tag{133}$$

Elastic coefficients: For single-porosity model the three independent coefficients h_{ij} are given by (cf. Eqs. (62), (75), (78))

$$h_{11} = \frac{1}{K},\ h_{12} = -\frac{\alpha}{K},\ h_{22} = \frac{\alpha}{B\,K}. \tag{134}$$

The equation of motion (131), mass and dissipation coefficients in equations (132), (133) and the constitutive equation derived using the independent coefficients h_{ij} in equation (134) are in good agreement with the results of single porosity theory described by Biot [1], Berryman and Wang [17].

9 Example and numerical discussions

The objective of this section is to elucidate the velocity profiles arising from compressional and rotational waves within a triple-porosity medium incorporating dissipation. Subsequently, our attention is directed towards the dispersion equation governing the propagation of compressional and rotational elastic waves, accounting for dissipation arising from solid-fluid interactions. Considering the very small or negligible fluid-fluid interaction, also observed by Berryman and Wang [17], the terms h_{23}, h_{24}, and h_{34} of this triple-porosity model are neglected.

In the context of a triple porosity medium ($n = 3$), the dispersion equation (99) governing compressional waves can be expressed in the form of a polynomial equation of V_d^2 as

$$V_d^8\mathbf{Q} - V_d^6(1+i\theta)^2\sum_{i=1}^{4}\mathbf{Q}_i + V_d^4(1+i\theta)^4\sum_{\substack{i,j=1\\ i<j}}^{4}\mathbf{Q}_{ij} - V_d^2(1+i\theta)^6\sum_{\substack{i,j,k=1\\ i<j<k}}^{4}\mathbf{Q}_{ijk} + (1+i\theta)^8\mathbf{P} = 0, \quad (135)$$

where, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{Q}_i$, $\mathbf{Q}_{ij}$, $\mathbf{Q}_{ijk}$ $(i, j, k = 1, 2, 3, 4)$, are the 4×4 determinants defined by

$$\mathbf{P} = \begin{vmatrix} C_{11} + \sum_{j=1}^{3}\gamma^{(j)}\nu^{(j)}\phi^{(j)} & -\gamma^{(1)}\nu^{(1)}\phi^{(1)} & -\gamma^{(2)}\nu^{(2)}\phi^{(2)} & -\gamma^{(3)}\nu^{(3)}\phi^{(3)} \\ -\nu^{(1)}\phi^{(1)}\left(\gamma^{(1)} + \sum_{j=1}^{3}\nu^{(j)}\phi^{(j)}C_{2(j+1)}\right) & {\nu^{(1)}}^2{\phi^{(1)}}^2C_{22} & \nu^{(1)}\nu^{(2)}\phi^{(1)}\phi^{(2)}C_{23} & \nu^{(1)}\nu^{(3)}\phi^{(1)}\phi^{(3)}C_{24} \\ -\nu^{(2)}\phi^{(2)}\left(\gamma^{(2)} + \sum_{j=1}^{3}\nu^{(j)}\phi^{(j)}C_{3(j+1)}\right) & \nu^{(1)}\nu^{(2)}\phi^{(1)}\phi^{(2)}C_{32} & {\nu^{(2)}}^2{\phi^{(2)}}^2C_{33} & \nu^{(2)}\nu^{(3)}\phi^{(2)}\phi^{(3)}C_{34} \\ -\nu^{(3)}\phi^{(3)}\left(\gamma^{(3)} + \sum_{j=1}^{3}\nu^{(j)}\phi^{(j)}C_{4(j+1)}\right) & \nu^{(1)}\nu^{(3)}\phi^{(1)}\phi^{(3)}C_{42} & \nu^{(2)}\nu^{(3)}\phi^{(2)}\phi^{(3)}C_{43} & {\nu^{(3)}}^2{\phi^{(3)}}^2C_{44} \end{vmatrix}, \quad (136)$$

$$\mathbf{Q} = \begin{vmatrix} \rho_{00} + \frac{i}{\omega}\sum_{j=1}^{3}b_{0j} & \rho_{01} - \frac{i}{\omega}b_{01} & \rho_{02} - \frac{i}{\omega}b_{02} & \rho_{03} - \frac{i}{\omega}b_{03} \\ \rho_{01} - \frac{i}{\omega}b_{01} & \rho_{11} + \frac{i}{\omega}b_{01} & 0 & 0 \\ \rho_{02} - \frac{i}{\omega}b_{02} & 0 & \rho_{22} + \frac{i}{\omega}b_{02} & 0 \\ \rho_{03} - \frac{i}{\omega}b_{03} & 0 & 0 & \rho_{33} + \frac{i}{\omega}b_{03} \end{vmatrix}, \quad (137)$$

and,

$$\left.\begin{aligned} &\mathbf{Q}_i = i\text{-th column of } \mathbf{Q} \text{ replaced by } i\text{-th column of } \mathbf{P}, \\ &\mathbf{Q}_{ij} = i,j\text{-th columns of } \mathbf{Q} \text{ replaced by } i,j\text{-th columns of } \mathbf{P}, \\ &\mathbf{Q}_{ijk} = i,j,k\text{-th columns of } \mathbf{Q} \text{ replaced by } i,j,k\text{-th columns of } \mathbf{P}. \end{aligned}\right\} \quad (138)$$

Equation (135) constitutes a fourth-degree polynomial involving V_d^2 and θ^2. The four solutions of this equation correspond to phase velocity (V_d) and attenuation coefficient (θ), characterizing four distinct compressional waves for each value of frequency (ω) associated with a solid and three fluid phases.

Also, in the context of a triple porosity medium ($n = 3$), the dispersion equation (116) governing rotational waves can be expressed as

$$\frac{V_r^2}{(1+i\theta)^2} = \frac{(C_{11} - C_{12})/2}{\left(\rho_{00} + \frac{i}{\omega}\sum_{j=1}^{3}b_{0j}\right) - \sum_{j=1}^{3}\frac{\left(\rho_{0j} - \frac{i}{\omega}b_{0j}\right)^2}{\rho_{jj} + \frac{i}{\omega}b_{0j}}}. \quad (139)$$

Equation (139) is a linear polynomial involving V_r^2 and θ^2, yielding a single solution that provides the phase velocity (V_r) and attenuation coefficient (θ) of rotational waves for each value of frequency (ω) associated with the solid phase.

For the purpose of numerical simulation, a set of data specific to the triple-porosity medium (cf. Berryman and Wang [17], Lewallen and Wang [77]) for water-saturated Berea sandstone is given in Table 1. All the figures are plotted for dimensionless phase velocity or attenuation coefficient against non-dimensional angular frequency. The characteristic velocities V_c and v_r (cf. Eqs. (105), (121)) are used to non-dimensionalize the phase velocity of compressional (V_d) and rotational (V_r) waves, respectively. Also, the characteristic angular frequency ω_c defined by $\omega_c = \frac{\sum_{j=1}^{3} b_{0j}}{\rho_f \phi}$ is used to non-dimensionalize the angular frequency ω.

Table 1: Numerical values of the material parameters

Parameter	Numerical value	Parameter	Numerical value
λ	0.481 GPa	η	$0.001\,\mathrm{Pa\cdot s}$
μ	0.857 GPa	K	1.052 GPa
$B\left[U^{(1)}\right]$	0.847	$K\left[U^{(1)}\right]$	1.151 GPa
$B\left[U^{(2)}\right]$	0.922	$K\left[U^{(2)}\right]$	1.058 GPa
$B\left[U^{(3)}\right]$	0.998	$K\left[U^{(3)}\right]$	7.920 GPa
$\nu^{(1)}$	0.80	$\phi^{(1)}$	0.064
$\nu^{(2)}$	0.1936	$\phi^{(2)}$	0.015
$\nu^{(3)}$	0.0064	$\phi^{(3)}$	1.000
ρ_s	$3000\,\mathrm{Kg/m^3}$	ρ_f	$1000\,\mathrm{Kg/m^3}$
$L^{(11)}$	$10^{-6}\,\mathrm{m^2}$	$L^{(33)}$	$10^{-12}\,\mathrm{m^2}$
$L^{(22)}$	$10^{-9}\,\mathrm{m^2}$		

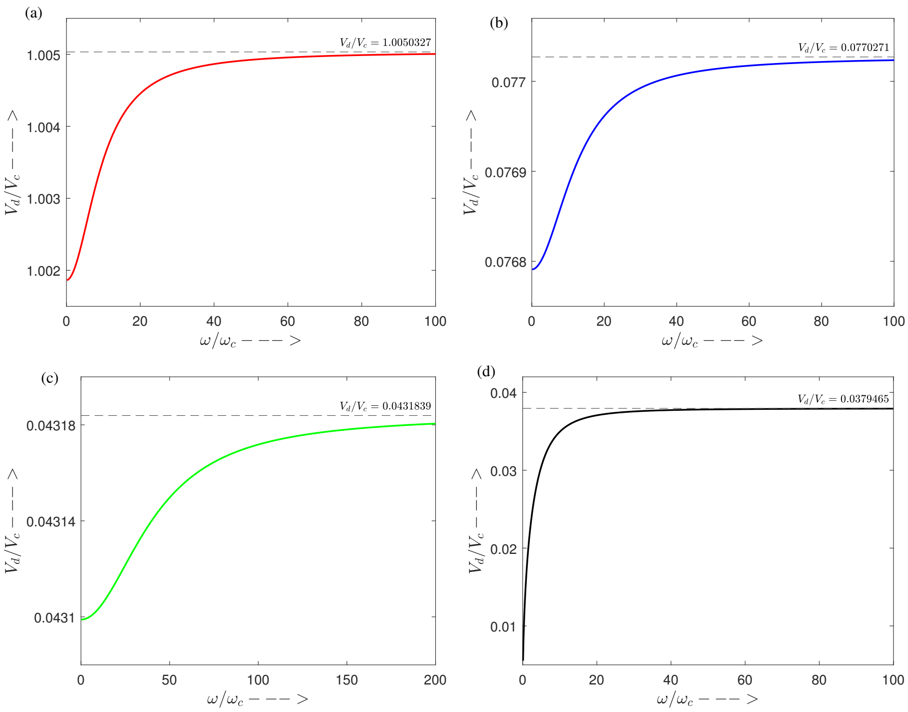

Figure 2: Variation of dimensionless phase velocity (V_d/V_c) against non-dimensional angular frequency (ω/ω_c) of four compressional waves **(a)** P1, **(b)** P2, **(c)** P3, and **(d)** P4

In Figure 2, an attempt has been made to show the variation of dimensionless phase velocity (V_d/V_c) of compressional waves with non-dimensional angular frequency (ω/ω_c) in a triple-porosity medium. In this context, the medium features four distinct compressional waves denoted as P1, P2, P3, and P4, according to their magnitude of the phase velocity. The higher magnitude wave P1 corresponds to the solid phase, while the remaining waves correspond to the three fluid phases. From this figure, it becomes evident that with increasing frequency, the phase velocity of the four compressional waves rises, but up to a finite limit. The limiting phase velocities of the four compressional waves are $V_d/V_c = 1.0050327$, 0.0770271, 0.0431839, and 0.0379465, respectively. These constant limiting velocities are the velocity of four non-dispersive compressional waves without dissipation, i.e., $b_{0j} = 0\,(j = 1, 2, 3)$. In other words, for a high-frequency compressional wave, the dispersive wave behaviour transforms into a non-dispersive wave as the frequency increases.

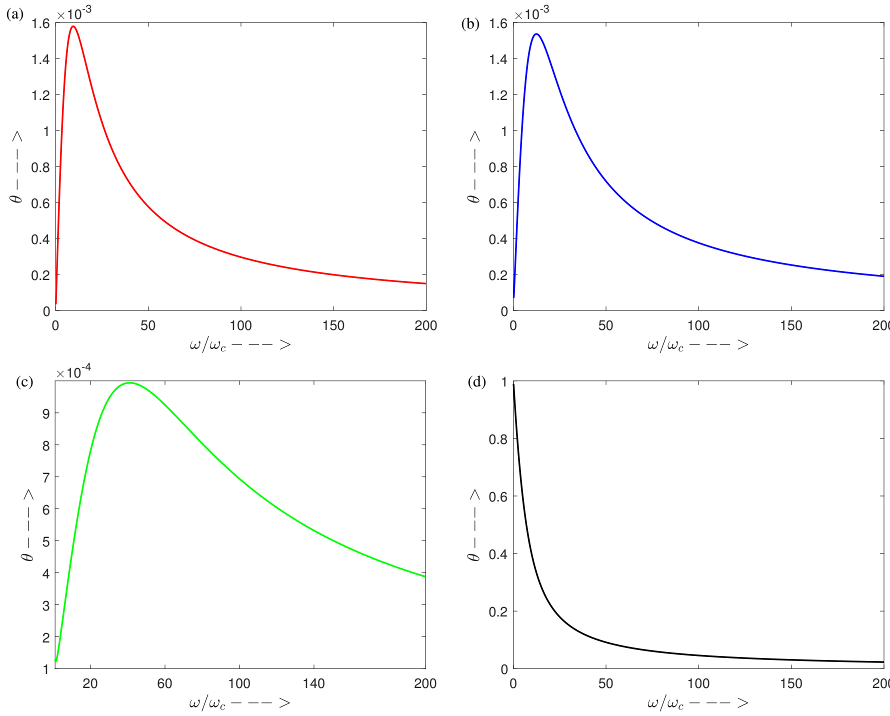

Figure 3: Variation of attenuation coefficient (θ) against non-dimensional angular frequency (ω/ω_c) of four compressional waves **(a)** P1, **(b)** P2, **(c)** P3, and **(d)** P4

Figure 3 illustrates the variation of attenuation coefficients (θ) with non-dimensional angular frequency (ω/ω_c) for the four compressional waves. This plot reveals that at low frequencies, the attenuation coefficients for waves P1, P2, and P3 initially increase significantly before reaching a finite limit, after which they begin to decrease as the frequency rises. However, wave P4 displays a decrease in attenuation coefficient as the frequency increases, with notably higher attenuation values compared to the other three waves. This may happen because the wave P4 corresponds to the fluid phase in the fractured pore $\left(\tau^{(3)} = 1 \text{ as } \phi^{(3)} = 1\right)$ with high permeability. When there is no dissipation, the wave will be non-dispersive, and so the attenuation coefficients of the four compressional waves will be zero. It is also evident that at high frequencies, the attenuation coefficients approach zero, indicative of non-dispersive behaviour.

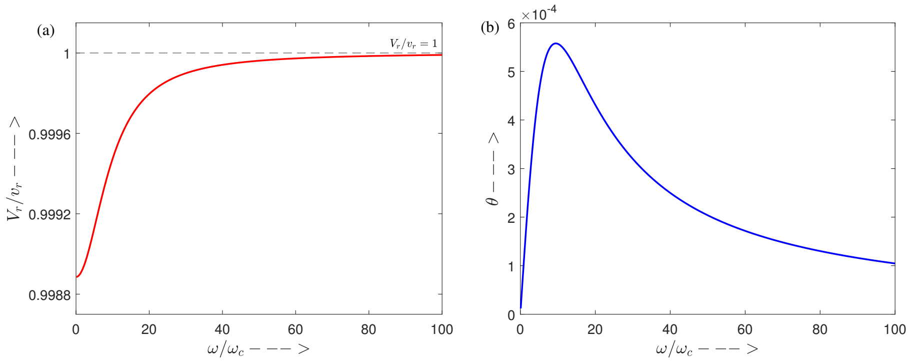

Figure 4: Variation of **(a)** phase velocity (V_r/v_r), and **(b)** attenuation coefficient (θ); against non-dimensional angular frequency (ω/ω_c) of rotational wave

Figure 4 demonstrates the variation of dimensionless phase velocity and attenuation coefficient with the non-dimensional angular frequency of the rotational wave. Here, the rotational wave in the triple-porosity medium signifies the rotational wave in the solid phase. Subfigure 4a reveals that with increasing frequency, the phase velocity of the rotational wave rises, but up to a finite limit. The limiting phase velocity of the rotational wave is $V_r/v_r = 1$, which is the constant velocity of the non-dispersive wave when there is no dissipation. This implies that for a higher frequency rotational wave, the dispersive wave behaviour transforms into non-dispersive as the frequency increases. Also, subfigure 4b illustrates that, for lower frequencies, the attenuation coefficient rapidly escalates to a finite value before diminishing as frequency rises. In the case of higher-frequency rotational waves, the attenuation coefficient approaches zero, signifying non-dispersive behaviour.

10 Summary and conclusion

In this study, we have introduced a comprehensive mathematical theory addressing the propagation of elastic waves within a multi-porosity medium with multi-permeability. This theory considers the presence of n distinct pore fluid phases and accounts for solid-fluid and fluid-fluid interactions. Lagrangian mechanics have been used to formulate the governing equation of elastic wave propagation in this complex multi-porous system. A notable contribution of this research, after the equation of motion, is the derivation of mass and dissipation coefficients in terms of known measurable parameters. The determination of dissipation coefficients involved utilising Darcy's law of multi-phase system. Furthermore, we conducted a series of Gedanken experiments to establish the constitutive equation of isotropic linear elastic multi-porous medium. Subsequently, equations of motion and dispersion for both compressional and rotational waves within the multi-porous medium are derived. The dispersion equations for these waves in a specific case, such as a triple porosity medium, are graphically represented using MATLAB software. Our comprehensive theoretical and numerical investigations have yielded several significant conclusions:

- The mathematical framework developed in this study provides a robust foundation for understanding and predicting elastic wave propagation within multi-porous media featuring complex fluid-solid interactions.
- The derived constitutive equation, along with the mass and dissipation coefficients, collectively enhances the practical utility by enabling the estimation of wave behaviour based on readily measurable parameters, enhancing the applicability of our model.
- In a multi-porous medium featuring n types of pore structures, there exist $(n+1)$ compressional waves exhibiting either dispersive or non-dispersive behaviour based on the presence or absence of dissipation within the medium.
- In a multi-porous medium with n types of pore structures, there exists a single rotational wave corresponding to the solid phase, and its dispersion behaviour depends on the presence or absence of dissipation within the medium.
- The phase velocities of compressional and rotational waves increase up to a finite limit as the angular frequency rises. These limits represent the phase velocities of their respective non-dispersive waves.
- In a dissipative triple-porous medium, all four compressional waves and one rotational wave are attenuated, and the attenuation approaches zero as the frequency increases.

This research not only advances our theoretical understanding of wave propagation in multi-porosity media but also offers practical tools for the analysis and prediction of such phenomena in real-world applications. The insights gained from this study hold promise for a wide range of fields, including geophysics, hydrology, and materials science.

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