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1 Hamiltonian/Stroh formalism for reversible poroelasticity
2 (and thermoelasticity)

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6 **Abstract**

Stroh's sextic formalism represents the equilibrium equations of anisotropic elasticity in a particularly attractive form, that is most suitable for studying layered and composite materials and time harmonic problems. Taking advantage of the fact that the Stroh formalism really amounts to the canonical form of the equations in the Hamiltonian sense, the case of Biot's reversible (i.e. no fluid dissipation) poroelasticity is here addressed, in the absence of a fluid pressure gradient. This framework is the same as thermoelasticity of perfect conductors. Two Hamiltonian formulations are developed: the first describes both the solid and the fluid phases and it exhibits, besides energy conservation, momentum conservation, as a result of pressure uniformity. The second is restricted to the solid skeleton and parallels anisotropic elasticity, although with Stroh matrices that account for fluid coupling. The case of weak fluid-solid coupling is also considered and it produces a perturbation from anisotropic elasticity with the same structure as incompressibility, although in an "opposing" manner. This comparison suggests that the incompressibility limit introduced by Biot should be revised. The energy conservation integral and the edge impedance matrix are also illustrated.

7 *Keywords:* Stroh formalism, reversible poroelasticity, thermoelasticity,
8 Hamiltonian form

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9 1. Introduction

10 The foundation of Stroh sextic formalism is laid out in a pair of celebrated pa-
11 pers concerning dislocations (Stroh, 1958) and harmonic motion (Stroh, 1962) in
12 generally anisotropic materials (in plane strain). This framework provides a sub-
13 stantial improvement over the already established Eshelby-Reid-Shockley form
14 of the equations of elastostatics (Ting, 1996). Indeed, although both methods
15 share the fact that mechanical features are interpreted and described under the
16 unifying lens of linear algebra, Stroh’s approach exhibits very distinctive features
17 for the involved matrices. In fact, solutions are built in terms of eigenvalues and
18 eigenvectors of a block matrix, the so-called Stroh fundamental elastic matrix,
19 endowed with many striking properties (Barnett, 2000). Since then, the method
20 has been extensively applied to composite materials, harmonic wave propaga-
21 tion, crack and dislocations, instability and many more topics (Ting, 1996).
22 Given its success, it is little wonder that extensions of the method have been
23 proposed outside its original domain, to address, for example, constrained ma-
24 terials (Chadwick and Smith, 1977), anisotropic plates (Fu, 2007) and internally
25 constrained micro-polar solids (Nobili and Radi, 2022). In general, the success
26 of the procedure hangs on the careful choice of the unknown variables, which
27 can be rather tricky unless somehow guided. In fact, many contributions ex-
28 ist in the literature where a trial-and-error approach was used (see Fu (2007)).
29 As an illustration, Hwu (2003) analyses coupled stretching-bending modes in
30 anisotropic laminates through a modification of the Lekhnitskii formalism (for
31 details on which see Ting (1996)) in an attempt to recover the properties specific
32 to the Stroh form. Recently, Fu (2003) developed a Stroh-like formulation for
33 determining the dispersion relation of edge waves in generally anisotropic plates
34 under the sole restriction that the mid-plane is a plane of material symmetry.
35 In that work, Fu capitalized on the observation that the Stroh formalism is re-
36 ally an Hamiltonian formulation where a space variable is treated in time-like
37 fashion, already available in the literature (Barnett, 2000), to develop a guid-
38 ing principle for the right choice of the unknown pairs, namely the principle of

39 energy conjugation. Successively, this approach was used in (Fu and Brookes,
40 2006) to study edge waves in asymmetrically laminated plates, for which in-
41 plane and out-of-plane deformations are coupled by anisotropy. By the same
42 method, Fu (2007) studies incompressible anisotropic materials and anisotropic
43 plates, and results are later extended by Edmondson and Fu (2009) to generally
44 constrained and pre-stressed anisotropic materials. The procedure paves the
45 way for the application of the surface-impedance matrix for studying localized
46 waves (Fu, 2005).

47 Thus far, a classical Stroh formalism could be retrieved, by which a right
48 eigenvalue problem is finally obtained (as presently explained). Yet, the Hamil-
49 tonization of any mechanical model may be carried out by the same principles
50 and the outcome, in general, may not correspond to a classical Stroh-like struc-
51 ture. As a case in point, Fu and Kaplunov (2012) study waves localized at the
52 edge of isotropic thin cylindrical shells and find that the fundamental elastic
53 matrix is in fact wavenumber dependent. This result, which is typical of dis-
54 persive systems, is also retrieved by Nobili and Radi (2022) in the context of
55 the indeterminate couple-stress theory of elasticity. The structure of the Stroh
56 formalism is now supplemented by a right hand side that is proportional to
57 the unknowns (i.e. the problem is still linear). Therefore, the very form of
58 the Stroh-like canonical system already reveals important informations on the
59 problem under scrutiny.

60 Biot's poroelasticity is a very successful phenomenological theory with enor-
61 mous practical implications in the fields of seismology and seismic exploration,
62 geology and geotechnical structures, soil testing and characterization, to name
63 only a few Dullien (2012). The literature on this topic is very extensive and
64 moves in many directions, for example, concerning wave propagation in porous
65 media, see the review paper by Corapcioglu and Tuncay (1996). Efforts in the
66 direction of connecting this theory to the theory of mixtures or to microme-
67 chanical theories have been long going, with mixed success, see, among many,
68 Lopatnikov and Cheng (2004). Extensions of the theory have been proposed
69 in the many directions, for example introducing double (Berryman and Wang,

70 2000) or multi- porosities (Pramanik et al., 2024), finite elastic deformations
 71 (Norris and Grinfeld, 1995) or even piezoelectric effects (Sharma, 2010). Still,
 72 no Stroh-like formalism may be traced in the literature, possibly on the grounds
 73 that multi-field theories may prove impervious to this framework.

74 In this paper, we Hamiltonize the equations of Biot’s poroelasticity in the
 75 absence of dissipation, i.e. in the context of reversible processes (thermostatistics)
 76 and in the absence of a fluid pressure gradient (Biot, 1955, 1956a). This same
 77 framework may be applied to thermoelastostatics of perfect conductors, where in
 78 fact temperature plays the role of the fluid pressure (Biot, 1956b). Inertia effects
 79 are only considered inasmuch as they may be incorporated into the material
 80 properties in the form of time-harmonic contributions. Focus is set on the
 81 determination of the canonical formalism and on the properties it reveals.

82 2. Reversible poroelasticity

83 Let \mathbf{u} and \mathbf{U} denote the displacement in the solid and in the fluid phase,
 84 respectively. Besides, the fluid-to-solid displacement per unit volume of the
 85 poroelastic medium reads

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}), \quad (1)$$

86 where f is the *effective porosity*, generally not uniform, that represents the
 87 interconnected pore space. In particular, the porosity is defined as the ratio
 88 between the volume of interconnected pores, V_p , over the bulk volume V_b , the
 89 latter being obtained by $V_b = V_p + V_s$, i.e. summing the pore volume to the
 90 volume occupied by the solid skeleton, see (Biot, 1955). Following Biot (1962),
 91 in this theory closed porosity is assumed to be part of the solid skeleton. Also,
 92 we let

$$e = \operatorname{div} \mathbf{u}, \quad \zeta = -\operatorname{div} \mathbf{w}, \quad (2)$$

93 that provide the volume *increment* for the solid and the fluid phase, respectively
 94 (indeed the fluid increment is obtained by the inflow of \mathbf{w}). In particular, we
 95 have the connection

$$-\zeta = (\mathbf{U} - \mathbf{u}) \cdot \operatorname{grad} f + f(\epsilon - e), \quad (3)$$

96 where we have let $\epsilon = \text{div } \mathbf{U}$. In the case of uniform porosity, we retrieve the
 97 result given in Biot (1962)

$$\epsilon = e - f^{-1}\zeta. \quad (4)$$

Let the rank-2 tensor \mathbf{T} denote the *total stress*, that is obtained summing the stress in the solid phase $\boldsymbol{\sigma}$ with the stress in the fluid phase $\boldsymbol{\sigma}_f = -fp_f\mathbf{1}$, where, here and after, $\mathbf{1}$ is the rank-3 identity tensor and p_f is the fluid pressure (positive when compressive) per unit area of the fluid phase. Sometimes, to refer pressure to the unit bulk area, the shorthand $\sigma_f = -fp_f$ is introduced. Let (O, x_1, x_2, x_3) denote an orthogonal reference frame and \mathbf{n} be the unit vector normal to any relevant directed surface S . Alongside the axis (x_1, x_2, x_3) , we introduce an orthonormal set of basis vectors, \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 , such that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, with the usual understanding that twice repeated subscripts are summed over in the set $\{1, 2, 3\}$. Here, δ_{ij} is zero for $i \neq j$ and 1 for $i = j$. We define the fundamental force vectors in a generally anisotropic medium with elastic constants c_{ijkl}

$$\mathbf{t}_1 = \mathbf{T}\mathbf{e}_1 = \mathbf{Q}\mathbf{u}_{,1} + \mathbf{R}\mathbf{u}_{,2} - r\zeta\mathbf{e}_1, \quad (5a)$$

$$\mathbf{t}_2 = \mathbf{T}\mathbf{e}_2 = \mathbf{R}^T\mathbf{u}_{,1} + \mathbf{T}\mathbf{u}_{,2} - r\zeta\mathbf{e}_2, \quad (5b)$$

98 where $Q_{ij} = c_{i1j1}$, $R_{ij} = c_{i1j2}$, $T_{ij} = c_{i2j2}$ are the usual Stroh matrices. In
 99 particular, \mathbf{Q} and \mathbf{T} are symmetric, i.e. $\mathbf{Q} = \mathbf{Q}^T$ and $\mathbf{T} = \mathbf{T}^T$, and positive def-
 100 inite, provided the strain energy is a positive function (Ting, 1996, §6.1). Here,
 101 r denotes the cross coupling term between volume changes in the solid and in
 102 the fluid (denoted by C in (Biot, 1962, Eq.(3.5)), and by Q/f in (Corapcioglu
 103 and Tuncay, 1996, Eq.(2.16))). In this paper, we assume that cross coupling oc-
 104 curs in isotropic fashion, for transverse anisotropy see, for example, Biot (1955,
 105 Eq.(3.2)). Besides, it is assumed that dependent variables are independent from
 106 x_3 , i.e. $\partial/\partial x_3(\) = 0$. In a steady-state motion with velocity v in the x_1 -
 107 direction, the matrix \mathbf{Q} is simply replaced by $\mathbf{Q} - \rho v^2\mathbf{I}$, where ρ is the density
 108 of the solid skeleton and \mathbf{I} is the identity matrix. Besides, for an isotropic solid,
 109 it is $c_{ijkl} = 2\mu\delta_{ik}\delta_{jl} + \lambda_c\delta_{kl}\delta_{ij}$, where μ and λ_c are the Lamé moduli (Nobili
 110 and Radi, 2022).

111 For a compressible fluid, it is

$$p_f = -re + m\zeta, \quad (6)$$

112 where m is the compressibility modulus for the fluid, defined as the fluid pressure
 113 required to force a unit volume of fluid into the pore structure while keeping the
 114 solid volume unchanged, i.e. $e = 0$. As pointed out by Biot and Willis (1957),
 115 the reversibility assumption, by which a stored elastic potential is admitted,
 116 identifies the coupling coefficient in the last term in (5) with that in (6). Also,
 117 following Biot and Willis (1957), it is

$$f \leq \alpha = r/m < 1. \quad (7)$$

118 Biot (1956b) showed that this framework parallels that of thermo-elasticity, with
 119 the pressure p_f playing the same role as temperature.

120 We are now in the position to write the potential elastic energy minus the
 121 work done by the applied external forces over the body B (i.e. the total energy
 122 in the sense of Eshelby)

$$\mathcal{L} = \int_B W dV - \int_{\partial B} (\mathbf{t}_0 \cdot \mathbf{u} - p_{f0} \mathbf{n} \cdot \mathbf{w}) dS, \quad (8)$$

123 where we have let the stored potential energy density

$$W = \frac{1}{2} (\mathbf{T} \cdot \text{grad } \mathbf{u} + p_f \zeta). \quad (9)$$

124 Here, \mathbf{t}_0 and p_{f0} are the prescribed surface force and fluid pressure over the
 125 body boundary ∂B with unit normal \mathbf{n} . For the sake of simplicity, no body
 126 force is considered. With a little abuse of notation yet in favour of tidiness, an
 127 interposed dot denotes the scalar product between both tensor and vector pairs,
 128 i.e. in components $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$ and $\mathbf{a} \cdot \mathbf{b} = a_i b_i$, respectively. By strong
 129 ellipticity, Q and T are positive definite and $m > 0$.

130 In a reversible process (i.e. thermostatics), the imposed boundary pressure
 131 should not trigger movement of the fluid phase and, therefore, p_{f0} is constant
 132 on the surface ∂B and p_f is equally constant *throughout the body* (Biot, 1962).
 133 This pressure distribution holds also in steady state motion of the solid provided

134 that we take no account of dissipation. The minus sign associated with the fluid
 135 pressure p_{f0} is a consequence of it being positive in compression. In Eq.(8), we
 136 have let the shorthand

$$\mathbf{T} \cdot \text{grad } \mathbf{u} = \mathbf{t}_1 \cdot \mathbf{u}_{,1} + \mathbf{t}_2 \cdot \mathbf{u}_{,2},$$

137 where it is understood that $\text{grad } \mathbf{u} = \mathbf{u}_{,1} \otimes \mathbf{e}_1 + \mathbf{u}_{,2} \otimes \mathbf{e}_2$ and a subscript comma
 138 denotes differentiation, e.g. $\mathbf{u}_{,1} = \partial \mathbf{u} / \partial x_1$. Here, we have used the vector
 139 dyadic that, for any pair of vectors \mathbf{a} and \mathbf{b} , yields the rank-2 tensor $\mathbf{a} \otimes \mathbf{b}$ such
 140 that, for any vector \mathbf{c} , $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.

The equilibrium equations read

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = \mathbf{o}, \quad (10a)$$

$$\text{grad } p_f = \mathbf{o}, \quad (10b)$$

141 where the last is the equilibrium version of Darcy's law¹, see Biot (1962, Eq.(7.2)).
 142 Without loss of generality, we assume that the boundary conditions are only ex-
 143 pressed in terms of forces

$$\mathbf{T}\mathbf{n} = \mathbf{t}_0 \quad \text{and} \quad p_f = p_{f0}, \quad \text{for } \mathbf{x} \in \partial B, \quad (11)$$

144 where ∂B is the frontier of the body B . By the divergence theorem, one can
 145 rewrite the total energy in terms of a single volume integral

$$\mathcal{L} = - \int_B L dV,$$

146 where we have introduced the Lagrangian density L

$$L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \text{div } \mathbf{w}) = \frac{1}{2} \mathbf{T} \cdot \text{grad } \mathbf{u} + \frac{1}{2} p_f \zeta. \quad (12)$$

In light of Eqs.(5) and (6), this may be rewritten as

$$\begin{aligned} L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{w}_{,1}, \mathbf{w}_{,2}) &= \frac{1}{2} \mathbf{u}_{,1} \cdot \mathbf{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \mathbf{R}\mathbf{u}_{,2} + \frac{1}{2} \mathbf{u}_{,2} \cdot \mathbf{T}\mathbf{u}_{,2} \\ &\quad - \frac{1}{2} r \zeta (\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \mathbf{u}_{,2}) + \frac{1}{2} (-re + m\zeta) \zeta, \end{aligned} \quad (13)$$

¹Darcy's law emerges from considering an irreversible process and the attached *dissipation function*, that is a quadratic form in $\partial \mathbf{w} / \partial t$

where $\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \mathbf{u}_{,2} \cdot \mathbf{e}_2 = \text{div } \mathbf{u} = e$. Clearly, this formulation admits the Stroh formalism because, unlike internally constrained solids (Nobili and Radi, 2022), both displacement vectors, \mathbf{u} and \mathbf{w} , appear only in differentiated form. The Lagrangian density becomes

$$L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{w}_{,1}, \mathbf{w}_{,2}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \mathbf{Q} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \mathbf{R} \mathbf{u}_{,2} + \frac{1}{2} \mathbf{u}_{,2} \cdot \mathbf{T} \mathbf{u}_{,2} + r(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \mathbf{w}_{,2})(\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \mathbf{u}_{,2}) + \frac{1}{2} m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \mathbf{w}_{,2})^2, \quad (14)$$

whence the Euler-Lagrange equations read

$$\frac{d}{dx_1} \frac{\partial L}{\partial \mathbf{u}_{,1}} + \frac{d}{dx_2} \frac{\partial L}{\partial \mathbf{u}_{,2}} = 0, \quad (15a)$$

$$\frac{d}{dx_1} \frac{\partial L}{\partial \mathbf{w}_{,1}} + \frac{d}{dx_2} \frac{\partial L}{\partial \mathbf{w}_{,2}} = 0, \quad (15b)$$

147 which is clearly in the Stroh form once we settle for either coordinate to act as
148 a time-like variable, say x_2 as in Fu (2007). Eq.(15a) gives

$$(\mathbf{Q} \mathbf{u}_{,1} + \mathbf{R} \mathbf{u}_{,2} - r\zeta \mathbf{e}_1)_{,1} + (\mathbf{R}^T \mathbf{u}_{,1} + \mathbf{T} \mathbf{u}_{,2} - r\zeta \mathbf{e}_2)_{,2} = \mathbf{o}, \quad (16)$$

149 that corresponds to the equilibrium equation (10a), provided that we account
150 for (5). Similarly, Eq.(15b) lends

$$(re - m\zeta)_{,1} \mathbf{e}_1 + (re - m\zeta)_{,2} \mathbf{e}_2 = \mathbf{o}, \quad (17)$$

151 that indeed amounts to Eq.(10b), once acknowledging for (6).

152 3. Hamiltonian formalism

153 We now introduce the Hamiltonian formalism by treating x_2 as a time-like
154 variable (Fu, 2007). Consequently, differentiation with respect to x_2 will be
155 denoted by a superscript dot. For reasons that shall be presently apparent, we
156 let

$$\bar{\mathbf{Q}} = \mathbf{Q} - \frac{r^2}{m} \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \bar{\mathbf{R}} = \mathbf{R} - \frac{r^2}{m} \mathbf{e}_1 \otimes \mathbf{e}_2, \quad \bar{\mathbf{T}} = \mathbf{T} - \frac{r^2}{m} \mathbf{e}_2 \otimes \mathbf{e}_2, \quad (18)$$

whence we may rewrite (5) as

$$\mathbf{t}_1 = \bar{\mathbf{Q}} \mathbf{u}_{,1} + \bar{\mathbf{R}} \dot{\mathbf{u}} - r \frac{p_f}{m} \mathbf{e}_1, \quad (19a)$$

$$\mathbf{t}_2 = \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \bar{\mathbf{T}} \dot{\mathbf{u}} - r \frac{p_f}{m} \mathbf{e}_2. \quad (19b)$$

Eq.(14) becomes

$$L(\mathbf{u}_{,1}, \dot{\mathbf{u}}, \mathbf{w}_{,1}, \dot{\mathbf{w}}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{\mathbf{Q}} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \bar{\mathbf{R}} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{u}} \cdot \bar{\mathbf{T}} \dot{\mathbf{u}} \\ + \frac{1}{2} m^{-1} [r(\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \dot{\mathbf{u}}) + m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \dot{\mathbf{w}})]^2. \quad (20)$$

from which conjugate momenta are immediately obtained

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{u}}} = \mathbf{t}_2, \quad (21a)$$

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{w}}} = (re - m\zeta) \mathbf{e}_2 = -p_f \mathbf{e}_2. \quad (21b)$$

157 Solving Eq.(21b) for ζ gives

$$\zeta = m^{-1} (p_f + re), \quad (22)$$

158 while solving Eq.(19b) for $\dot{\mathbf{u}}$ gives

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} \left(\mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right). \quad (23)$$

159 Scalar multiplication of (23) throughout by \mathbf{e}_2 lends

$$\dot{\mathbf{u}} \cdot \mathbf{e}_2 = \zeta_1 \bar{\mathbf{T}}^{-1} \bar{\mathbf{t}}_2 \cdot \mathbf{e}_2, \quad (24)$$

160 where we have let the shorthand

$$\bar{\mathbf{t}}_2 = \mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + r \left(\zeta - \frac{r}{m} \mathbf{u}_{,1} \cdot \mathbf{e}_1 \right) \mathbf{e}_2,$$

161 and, as in Fu (2007, Eq.(3.12)), it is

$$\zeta_1^{-1} = 1 + \frac{r^2}{m} \mathbf{e}_2 \cdot \bar{\mathbf{T}}^{-1} \mathbf{e}_2 > 1, \quad (25)$$

162 whose last term is always positive by virtue of strong ellipticity (see the Ap-

163 pendix). Hence, plugging (24) into (23), it is finally

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} \mathbf{P} \bar{\mathbf{t}}_2, \quad (26)$$

164 having let the projector (we have used the symmetry of $\bar{\mathbf{T}}$)

$$\mathbf{P} = \mathbf{1} - \frac{r^2}{m} \zeta_1 \mathbf{e}_2 \otimes \bar{\mathbf{T}}^{-1} \mathbf{e}_2. \quad (27)$$

165 We note that

$$\mathbf{P}\mathbf{e}_2 = \zeta_1\mathbf{e}_2, \quad \text{and} \quad \bar{\mathbf{T}}^{-1}\mathbf{P} \in \text{Sym}, \quad (28)$$

166 whence Eq.(26) may be rewritten as

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1}\mathbf{P}(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) + r\zeta_1\left(\zeta - \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1\right)\bar{\mathbf{T}}^{-1}\mathbf{e}_2. \quad (29)$$

167 Indeed, scalar multiplication by \mathbf{e}_2 , in view of the properties (28), immediately
168 lends (24).

169 In similar fashion, in light of Eqs.(2,24), Eq.(22) yields

$$-\dot{\mathbf{w}} \cdot \mathbf{e}_2 = \mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m} + \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \frac{r}{m}\bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \mathbf{e}_2. \quad (30)$$

We introduce the Hamiltonian density

$$\begin{aligned} H &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} + \mathbf{p}_2 \cdot \dot{\mathbf{w}} - L \\ &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} - p_f\mathbf{e}_2 \cdot \dot{\mathbf{w}} - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} - \mathbf{u}_{,1} \cdot \bar{\mathbf{R}}\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}} \cdot \bar{\mathbf{T}}\dot{\mathbf{u}} - \frac{1}{2}m^{-1}p_f^2, \end{aligned}$$

whence

$$\begin{aligned} H &= \frac{1}{2}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \\ &\quad + p_f\left(\mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m} + \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1\right) - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} - \frac{1}{2}m^{-1}p_f^2, \end{aligned}$$

and finally

$$\begin{aligned} H &= \frac{1}{2}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \\ &\quad + p_f\left(\mathbf{w}_{,1} + \frac{r}{m}\mathbf{u}_{,1}\right) \cdot \mathbf{e}_1 - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} + \frac{1}{2}\frac{p_f^2}{m}. \quad (31) \end{aligned}$$

170 As well known, the canonical equations may be grouped in two sets, described
171 by the vector canonical equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \text{and} \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \quad (32)$$

172 In the first group we have

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2} = \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right), \quad (33)$$

173 and

$$\dot{\mathbf{w}} \cdot \mathbf{e}_2 = -\frac{\partial H}{\partial p_f} = -\mathbf{w}_{,1} \cdot \mathbf{e}_1 - \frac{p_f}{m} - \frac{r}{m} \mathbf{u}_{,1} \cdot \mathbf{e}_1 - \frac{r}{m} \mathbf{e}_2 \cdot \bar{\mathbf{T}}^{-1} \left(\mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right), \quad (34)$$

174 that correspond to Eq.(23) and to (30), respectively. The second group provides
175 the equilibrium equations. Indeed, one gets

$$\dot{\mathbf{t}}_2 = -\frac{\partial H}{\partial \mathbf{u}} = -\left[\bar{\mathbf{R}} \bar{\mathbf{T}}^{-1} \left(\mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right) + \bar{\mathbf{Q}} \mathbf{u}_{,1} - \frac{r}{m} p_f \mathbf{e}_1 \right]_{,1} \quad (35)$$

176 that, accounting for (23), whereby $\bar{\mathbf{T}}^{-1}$ times the term in round brackets gives
177 $\dot{\mathbf{u}}$, and in light of the first of (5), amounts to (10a). By the same token,

$$-p_f \mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}} = (p_f \mathbf{e}_1)_{,1} \quad (36)$$

178 that is immediately (10b). Incorporating the dissipation function into this for-
179 mulation, may provide the starting point for addressing the general case of
180 irreversible poroelasticity.

181 3.1. Reduced Hamiltonian

182 Looking at Eq.(21b) and recalling that p_f is constant throughout the body,
183 as a result of the equilibrium equation (10b), one realises that, besides energy
184 conservation, another motion invariant is available. Indeed, this formulation
185 possesses translational invariance with respect to $\dot{\mathbf{w}}$. This is an outcome of the
186 fact that, unlike \mathbf{u} , \mathbf{w} appears in the Lagrangian only through its divergence
187 ζ , and therefore one may assume $\mathbf{w} = \text{grad } \varphi$ without loss of generality, the
188 solenoidal contribution to \mathbf{w} being irrelevant to the present purposes, see (Biot,
189 1962, Eq.(7.13)). This feature is specific to reversible poroelasticity and it is
190 lost when encompassing for dissipation. Consequently, $\mathbf{w}_{,1} \cdot \mathbf{e}_1$ and $\dot{\mathbf{w}} \cdot \mathbf{e}_2$
191 are not (globally) independent from one another. To avoid dealing with this
192 constraint, a more convenient approach consists of replacing ζ in (20) through
193 the connection (22) to get

$$\hat{L}(\mathbf{u}_{,1}, \dot{\mathbf{u}}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{\mathbf{Q}} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \bar{\mathbf{R}} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{u}} \cdot \bar{\mathbf{T}} \dot{\mathbf{u}}, \quad (37)$$

194 having dispensed with the irrelevant constant term $\frac{1}{2} p_f^2 / m$. In this form, the
195 system matches anisotropic elasticity, provided that the Stroh matrices (18) are

196 used. It is also emphasized that, in this reduced formulation (37), only the solid
 197 skeleton is represented. The Euler-Lagrange equation reads

$$\hat{\mathbf{t}}_{1,1} + \hat{\mathbf{t}}_2 = 0, \quad (38)$$

198 having let the force vectors

$$\hat{\mathbf{t}}_1 = \bar{\mathbf{Q}}\mathbf{u}_{,1} + \bar{\mathbf{R}}\dot{\mathbf{u}}, \quad \hat{\mathbf{t}}_2 = \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \bar{\mathbf{T}}\dot{\mathbf{u}}. \quad (39)$$

199 This amounts to defining the new stress tensor $\hat{\mathbf{T}}$, which differs from the total
 200 stress \mathbf{T} by the constant hydrostatic pressure $\frac{r}{m}p_f\mathbf{1}$, and corresponds to Biot's
 201 *effective stress* σ_{ij} , that is the force in excess to pressure applied to the solid per
 202 unit surface of the bulk material, see (Biot, 1956a, Eq.(3.2)) and (Biot, 1962,
 203 Eq.(3.9)). The corresponding momentum immediately follows

$$\hat{\mathbf{p}} = \frac{\partial L}{\partial \dot{\mathbf{u}}} = \hat{\mathbf{t}}_2, \quad (40)$$

204 and it can be solved for the conjugate coordinate $\dot{\mathbf{u}}$ giving again (23), yet as-
 205 suming that $p_f = 0$, i.e.

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}). \quad (41)$$

206 The possibility to invert $\bar{\mathbf{T}}$ is granted by strong ellipticity, as discussed in the Ap-
 207 pendix. The corresponding Hamiltonian is similarly obtained from (31) letting
 208 $p_f = 0$,

$$\hat{H} = \hat{\mathbf{p}} \cdot \dot{\mathbf{u}} - \hat{L} = \frac{1}{2} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) \cdot \bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1}. \quad (42)$$

209 The canonical equations are

$$\dot{\mathbf{u}} = \frac{\partial \hat{H}}{\partial \hat{\mathbf{t}}_2}, \quad (43)$$

210 that indeed gives (41), and

$$\dot{\hat{\mathbf{t}}}_2 = -\frac{\partial \hat{H}}{\partial \mathbf{u}} = -[\bar{\mathbf{R}}\bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) - \bar{\mathbf{Q}}\mathbf{u}_{,1}]_{,1}, \quad (44)$$

211 that corresponds to (38).

212 For a homogeneous material, letting the stress potential $\hat{\phi} = \int \hat{\mathbf{t}}_2 dx_1$ and
 213 the vector of unknowns

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{u} \\ \hat{\phi} \end{bmatrix}, \quad (45)$$

214 we can write the Stroh formalism

$$\frac{\partial}{\partial x_2} \boldsymbol{\xi} = \mathbf{N} \frac{\partial}{\partial x_1} \boldsymbol{\xi}, \quad (46)$$

215 where \mathbf{N} is the *fundamental elasticity block-matrix* (Ting, 1996, §6)

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad (47)$$

216 and we have let the 3 by 3 block-matrices

$$\mathbf{N}_1 = -\bar{\mathbf{T}}^{-1} \bar{\mathbf{R}}^T, \quad \mathbf{N}_2 = \bar{\mathbf{T}}^{-1}, \quad \mathbf{N}_3 = \bar{\mathbf{R}} \bar{\mathbf{T}}^{-1} \bar{\mathbf{R}}^T - \bar{\mathbf{Q}}. \quad (48)$$

217 We observe that $\boldsymbol{\xi}$ has mixed dimensions, namely length and force over length
 218 for the first and for the second vector component, respectively. Consequently,
 219 \mathbf{N}_1 is dimensionless, while \mathbf{N}_3 and \mathbf{N}_2 have dimension of stress and inverse of
 220 stress (compliance), respectively.

221 Letting the 6 by 6 constant matrix (Ting, 1996, Eq.(5.5-7))

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{bmatrix}, \quad (49)$$

222 and in view of the symmetry of \mathbf{N}_2 and \mathbf{N}_3 , one retrieves the fundamental
 223 symmetric matrix

$$\hat{\mathbf{I}} \mathbf{N} = \begin{bmatrix} \mathbf{N}_3 & \mathbf{N}_1^T \\ \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix} = (\hat{\mathbf{I}} \mathbf{N})^T. \quad (50)$$

224 Following Ting (1996, §5.5), \mathbf{N}_2 is positive definite and $-\mathbf{N}_3$ is positive semidef-
 225 inite. When looking for travelling solutions of the form $\boldsymbol{\xi} = \boldsymbol{\Xi} f(x_1 + px_2)$, a
 226 right eigenvalue problem is retrieved

$$\mathbf{N} \boldsymbol{\Xi} = p \boldsymbol{\Xi}, \quad (51)$$

227 The Hamiltonian density (42) may be rewritten as the quadratic form asso-
 228 ciated with the fundamental matrix

$$\hat{H} = \frac{1}{2} \boldsymbol{\xi} \cdot \hat{\mathbf{N}} \boldsymbol{\xi}, \quad (52)$$

229 whence the first integral associated with energy conservation (given that the
 230 Lagrangian is x_2 -independent) reads

$$\int_{\Sigma} \boldsymbol{\xi} \cdot \hat{\mathbf{N}} \boldsymbol{\xi} dx_1 dx_3 = \text{const}, \quad (53)$$

231 in the assumption that we may decompose the domain as $B = \Sigma \times I$, where I
 232 is an interval in the x_2 coordinate. Finally, we may define the *edge impedance*
 233 *matrix* \mathbf{M} as

$$\hat{\boldsymbol{\phi}} = \imath \mathbf{M} \mathbf{u}, \quad (54)$$

234 and in light of (45,46), we may write

$$\dot{\hat{\boldsymbol{\phi}}} = \imath \mathbf{M} \dot{\mathbf{u}} = (\imath \mathbf{M} \mathbf{N}_1 - \mathbf{M} \mathbf{N}_1 \mathbf{M}) \mathbf{u}_{,1} = (\mathbf{N}_3 + \imath \mathbf{N}_1^T \mathbf{M}) \mathbf{u}_{,1},$$

235 whence \mathbf{M} satisfies the matrix equation (Fu, 2007, Eq.(4.40))

$$\mathbf{N}_3 + \imath \mathbf{N}_1^T \mathbf{M} - \imath \mathbf{M} \mathbf{N}_1 + \mathbf{M} \mathbf{N}_1 \mathbf{M} = \mathbf{O}. \quad (55)$$

236 This matrix provides a very simple procedure to determine localized waves, for
 237 which $\mathbf{t}_2 \equiv \mathbf{o}$ on the body surface $x_2 = 0$. Indeed, the dispersion relation is
 238 simply obtained by admitting non-trivial solutions to the system

$$\mathbf{M} \mathbf{u} = \mathbf{o},$$

239 hence the major obstacle lying in the way is the determination of the impedance
 240 matrix through the connection (55). This result is most simply achieved through
 241 the integral representation originally introduced by Barnett and Lothe (1974).

242 4. Weak reversible poroelasticity and the incompressible limit

243 We shall now consider the limit where the coupling effect is weaker than the
 244 elastic response. For this, we let $\tau_0 = \|\mathbf{T}\|$ be the norm of the matrix \mathbf{T} , and we

245 assume that $\tau_0^{-1}r^2/m = \varepsilon \ll 1$ is small. We name this condition *weak reversible*
 246 *poroelasticity*. In this case,

$$\bar{\mathbf{T}}^{-1} = (\mathbf{1} + \varepsilon\tau_0\mathbf{T}^{-1}\mathbf{e}_2 \otimes \mathbf{e}_2) \mathbf{T}^{-1} + O(\varepsilon^2), \quad (56)$$

247 and

$$\zeta_1 = 1 - \varepsilon\tau_0\zeta_0^{-1} + O(\varepsilon)^2, \quad \zeta_0 = 1/(\mathbf{e}_2 \cdot \mathbf{T}^{-1}\mathbf{e}_2),$$

whence $\zeta_1 \approx 1$. Then, expanding to first order terms in ε , one gets (collecting dimensionality terms)

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T + \varepsilon\tau_0\mathbf{T}^{-1}\mathbf{e}_2 \otimes \mathbf{e}_T \quad (57a)$$

$$\mathbf{N}_2 = \mathbf{T}^{-1}(\mathbf{I} + \varepsilon\tau_0\mathbf{e}_2 \otimes \mathbf{T}^{-1}\mathbf{e}_2), \quad (57b)$$

$$\mathbf{N}_3 = \mathbf{R}(\mathbf{T}^{-1} + \varepsilon\tau_0\mathbf{R}^{-1}\mathbf{e}_T \otimes \mathbf{R}^{-1}\mathbf{e}_T)\mathbf{R}^T - \mathbf{Q}, \quad (57c)$$

248 with $\mathbf{e}_T = \mathbf{e}_1 - \mathbf{R}\mathbf{T}^{-1}\mathbf{e}_2$. It is pointed out that Eqs.(57) are indeed valid asymp-
 249 totic expansions inasmuch as $\tau_0^{-1}\|\mathbf{R}\| = O(1)$ and $\tau_0^{-1}\|\mathbf{Q}\| = O(1)$ or bigger.
 250 Physically, this amounts to requiring that all elastic constants are of the same
 251 order, i.e. contrast is excluded. Formally, Eqs.(57) match the corresponding ma-
 252 trices in incompressible anisotropic elasticity (Fu, 2007, Eqs.(3.14-16)), provided
 253 that $\varepsilon\tau_0$ is replaced by ζ_0 and the opposite sign is taken in the incompressibility
 254 contributions, that are given by the correction term in each of Eqs.(57). Indeed,
 255 a similar expansion of Eq.(41) yields

$$\dot{\mathbf{u}} = \mathbf{T}^{-1}(\hat{\mathbf{t}}_2 - \mathbf{R}^T\mathbf{u}_{,1} - p_0\mathbf{e}_2) \quad (58)$$

256 with

$$p_0 = \varepsilon\tau_0 \{-\mathbf{T}^{-1}(\hat{\mathbf{t}}_2 - \mathbf{R}^T\mathbf{u}_{,1}) \cdot \mathbf{e}_2 - \mathbf{u}_{,1} \cdot \mathbf{e}_1\}. \quad (59)$$

257 Providing again that $\varepsilon\tau_0 = \zeta_0$ and p_0 is sign reversed, such equations are for-
 258 mally equivalent to (3.7) and (3.11) of Fu (2007), respectively giving $\dot{\mathbf{u}}$ and the
 259 Lagrange multiplier enforcing incompressibility for incompressible anisotropic
 260 solids. This analysis reveals that the weak poroelastic limit is similar to incom-
 261 pressible anisotropic elasticity, with yet two important differences. First, given
 262 that $\tau_0 \sim \zeta_0$, the condition $\varepsilon\tau_0 = \zeta_0$ can only be achieved in a correction sense

263 and therefore incompressibility is to be intended as a perturbation from the
 264 unconstrained leading solution. Second, the sign reversal of p_0 reveals that this
 265 perturbation is taken in the opposite direction, i.e. the role of the fluid phase
 266 in the weak limit is opposite to that of the incompressibility constraint. At any
 267 rate, incompressibility cannot be achieved for the solid skeleton in the general
 268 sense.

269 Biot, on heuristic grounds, claims that the incompressible limit is obtained
 270 letting $m \rightarrow +\infty$ and $\alpha = r/m = 1$, see for example Biot (1962). Although,
 271 just looking at (6), it is manifest that the former condition is sufficient for fluid
 272 incompressibility, the latter needs some revision. Indeed, the condition $\alpha = 1$
 273 merely demands that the fluid response is the same under fluid and solid vol-
 274 umetric changes, and therefore one may deduce that, for a given pressure p_f ,
 275 it must be $\zeta - e = -\operatorname{div}(\mathbf{u} + \mathbf{w}) = p_f/m$. When the fluid phase becomes in-
 276 compressible, i.e. $m \rightarrow +\infty$, one needs to specify how the pressure p_f behaves
 277 compared to m . If $p_f/m \rightarrow 0$, then zero net flow of both fluid and solid out of
 278 the control volume is approached and this limit amounts to an isochoric trans-
 279 formation. This line of reasoning led Biot to the concept of incompressible limit,
 280 as in Biot (1955). However, while the fluid may behave as incompressible, the
 281 foregoing analysis shows that the solid does not. In fact, the solid behaves just
 282 like an anisotropic elastic solid whose Stroh matrices (46) become unbounded
 283 as $r \sim m \rightarrow +\infty$. Besides, to support strong ellipticity (A.2), the elastic con-
 284 stants must also become unbounded, hence it is concluded that this limit is
 285 questionable. In fact, the actual physical regime is determined by the ratio $\tau_0\varepsilon$
 286 of the poroelastic effect to the elastic effect. In general, when $\tau_0\varepsilon = O(1)$, the
 287 solid behaves like an ordinary anisotropic solid whose material properties are
 288 affected by the fluid phase. Instead, in the weak limit $\tau_0\varepsilon \ll 1$, the fluid acts as
 289 a perturbation to the anisotropic solid and this perturbation operates similarly
 290 to incompressibility, yet in opposing fashion, i.e. a positive pressure accompa-
 291 nies positive volumetric changes. Finally, when $\tau_0\varepsilon \gg 1$, the solid behaves like a
 292 perturbation of an ideal liquid with small viscosity $O(\tau_0\varepsilon)^{-1}$ given by the elastic
 293 phase.

294 5. Conclusions

295 When deriving the Stroh-like formulation of a mechanical system, one is
296 confronted with the crucial step of designating the right variable pairs, which
297 unlock the full potential of the formalism. Recently, Fu (2007) pointed out that
298 energy conjugation is really the guiding tool which drives such designation, thus
299 getting away from guess-working and problem intuition, which may not suffice in
300 complex situations. Indeed, the Stroh formalism is really a canonical formalism
301 in the Hamiltonian sense, where a coordinate is treated in time-like fashion. In
302 this paper, we adopt this viewpoint to deal with Biot's reversible poroelasticity,
303 that dispenses with dissipation and occurs in the absence of a fluid pressure
304 gradient. This is the same framework as thermoelasticity of perfect conductors,
305 the pressure playing the role of temperature. Although this framework is insuf-
306 ficient to deal with any poroelastic problem, it may well provide the starting
307 point for the general formulation. Also, it investigates the most useful setting for
308 specimen testing. Spotlight is here set on emphasizing the canonical approach
309 and the features it brings out. Two formulations are derived: the first accounts
310 for both the solid and the fluid and it possesses, besides energy conservation,
311 translational invariance with respect to the fluid velocity. This feature, that is
312 a result of the absence of a pressure gradient, reveals constraints on the con-
313 jugate variables. To avoid dealing with such constraints, a second approach is
314 developed that is restricted to the solid skeleton only. The corresponding Stroh
315 formulation matches anisotropic elasticity where, however, the Stroh matrices
316 incorporate fluid coupling. Besides, strong ellipticity warrants their positive def-
317 inite character. Energy conservation and the impedance matrix follow naturally.
318 The special case of weak poroelasticity, whereby fluid-solid coupling is weaker
319 than the elastic response, is also investigated and shows remarkable similarities
320 with incompressible anisotropic elasticity with yet two important differences,
321 namely incompressibility acts as a small perturbation with opposite sign. This
322 analysis leads to reconsider the incompressible limit originally introduced by
323 Biot, that seems to show some inconsistencies.

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334 Conflict of Interest statement

335 The author has no conflict of interest to declare.

336 Data Availability

337 This paper makes use of no data.

338 Appendix: Strong ellipticity in reversible poroelasticity

339 We now discuss the role of strong ellipticity in poroelasticity in the absence
340 of dissipation. For a reversible process, we have a uniform pressure distribution
341 p_f and the motion equation for the solid skeleton is given by Eq.(10a) where we
342 write inertia explicitly (i.e. it is not hidden inside the Stroh matrices)

$$c_{ijkl}u_{k,lj} - r\zeta_{,j}\delta_{ij} = \rho\ddot{u}_i. \quad (\text{A.1})$$

As well known (Edmondson and Fu, 2009), strong ellipticity may be equally
retrieved demanding that the speed v of *any amplitude* propagating body wave
in *any direction* is real (and positive, without loss of generality). To this aim,
let's assume $\mathbf{u} = \boldsymbol{\alpha}e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)}$, whence

$$e = \alpha_k\beta_k e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)}, \quad \zeta = m^{-1} \left(p_f + r\alpha_k\beta_k e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)} \right).$$

Then, Eq.(A.1) becomes

$$c_{ijkl}\alpha_k\beta_l\beta_j - \frac{r^2}{m}\delta_{ij}\alpha_k\beta_k\beta_j = \rho v^2\alpha_i,$$

which, multiplied through by α_i and summed over i , gives

$$\left(c_{ijkl} - \frac{r^2}{m}\delta_{ij}\delta_{kl}\right)\alpha_i\alpha_k\beta_l\beta_j = \rho v^2\alpha_i\alpha_i > 0,$$

343 for any α, β different from zero. This is a variant of the incompressibility
344 constraint. In particular, letting $\beta = \mathbf{e}_2$, one gets that

$$\bar{\mathbf{T}} = \mathbf{T} - \frac{r^2}{m}\mathbf{e}_2 \otimes \mathbf{e}_2 \text{ is positive definite,} \quad (\text{A.2})$$

345 and therefore Eq.(40) may be solved.

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