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A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

--Manuscript Draft--

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Corresponding Author:	Andrea Nobili, PhD University of Modena and Reggio Emilia Modena, ITALY	
First Author:	Andrea Nobili, PhD	
Order of Authors:	Andrea Nobili, PhD	
Abstract:	<p>When investigating nonlinear wave propagation in slender hyperelastic rods, the usual stance is to construct a reduced kinematics and then derive a system of coupled nonlinear PDEs for the unknown functions. To make further analytical progress, the linear Love hypothesis, that connects longitudinal and transversal strain, is often reverted to. The viability of this assumption, that was originally proposed within the framework of linear elasticity, remains uncertain. In this paper, a refined Love hypothesis is derived in the weakly nonlinear regime by slow-time perturbation of the motion equations. For the sake of illustration, the simplest two-modal setting is adopted. This refined Love assumption is not equivalent, not even in principle, to that derived by Porubov (1993) by accommodating for the free boundary conditions at the rod mantle. Besides, the perturbation process lends a uni-dimensional model equation which parallels that obtained by Ostrovskii (1977) with the help of the linear Love hypothesis, with yet different coefficients in the dispersive term. The corresponding longitudinal motion is compared numerically against the solution of the bimodal nonlinear system and the transversal motion is contrasted with the linear Love hypothesis. For both motions, excellent agreement is found and the quality of the approximation extends to a wide range of values for the small parameter. Finally, within this setting, the corresponding unimodal Lagrangian is also derived, and it remains accurate regardless of the first correction terms to the linear Love hypothesis.</p>	
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Suggested Reviewers:	<p>Julius Kaplunov, PhD full professor, Keele University j.kaplunov@keele.ac.uk Established expert in the field of asymptotic methods</p> <p>Gennady Mishuris, PhD full professor, Aberystwyth University ggm@aber.ac.uk Expert in continuum mechanics</p> <p>Yibin Fu, PhD full professor, Keele University y.fu@keele.ac.uk Established expert in wave propagation</p> <p>Davide Bigoni, PhD full professor, University of Trento</p>	

davide.bigoni@unitn.it
Established expert in the field of reduced models

Gaetano Napoli, PhD
full professor, Università degli Studi di Napoli Federico II
gaetano.napoli@unina.it
Expert in the field of microstructured materials



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UNIVERSITÀ DEGLI STUDI DI
MODENA E REGGIO EMILIA

April 9th, 2024

Dear Editor,

Please find enclosed, the revised version of the research paper NLM-D-24-00069

“A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods”

by

A. Nobili

submitted for consideration in the International Journal of Non-Linear Mechanics, for the special issue “Multiscale and microstructure-inspired constitutive models for soft materials”.

The paper has been revised according to the Reviewers’ suggestions. Changes are highlighted in red color.

The material is original and it has not been submitted for publications in other journals.

The author is aware of the ethical guidelines put forward by the journal and fully adhere to it.

The author is in no conflict of interest regarding the material presented in this paper.

Best regards,

Andrea Nobili

(Corresponding author)

Research paper NLM-D-24-00069

A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

Reply to Reviewer #1

The author is grateful to the reviewer for his/her scrutiny of the manuscript and for his/her support.

Following upon the Reviewer's suggestion, in the Conclusion section the opportunity to extend the present analysis to rods that are reinforced with fibres, possibly made of soft material. For the Reviewer's convenience, corrections in the manuscript are marked in red.

Reply to Reviewer #2

The author is much appreciative for the honest assessment of the paper and for the many suggestions that are proposed by the Reviewer, especially in terms of literature review. In fact, it clearly appears that the Reviewer competence adds significant value to this work and I tried to make the most out of it. The author has put his best effort in addressing the Reviewer's objections with a fair and honest mindset, while collecting as much literature background as he could manage. The manuscript has been modified according to the Reviewer's suggestion as hereinafter detailed:

1. Overall, the approach is similar to that in

L. Ostrovsky, A. Sutin, 1977. Nonlinear elastic waves in rods. *J. Appl. Math. Mech. (PMM)* 41, 543-549.

In the 1977 paper the full Lagrangian of the problem was simplified using the Love hypothesis, and the reduced Lagrangian was used to derive a two-directional Boussinesq equation, and then the KdV approximation for uni-directional waves..

Answer.

Indeed, in this paper the Lagrangian approach is undertaken, much like in Ostrovsky, A. Sutin (1977) or Samsonov (2001) or Porubov (2003) or Dai and Fan (2004) or many more, but other than this the similarity is not great. In fact, it was precisely the point of this paper NOT to adopt the Love hypothesis firsthand but, rather, to (possibly) motivate it, specifically by a slow time perturbation. Indeed, the Love hypothesis is derived from the two-modal kinematics and in this sense it is no longer a hypothesis. Of course, in a weakly nonlinear setting the leading order solution is the linear Love hypothesis and therefore a two-directional Boussinesq equation is arrived at similarly to Ostrovsky, A. Sutin's. Yet, coefficients in the dispersive term (and also in the nonlinear term but this may be a typo in Ostrovsky, A. Sutin's paper) are different.

2. This was then followed by research in A. Samsonov's group, most notably Porubov and Samsonov, see

A. M. Samsonov, *Strain Solitons in Solids and How to Construct Them* (Monogr. Surv. Pure Appl. Math., Vol. 117), Chapman and Hall CRC, Boca Raton, Fla. (2001).

A. V. Porubov, *Amplification of Nonlinear Strain Waves in Solids* (Ser. Stab. Vibr. Control of Systems Ser. A, Vol. 9), World Scientific, Singapore (2003).

Porubov and Samsonov have refined the Love hypothesis, using the weakly-nonlinear

boundary conditions of the problem formulation, and used this weakly-nonlinear Love hypothesis in order to derive a refined model.

Answer

As pointed out by the Reviewer, the motivation behind this paper parallels that put forward in Samsonov (2001), Sec.2.3.2, where the need for a refined Love assumption is expressed. However, as confirmed by the Reviewer, in Samsonov's work this need is satisfied by matching the free boundary conditions, that is an altogether different approach than it is considered here. Indeed, this paper makes use of slow time perturbation of the motion equations and utterly disregards the free boundary conditions. In fact, it provides an alternative and different expression for the refinement of the Love hypothesis. This is now stressed in the manuscript to avoid confusion and a comparison is drawn. It should be also stressed that the fact that the free boundary conditions are disregarded is not a problem by itself, because, following Samsonov (2001), "generally speaking the identity is not required because an asymptotic solution is to be found". It is therefore important to emphasize that indeed a refined Love assumption is derived in Samsonov (2001) and yet this refinement is very different from the one that is proposed here, which purposefully makes no use of the boundary conditions for it aims to remain within a fully asymptotic solution.

3. Moreover, Garbuzov et al. (2019), mentioned in the paper, have derived Boussinesq-type equations using the multiple-scale analysis of the full problem formulation for a Murnaghan material, having again derived a weakly-nonlinear generalisation of the Love hypothesis.

Answer

Garbuzov et al. (2019) proceed in a similar way as Dai and Fan (2004) (and to an extent as Samsonov (2001) and Prubov (2003)) by introducing a multi-modal representation in the radius, then enforcing the motion equation in an asymptotic sense (i.e. term-wise) and finally accounting for force-type boundary conditions, either asymptotically or to linear terms. A second derivation is also presented which solves a set of nonlinear ODEs obtained by asymptotic solution of the boundary conditions.

Either way, this approach differs under many respects from that considered in this paper (for example and above all the fact that no mention is here made of the boundary conditions) and, indeed, it provides a model equation (in fact two of them) different than Ostrovsky, A. Sutin's in the dispersive term. Still, two interesting observations emerge from comparing Garbuzov et al. (2019)'s work with this paper. First, as observed by the Reviewer, despite its simplicity, this paper presents results that are qualitatively similar to those of other, much more involved works. Second, it appears that, to within this level of approximation, no need is felt to introduce the fulfillment of the boundary conditions.

4. The difference with the aforementioned studies is that in the present manuscript approximations have been built as a two-step process. First, a two-modal approximation (1) was assumed, hugely simplifying the problem at hand, and then the subsequent analysis has followed that in Dai and Fan (2004) and Garbuzov et al. (2019). It will be fair to comment that the derivation of the KdV equation in this hugely simplified setting could be performed by a student.

Answer

As stressed in the manuscript, "Spotlight is set on elucidating the procedure as well as on assessing the quality of the approximation in the simplest setting, leaving more sophisticated mechanical models for future developments." Indeed, the simplest possible setting is adopted, with the aim of communicating the method deprived of unnecessary details. For the same reason, the introduction of the boundary conditions is excluded, because, as well-

known and also pointed out by Samsonov (2001), they are not required in an asymptotic solution. Clearly, this by no means precludes that more involved situations may be addressed by this approach, also accounting for the boundary conditions. It is nonetheless emphasized that none of the literature contributions, to the best of my knowledge, rely on slow-time perturbation to reduce one of the motion equations to a Love-like assumption. Instead, the general approach is to either introduce some form of Love hypothesis, somehow guessed, or to adopt the reductive perturbation method to the full system, like, for example, in Dai and Fan (2004).

5. Finally, numerical comparisons were made between solutions of the two-modal approximation and the reduced model based on the Love hypothesis, showing good agreement in the range of validity of the model.

This last part should be put in the context of the study in

F.E. Garbuzov, Y.M. Beltukov, K.R. Khusnutdinova, Longitudinal bulk strain solitons in a hyperelastic rod with quadratic and cubic nonlinearities, *Theor. Math. Phys.* 202 (2020) 319-333,

where numerics was performed within the scope of the full nonlinear problem formulation for a Murnaghan material, and results of the weakly-nonlinear modelling with the KdV and extended KdV (i.e. higher-order model) were compared to that solution, showing excellent agreement for weakly-nonlinear waves, with the extended KdV equation extending the range of validity to the waves of moderate amplitude.

Answer

I thank the Reviewer for this reference. The procedure followed in Garbuzov et al (2020) exactly parallels that in Garbuzov et al. (2019) and therefore, on the basis of the previous comments, it is significantly different than that used in this paper. In terms of numerics, the soliton solution is investigated that is not the focus of interest of this paper, which instead compares the longitudinal motion against the wave equation and the nonlinear Mindlin-Herrmann system and the transversal motion against the linear Love hypothesis (which is the focus of this paper). None of the above is related in Garbuzov et al (2020). Still, this paper is now referred to in the manuscript.

6. Minor points: Boussinesq equation can not be integrated exactly, this wording is incorrect. We can construct certain classes of exact solutions, e.g. travelling waves.

Answer

This is indeed correct. The incorrect wording is taken verbatim from Soerensen (1984) “The BE contains U_{xxxx} and U^2_{xx} and is exactly integrable”. This is now better specified in the paper.

7. The other minor points raised by the Reviewer have been dealt with.

Finally, some rearrangements, rewriting and error corrections have been introduced in the manuscript.

For the Reviewer’s convenience, corrections in the manuscript are marked in red color.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The author is an Editorial Board Member/Editor-in-Chief/Associate Editor/Guest Editor for *[Journal name]* and was not involved in the editorial review or the decision to publish this article.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Highlights of the paper

A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

- A weakly nonlinear Love hypothesis is derived from a two-modal kinematics by the method of multiple scales;
- The corresponding model equation is the Boussinesq equation;
- Comparison with numerical integration reveals that the quality of the approximation is excellent;
- The same unimodal Lagrangian is derived irrespective of the correction to the linear Love hypothesis.

1 A weakly nonlinear Love hypothesis for longitudinal
2 waves in elastic rods

3 Andrea Nobili¹

4 ^a*Department of Engineering Enzo Ferrari, University of Modena and Reggio Emilia, via*
5 *Vivarelli 10, 41125 Modena, Italy*

6 **Abstract**

When investigating nonlinear wave propagation in slender hyperelastic rods, the usual stance is to construct a reduced kinematics and then derive a system of coupled nonlinear PDEs for the unknown functions. To make further analytical progress, the linear Love hypothesis, that connects longitudinal and transversal strain, is often reverted to. The viability of this assumption, that was originally proposed within the framework of linear elasticity, remains uncertain. In this paper, a refined Love hypothesis is derived in the weakly nonlinear regime by slow-time perturbation of the motion equations. For the sake of illustration, the simplest two-modal setting is adopted. This refined Love assumption is not equivalent, not even in principle, to that derived by Porubov and Samsonov (1993) by accommodating for the free boundary conditions at the rod mantle. Besides, the perturbation process lends a uni-dimensional model equation which parallels that obtained by Ostrovskii and Sutin (1977) with the help of the linear Love hypothesis, with yet different coefficients in the dispersive term. The corresponding longitudinal motion is compared numerically against the solution of the bimodal nonlinear system and the transversal motion is contrasted with the linear Love hypothesis. For both motions, excellent agreement is found and the quality of the approximation extends to a wide range of values for the small parameter. Finally, within this setting, the corresponding unimodal Lagrangian is also derived, and it remains accurate regardless of the first correction terms to the linear Love hypothesis.

7 *Keywords:* Love hypothesis, Murnaghan materials, Multiple scales, Nonlinear

¹Email: andrea.nobili@unimore.it, Tel.+39 059 2056217, Fax.+39 059 2056126
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9 **1. Introduction**

10 When developing models to describe wave propagation in elastic rods, the
11 starting point is usually represented by a restricted kinematics in which some
12 unknown functions are introduced expressing the longitudinal and the radial
13 strain in the body, see Shatalov et al. (2011) and references therein. Besides
14 the classical wave equation, the simplest case in point is the so-called bimodal
15 representation, where this restricted kinematics takes the form of the Navier–
16 Bernoulli (NB) assumption of plane cross-sections, in which only two functions
17 are used, respectively W for longitudinal and RU for transversal displacement,
18 R being the rod radial position. In the special case $U = 0$, this procedure
19 produces the classical non-dispersive wave equation. Moving from some original
20 intuitions of Rayleigh (1894), this approach was later refined by Love (1927)
21 to encompass for transversal inertia through the well-know *Love hypothesis*,
22 $U = -\nu_0 W_{,Z}$, that relates longitudinal and transversal strain via the Possion’s
23 ratio ν_0 . In Love’s original formulation (Love, 1927, Sec.278), this connection
24 is purposefully assumed *for inertia terms only*, in the so-called Rayleigh-Love
25 theory (on this point see also Hutchinson and Percival (1968)). This ad-hoc
26 procedure, which proves very effective, was later relaxed by Bishop (1952) to
27 include shear deformations, thus leading to the Bishop-Love theory. It may be
28 worth noting that the opposite pathway was taken by Sørensen et al. (1984) to
29 derive the improved Boussinesq equation in a simplified weakly nonlinear setting
30 where transversal inertia is neglected in favor of shear deformation. Mindlin
31 (1951) presents, for the first time, a full bimodal plane section approach and the
32 resulting pair of coupled PDEs is in fact named the Mindlin-Herrmann system
33 (Graff, 2012, Sec.8.3.3). In general, the bimodal approach is especially attractive
34 for its simplicity, although it remains limited in that it cannot accommodate
35 for all three boundary conditions on the mantle. Later, McNiven and Perry
36 (1962) remediated this shortcoming at the expense of introducing extra degrees

37 of freedom, by extending the transversal kinematics through successive odd
38 powers of the radius R multiplied by extra unknown functions.

39 In the linear framework, the quality of these approximations is usually as-
40 sessed by comparison with the well-known Pochhammer (1876) solution of the
41 3D elasticity problem for a rod with circular cross-section. Indeed, as already
42 observed by Graff (2012), the Love hypothesis is the leading order approxima-
43 tion of the Pochhammer-Chree solution in the long-wave low-frequency (LWLF)
44 approximation. This specific feature, that is further detailed in Nobili and Sac-
45 comandi (2024), cannot be pursued in the nonlinear framework because of the
46 insurmountable difficulties attached to developing any analytical solution within
47 the full 3D theory. In fact, avoiding such difficulties is precisely the main reason
48 why reduced-dimensional models are introduced in the first place. Still, while
49 facing the formidable task of solving complicated systems of coupled nonlinear
50 PDEs, many contributions appear in the literature that appeal to the original
51 linear Love hypothesis outside the linear framework where it properly belongs.

52 Sørensen et al. (1984) numerically study soliton interaction in nonlinear elas-
53 tic rods under many approximations that include the original Love hypothesis.
54 His interest lies in developing soliton solutions of nearly integrable systems.
55 Wright (1985) develops a purely axiomatic (in his words “intrinsic”) 1D theory
56 of straight elastic incompressible rods and is able to connect longitudinal and
57 transversal deformation through the incompressibility constraint. As pointed
58 out in Amendola and Saccomandi (2021), the incompressibility constraint is
59 compatible with the Love’s hypothesis only to leading order, while the analysis
60 in Wright (1985) is extended to the first correction term in the small deforma-
61 tion, i.e. it is weakly nonlinear. It is noted that a similar “intrinsic” 1D theory
62 was used very recently by Li et al. (2023) to derive nonlinear dispersion curves.

63 **Much research on the topic of nonlinear waves in rods is contributed by**
64 **Samsonov and his collaborators.** Samsonov (1994) assumes the linear Love
65 hypothesis to be valid in a fully nonlinear framework and derives the improved
66 Boussinesq equation. **Porubov and Samsonov (1993) propose a multi-modal**
67 **solution that satisfies the motion equation in the asymptotic sense, as well as**

68 the free boundary conditions in the Piola stress. In this process, a refined
69 Love assumption is introduced. According to Samsonov (2001), this analysis is
70 motivated by the desire to “confirm the Love hypothesis formally”, whose limit
71 “is that the boundary conditions on a free lateral surface were not properly
72 taken into account”, although “generally speaking, the identity is not required
73 because an asymptotic solution is to be found”. Precisely in this sense, and as
74 an alternative to Porubov and Samsonov’s approach, this paper adopts a slow
75 time perturbation of the motion equations to derive an asymptotically refined
76 Love hypothesis that is valid regardless of the boundary conditions.

77 Also Dai and collaborators have much contributed to this topic. Dai and
78 Huo (2000) study propagation of small-but-finite-amplitude (i.e. weakly nonlin-
79 ear) longitudinal waves in compressible rods by employing an asymptotic form
80 of the Love hypothesis, that is suggested by comparison with the incompress-
81 ible case. The reductive perturbation method of Jeffrey and Kawahara (1982)
82 is then adopted to derive the model equation valid in the far-field. This pro-
83 cedure is in essence a multiscale analysis similar to that used in this paper. In
84 Dai and Huo (2002), the reductive perturbation method is used in the incom-
85 pressible context to support the validity of the Navier-Bernoulli approximation
86 through comparing the resulting model equations. Dai and Fan (2004) intro-
87 duce four different 1D models to study longitudinal waves in a weakly nonlinear
88 Murnaghan material with an enriched kinematics that, departing from the NB
89 assumption, accommodates all free boundary condition on the mantle. How-
90 ever, when developing the far-field model, the linear Love’s assumption is again
91 reverted to. Dai and Fan (2004) are especially critical of Porubov and Samsonov
92 (1993)’s refined model because “still boundary conditions cannot be completely
93 satisfied even for linear terms”. Besides, while discussing Samsonov et al. (1998)
94 and Porubov et al. (1998), they observe “some serious algebraic errors in their
95 derivations [that] led the model equations to be unacceptable”.

96 More recently, Garbuzov et al. (2019) extend Samsonov (2001)’s approach
97 to the case of general force conditions at the rod mantle and longitudinal pre-
98 stretch. A family of Boussinesq-type model equations is obtained. The same

99 approach is adopted in Garbuzov et al. (2020) where a stable propagating table-
100 top soliton is observed numerically.

101 From the above literature review, it appears that several approaches to the
102 problem are possible which lead to widely different model equations. As pointed
103 out by Amendola and Saccomandi (2021), the adoption of the Love hypothesis
104 outside the realm of linear elasticity is questionable and calls for further investi-
105 gation. In particular, Nobili and Saccomandi (2024) make some progress in this
106 direction by showing that the original Love hypothesis may be equally obtained
107 from slow-time perturbation of the Mindlin-Herrmann model. The advantage
108 of this observation lies in that it provides a pathway to developing the equiva-
109 lent of the Love hypothesis outside the linear regime **and within the asymptotic**
110 **approach, i.e. without recourse to the boundary conditions.** This is precisely
111 the aim of this paper, which sets the spotlight on elucidating the procedure as
112 well as on assessing the quality of the approximation in the simplest possible
113 setting, namely the NB assumption and a bimodal representation (Sec.2). The
114 multiscale analysis is carried out in Sec.3 and the quality of the approxima-
115 tion is numerically investigated in Sec.4. A unimodal asymptotic Lagrangian is
116 illustrated in 5 and conclusions are finally drawn in Sec.6.

117 2. Mathematical background

118 Let us consider a rod that, in a reference configuration, is a circular cylinder
119 of radius A and let us introduce cylindrical coordinates in the current configu-
120 ration $\mathbf{x} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$ and, equally, cylindrical coordinates in the reference
121 configuration $\mathbf{X} = R\mathbf{E}_R + \Theta\mathbf{E}_\Theta + Z\mathbf{E}_Z$, with $0 \leq R \leq A$. Within this frame-
122 work **and in the absence of torsion**, the NB hypothesis consists of assuming the
123 following axisymmetric time dependent two-modal deformation (Wright, 1981,
124 Eq.(12))

$$125 \quad r = R + RU(Z, T), \quad \theta = \Theta, \quad z = Z + W(Z, T). \quad (1)$$

126 Letting the displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X}$ with components in our cylindrical
 127 reference system

$$128 \quad u_r(R, Z, t) = RU(Z, T), \quad u_\theta = 0, \quad u_z(Z, T) = W(Z, T), \quad (2)$$

129 the Lagrangian strain tensor immediately follows $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, namely

$$130 \quad \mathbf{E} = \begin{pmatrix} \frac{1}{2}U(U+2) & 0 & \frac{1}{2}R(U+1)U_{,Z} \\ 0 & \frac{1}{2}U(U+2) & 0 \\ \frac{1}{2}R(U+1)U_{,Z} & 0 & \frac{1}{2}(R^2U_{,Z}^2 + (W_{,Z} + 1)^2 - 1) \end{pmatrix}, \quad (3)$$

131 the linearized version of which is the infinitesimal strain tensor given, among
 132 many, in Nobili and Saccomandi (2024)

$$133 \quad \boldsymbol{\epsilon} = \begin{bmatrix} U & 0 & \frac{1}{2}RU_Z \\ 0 & U & 0 \\ \frac{1}{2}RU_Z & 0 & W_Z \end{bmatrix}. \quad (4)$$

134 Here, the identity (rank 2) tensor \mathbf{I} has been introduced as the invariant element
 135 in tensor composition, and it is understood that a comma subscript implies
 136 differentiation with respect to the following coordinate, i.e. $U_{,Z} = \partial U / \partial Z$. We
 137 let the quadratic (in the deformation components) invariants

$$138 \quad I_{11} = \text{tr}(\mathbf{E}), \quad I_{21} = \text{tr}(\mathbf{E}^2), \quad (5)$$

139 alongside the cubic deformation invariants

$$140 \quad I_{31} = I_{21}I_{11}, \quad I_{32} = I_{11}^3, \quad I_{34} = \text{tr}(\mathbf{E}^3). \quad (6)$$

141 Here, $\text{tr} \mathbf{E} = \frac{1}{2}(R^2U_{,Z}^2 + (W_{,Z} + 1)^2 - 1) + U(U + 2)$ denotes the usual trace
 142 operator. In the nonlinear framework, we take the isotropic Murnaghan strain-
 143 energy density per unit volume, see (Dai and Fan, 2004, Eq.(21))

$$144 \quad \mathcal{W} = \mu I_{21} + \frac{1}{2}\lambda I_{11}^2 + \mu \left(\kappa_1 I_{31} + \frac{1}{3}\kappa_2 I_{32} + \frac{1}{3}\kappa_4 I_{34} \right), \quad (7)$$

145 where μ and λ are the usual Lamé parameters and $\mu\kappa_i$, $i \in \{1, 2, 4\}$ are Mur-
 146 naghan material constants (the reason why it is three of them, instead of four,

147 is discussed in Ciarlet (2021)). The form (7) corresponds to the isotropic third-
 148 order Landau-Lifshitz constitutive relation.

149 Hamilton's principle is employed to determine the functions $U(Z, T)$ and
 150 $W(Z, T)$ through the Euler-Lagrange equations associated with the Lagrangian
 151 density $\mathcal{L} = \mathcal{T} - \mathcal{V}$. The kinetic energy density per unit length remain the same
 152 as that given by (Graff, 2012, Eq.(2.5.49)) or (Dai and Fan, 2004, Eq.(24))

$$153 \quad \mathcal{T} = \int_0^A \int_0^{2\pi} \frac{\rho}{2} (W_T^2 + R^2 U_T^2) R d\Theta dR = \frac{\pi A^2 \rho}{4} (2W_T^2 + A^2 U_T^2), \quad (8)$$

154 where ρ is the mass density in the reference configuration. In light of the as-
 155 sumed deformation (4), the strain-energy density (7) is given by Eq.(A.1) in the
 156 Appendix. The weakly nonlinear assumption affords great simplification of this
 157 otherwise cumbersome energy density. Within such framework, the displace-
 158 ment is small, i.e. $U \sim W_{,Z} \sim O(\varepsilon)$, with $\varepsilon \ll 1$, and only quadratic and cubic
 159 terms in ε are retained in the Lagrangian. We introduce the shorthands

$$160 \quad k_1 = 2(2\kappa_1 + \frac{4}{3}\kappa_2 + \frac{1}{3}\kappa_4 + \kappa^2 - 1), \quad k_2 = \kappa^2 - 1 + \kappa_1 + \frac{\kappa_4}{4}, \quad (9)$$

$$161 \quad k_3 = \kappa^2 - 2 + 2(\kappa_1 + 2\kappa_2), \quad k_4 = \frac{1}{2}\kappa^2 + \frac{1}{2}\kappa_1 + \frac{\kappa_4}{4}, \quad (10)$$

$$162 \quad k_5 = \kappa^2 - 2 + 2(\kappa_1 + \kappa_2), \quad k_6 = \frac{1}{2}\kappa^2 + \kappa_1 + \frac{\kappa_2 + \kappa_4}{3}, \quad (11)$$

163 where $\kappa = c_L/c_S$ is the ratio of $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ over $c_S = \sqrt{\mu/\rho}$, that are,
 164 respectively, the longitudinal and the transversal (shear) wave speed of linear
 165 elasticity. It is emphasized that the following constraints hold

$$166 \quad k_1 - k_5 - 2k_6 = 0, \quad -3k_1 + 8k_2 + 2k_3 = 2(1 + 2\lambda/\mu), \quad -3k_1 + 8k_4 + 4k_5 = 2(1 + \lambda/\mu). \quad (12)$$

167 Thus, the elastic energy density reads (Dai and Fan, 2004, Eq.(22))

$$168 \quad \mathcal{W} = \mu \left(\frac{1}{2} R^2 U_{,Z}^2 + 2U^2 + W_{,Z}^2 \right) + \frac{1}{2} \lambda (2U + W_{,Z})^2 \\
 169 \quad + \mu [k_1 U^3 + k_2 R^2 U U_{,Z}^2 + k_3 W_{,Z} U^2 + k_4 R^2 W_{,Z} U_{,Z}^2 + k_5 U W_{,Z}^2 + k_6 W_{,Z}^3], \quad (13)$$

170 which differs from the form adopted by Dai and Huo (2000) in Eq.(2.9). Inte-

171 grating over the cross section

$$172 \quad \mathcal{V} = \int_0^A \int_0^{2\pi} \mathcal{W} R d\Theta dR, \quad (14)$$

173 we obtain the potential energy per unit length (Dai and Fan, 2004, Eq.(23))

$$174 \quad \mathcal{V} = \pi A^2 \left\{ 2(\mu + \lambda)U^2 + 2\lambda U W_{,Z} + \left(\mu + \frac{1}{2}\lambda\right) W_{,Z}^2 + \frac{\mu A^2}{4} U_{,Z}^2 \right. \\ 175 \quad \left. + \mu \left[\frac{1}{2} A^2 U_{,Z}^2 (k_2 U + k_4 W_{,Z}) + k_3 U^2 W_{,Z} + k_5 U W_{,Z}^2 + k_1 U^3 + k_6 W_{,Z}^3 \right] \right\}. \quad (15)$$

176 We then form the Lagrangian

$$177 \quad \mathcal{L} = \mathcal{T} - \mathcal{V}, \quad (16)$$

178 and the corresponding Euler-Lagrange equations form a pair of nonlinear PDEs

179

$$180 \quad A^2 k_4 U_{,Z} U_{,ZZ} + 2U_{,Z} (k_3 U + k_5 W_{,Z} + \kappa^2 - 2) + W_{,ZZ} (2k_5 U + 6k_6 W_{,Z} + \kappa^2) \\ 181 \quad = c_S^{-2} W_{,TT} \quad (17a)$$

$$182 \quad -3k_1 U^2 - 2U (k_3 W_{,Z} + 2\kappa^2 - 2) - W_{,Z} (k_5 W_{,Z} + 2\kappa^2 - 4) \\ 183 \quad + \frac{1}{2} A^2 [k_2 U_{,Z}^2 + 2k_4 U_{,Z} W_{,ZZ} + U_{,ZZ} (2k_2 U + 2k_4 W_{,Z} + 1)] = \frac{1}{2} c_S^{-2} A^2 U_{,TT}, \quad (17b)$$

184 which provide a nonlinear generalization of the well-known Mindlin-Herrmann
185 system, where the Murnaghan strain energy density is used and small terms
186 in the deformation are retained up to $O(\varepsilon^2)$. These equations seem to differ
187 somehow from the corresponding equations (27) given by Dai and Fan (2004)
188 and specifically Eq.(27a) misses the term multiplying k_4 in (17a), while (27b)
189 misses several terms, such as those multiplying k_2 and k_4 . Clearly, the original
190 Mindlin-Herrmann system (Mindlin, 1951)

$$191 \quad (\lambda + 2\mu)W_{,ZZ} + 2\lambda U_{,Z} = \rho W_{,TT}, \quad (18) \\ \mu A^2 U_{,ZZ} - 8(\lambda + \mu)U - 4\lambda W_{,Z} = \rho A^2 U_{,TT},$$

192 is immediately retrieved when retaining only linear terms in (17). This reveals
193 that the system is a perturbation of a pair of coupled wave equations. This

194 coupled system provides the time evolution of U and W once suitable initial
 195 and boundary conditions are given. Since only second order time derivatives
 196 appear (linearly), initial conditions take the form

$$197 \quad U(Z, 0) = U_0(Z), W(Z, 0) = W_0(Z) \quad (19a)$$

$$198 \quad U_{,T}(Z, 0) = \dot{U}_0(Z), W_{,T}(Z, 0) = \dot{W}_0(Z). \quad (19b)$$

199 Also, in light of the fact that the highest space derivative is two, a pair of
 200 boundary conditions is equally required. In general, solving Eqs.(17) calls for
 201 numerical methods, as it occurs in Sec.4. A way around the solution of the full
 202 system is to impose the Love (L-) *hypothesis*, that assumes a linear relationship
 203 between the radial displacement and the longitudinal strain i.e

$$204 \quad U = -\nu_0 W_{,Z}, \quad (20)$$

205 where

$$206 \quad \nu_0 = \frac{\lambda}{2(\lambda + \mu)} = \frac{1}{2} \frac{\kappa^2 - 2}{\kappa^2 - 1},$$

207 is the Poisson's ratio. A nice derivation of the Love model in the context of in-
 208 variant manifolds from nonlinear dynamical systems theory is given by Roberts
 209 (1993). As it was originally proposed by [Porubov and Samsonov \(1993\)](#), [Sam-](#)
 210 [sonov \(2001\)](#) and later argued by Amendola and Saccomandi (2021), Love's
 211 hypothesis belongs to the linear framework and should be suitably generalized
 212 to the nonlinear setting. This is precisely the aim of the next Section.

213 **3. Love hypothesis for weakly nonlinear elasticity**

214 Let's begin by assuming that the deformation is small and yet not so small
 215 that third order terms in the Lagrangian may be neglected. Next, let's introduce
 216 the dimensionless coordinates

$$217 \quad \zeta = Z/l, \quad t = T/\mathbb{T},$$

218 where l is a typical wavelength and $\mathbb{T} = l/c_S$ is a reference time. Hence, the
 219 dimensionless small parameter $\delta = A^2/l^2$ naturally emerges, cf. Garbuzov et al.

220 (2019). Next, we need to introduce the weakly nonlinear hypothesis, and specif-
 221 ically its connection to the idea that the solid is slender, namely that $\delta \ll 1$.
 222 For this we introduce a second small parameter, ε , which is a measure of the
 223 magnitude of the deformation, i.e. $U \sim W_{,Z} = O(\varepsilon)$. As customary, we also as-
 224 sume that differentiation does not affect the asymptotic order of the unknowns.
 225 In this form, the problem is multi-parametric and to make further progress we
 226 need to assume the reciprocal relation between the small parameters δ and ε . In
 227 this paper, we assume $\varepsilon = \delta$, meaning that the deformation gets smaller as the
 228 rod becomes slender in linear fashion. **This distinct limit amounts to assuming**
 229 **that nonlinearity and dispersion are in balance and small enough (Samsonov,**
 230 **2001)**. Indeed, according to Ablowitz (2011), this is the “maximum balance
 231 model” wherein nonlinear and dispersive effects are equally important. Other
 232 choices are of course possible but we don’t pursue them in here. Thus, we let

$$233 \quad U = \delta u(\zeta, \tau) \quad \text{and} \quad W = \delta l w(\zeta, \tau). \quad (21)$$

234 The E-L equations may be given to first order in δ

$$235 \quad -4(\kappa^2 - 1)u - 2(\kappa^2 - 2)w_{,\zeta} + \frac{1}{2}\delta [u_{,\zeta\zeta} - 6k_1u^2 - 4k_3uw_{,\zeta} - 2k_5w_{,\zeta}^2] + O(\delta^2) = \frac{1}{2}\delta u_{,tt}, \quad (22a)$$

$$236 \quad 2(\kappa^2 - 2)u_{,\zeta} + \kappa^2 w_{,\zeta\zeta} + \delta [2u_{,\zeta}(k_3u + k_5w_{,\zeta}) + (2k_5u + 6k_6w_{,\zeta})w_{,\zeta\zeta}] + O(\delta^2) = w_{,tt}. \quad (22b)$$

237
 238 Following Nobili (2021), the motion equations are perturbed in the slow time
 239 $\tau = t\delta$, having let the moving axial coordinate $\xi = \zeta - ct$. Consequently, a
 240 straightforward expansion in δ is introduced

$$241 \quad u = \phi_0(\xi, \tau) + \delta\phi_1(\xi, \tau) + \dots, \quad w = \psi_0(\xi, \tau) + \delta\psi_1(\xi, \tau) + \dots$$

242 To leading order, the system already obtained in Nobili and Saccomandi (2024)
 243 is obtained,

$$244 \quad (4 - 2\kappa^2)\psi_{0,\xi} - 4(\kappa^2 - 1)\phi_0 = 0, \quad (23)$$

$$245 \quad (\kappa^2 - c^2)\psi_{0,\xi\xi} + 2(\kappa^2 - 2)\phi_{0,\xi} = 0. \quad (24)$$

246 This system lends the trivial solution unless

$$247 \quad c = \pm \hat{c}_B, \quad \text{with} \quad \hat{c}_B = \sqrt{\frac{3\kappa^2 - 4}{\kappa^2 - 1}},$$

248 that, multiplied by c_S , gives the dimensional longitudinal wave speed in rods
 249 $c_B = \sqrt{E/\rho}$, where $E = \mu \frac{3\lambda+2\mu}{\lambda+\mu} > 0$ is Young's modulus. Then, the leading
 250 order eigenform is obtained

$$251 \quad \phi_0 = -\nu_0 \psi_{0,\xi}, \quad (25)$$

252 that is precisely the Love assumption, which remains valid at leading order,
 253 with the understanding that a moving coordinate frame is considered.

254 Moving to the next order, we find

$$255 \quad \phi_1 + \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} \psi_{0,\xi}^2 - \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} \psi_{0,\xi\xi\xi} + \nu_0 \psi_{1,\xi} = 0, \quad (26a)$$

$$256 \quad (\kappa^2 - 2) \phi_{1,\xi} + \hat{c}_B \psi_{0,\xi\tau} + (3k_6 - 2\nu_0 k_5 + \nu_0^2 k_3) \psi_{0,\xi} \psi_{0,\xi\xi} + \nu_0 (\kappa^2 - 2) \psi_{1,\xi\xi} = 0. \quad (26b)$$

257 Remarkably, this system shows no dependence on k_2 and k_4 which only come at
 258 higher order and are therefore disregarded. Clearly, Eq.(26a) immediately lends
 259 the first correction to Love hypothesis

$$260 \quad \phi_1 = -\nu_0 \psi_{1,\xi} + \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} \psi_{0,\xi\xi\xi} - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} \psi_{0,\xi}^2, \quad (27)$$

261 which consists of the linear terms, already appreciated in Nobili and Saccomandi
 262 (2024), together with a nonlinear contribution which depends on Murnaghan's
 263 moduli. Together, Eq.(25) and (27) give the refined Love hypothesis in the
 264 weakly nonlinear setting

$$265 \quad \phi = -\nu_0 \psi_{,\xi} + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} \psi_{,\xi\xi\xi} - \delta \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} \psi_{,\xi}^2 + O(\delta^2), \quad (28)$$

266 and, in particular, we see that the linear Love hypothesis, in its simplicity, only
 267 conveys the first of such terms, and certainly misses the nonlinear correction.
 268 Plugging Eq.(27) into Eq.(26b), yields the governing equation for the perturba-
 269 tion to leading order,

$$270 \quad \frac{1}{4} \nu_0^2 (\hat{c}_B^2 - 1) \Psi_{,\xi\xi\xi} + \hat{c}_B \Psi_{,\tau} + 3\beta_1 \Psi \Psi_{,\xi} = 0, \quad \beta_1 = -\nu_0^3 k_1 + \nu_0^2 k_3 - \nu_0 k_5 + k_6, \quad (29)$$

271 that is the well-known Korteweg-de Vries (KdV) equation for **the longitudinal**
 272 **strain** $\Psi = \psi_{0,\xi}$. This equation corresponds to **the far-field model (4.18)** of Dai
 273 and Huo (2002) **that is valid for an incompressible elastic rod, whence coefficients**
 274 **are different**. Space differentiation of the KdV and exploiting the connection

$$275 \quad \frac{\partial^2}{\partial \xi \partial \tau} = \frac{1}{2} \delta^{-1} \hat{c}_B \left(\frac{\partial^2}{\partial \zeta \partial \zeta} - \hat{c}_B^{-2} \frac{\partial^2}{\partial t \partial t} \right) + O(\delta),$$

276 lends the *Boussinesq equation* (BE)

$$277 \quad \Psi_{,\zeta\zeta} - \hat{c}_B^{-2} \Psi_{,tt} + \frac{\delta}{\hat{c}_B^2} \left[\frac{1}{2} \nu_0^2 (\hat{c}_B^2 - 1) \Psi_{,\zeta\zeta} + 3\beta_1 \Psi^2 \right]_{,\zeta\zeta} = 0, \quad (30)$$

278 that **admits soliton-like travelling solutions**, see Bullough and Caudrey (1980).
 279 **The BE features poor existence and uniqueness properties and, in particular,**
 280 **no local well-posedness result is available**. To the same order of approximation,
 281 the asymptotically equivalent form may be obtained (this process, sometimes,
 282 is called "regularization")

$$283 \quad \Psi_{,\zeta\zeta} - \hat{c}_B^{-2} \Psi_{,tt} + \frac{\delta}{\hat{c}_B^2} \left[\frac{1}{2} \nu_0^2 (1 - \hat{c}_B^{-2}) \Psi_{,tt} + 3\beta_1 \Psi^2 \right]_{,\zeta\zeta} = 0, \quad (31)$$

284 that goes under the name of *improved Boussinesq equation* (IBE). **The IBE is**
 285 **far superior to the BE in that it is well-posed**.

286 Eq.(29) may be put to advantage in order to eliminate the third derivative in
 287 (28). Indeed, differentiating (28) with respect to ξ and using (29), one finds an
 288 asymptotically equivalent form of the refined weakly-nonlinear Love hypothesis

$$289 \quad \phi_{,\xi} = -\nu_0 \psi_{,\xi\xi} - \delta \frac{\hat{c}_B}{\kappa^2 - 2} \psi_{,\xi\tau} - \delta \frac{\nu_0 (k_3 \nu_0 - 2k_5) + 3k_6}{\kappa^2 - 2} \psi_{,\xi} \psi_{,\xi\xi} + O(\delta^2). \quad (32)$$

290 Going back to the original variables, the KdV equation (29) becomes

$$291 \quad w_{,\zeta\zeta} - \hat{c}_B^{-2} w_{,tt} + \delta \hat{c}_B^{-2} \left[\frac{1}{2} \nu_0^2 (\hat{c}_B^2 - 1) w_{,\zeta\zeta\zeta\zeta} + 6\beta_1 w_{,\zeta} w_{,\zeta\zeta} \right] = O(\delta^2), \quad (33)$$

292 that, to first correction terms, gives

$$293 \quad W_{,ZZ} - \hat{c}_B^{-2} W_{,TT} + \nu_0^2 K^2 (1 - \hat{c}_B^{-2}) W_{,ZZZZ} + 3\hat{c}_B^{-2} \beta_1 (W_{,Z})_{,Z}^2 = 0, \quad (34)$$

294 where $K = A/\sqrt{2}$ is the polar radius of gyration of the cross-section. Further-
 295 more, Eq.(33) may be rewritten, within the same order of accuracy, as

$$296 \quad w_{,\zeta\zeta} - \hat{c}_B^{-2} w_{,tt} + \delta \hat{c}_B^{-2} \left[\frac{1}{2} \nu_0^2 \hat{c}_B^{-2} (\hat{c}_B^2 - 1) w_{,\zeta\zeta tt} + 6\beta_1 w_{,\zeta} w_{,\zeta\zeta} \right] = O(\delta^2), \quad (35)$$

297 which has the undoubted advantage of being of order 2 in space as well as in
 298 time and therefore it requires only one pair of spatial boundary condition and
 299 one pair of initial conditions. In the original variables and to first correction, it
 300 becomes

$$301 \quad W_{,ZZ} - c_B^{-2} W_{,TT} + \nu_0^2 K^2 c_B^{-2} (1 - \hat{c}_B^{-2}) W_{,ZZTT} + 3\hat{c}_B^{-2} \beta_1 (W_{,Z})_{,Z}^2 = 0. \quad (36)$$

302 Recalling that

$$303 \quad \beta_1 = E/\rho + \kappa_1(1 - 2\nu_0)(1 + 2\nu_0^2) + \frac{1}{3}\kappa_2(1 - 2\nu_0)^3 + \frac{1}{3}\kappa_4(1 - 2\nu_0^3), \quad (37)$$

304 Eq.(36) corresponds to Eq.(1.4) of Ostrovskii and Sutin (1977), provided the
 305 coefficient $1 - \hat{c}_B^{-2}$ in the dispersion term is suppressed and the factor 1/2 added
 306 in the nonlinear term (the latter appears in Garbuzov et al. (2019)'s writing
 307 of Ostrovskii and Sutin's result). This partial correspondence may not be too
 308 surprising, given that Ostrovskii and Sutin obtain their equation via the linear
 309 Love hypothesis and therefore miss the correction terms. Conversely, the model
 310 equation (34) in Dai and Fan (2004), that is obtained using the linear Love hy-
 311 pothesis, also appears in Porubov and Samsonov (1993) and it is a Boussinesq-
 312 type combination of the BE and of the IBE, that is named by Garbuzov et al.
 313 (2019) the "doubly dispersive equation" (DDE). In fact, Eq.(45) in Garbuzov
 314 et al. (2019) provides yet another model equation. As pointed out by Garbuzov
 315 et al. (2019), "the models [by Porubov and Samsonov (1993) and by Ostrovskii
 316 and Sutin (1977) (adding the missing 1/2 coefficient)] have different dispersive
 317 properties", although, after "regularization", they are all asymptotically equiv-
 318 alent. Furthermore, "it would be interesting to compare the performance of
 319 these four nonlinear models with the exact (numerical) solution of the nonlinear
 320 problem", that is indeed what is carried out in Sec.4 for the present model. Be-
 321 sides being equivalent, it is also clear that these models are also asymptotically
 322 consistent up to $O(\delta)$ terms. In contrast, however, they are not the same to
 323 those in Dai and Huo (2000) and Dai and Fan (2004) (where, presumably in
 324 Eq.(33) and (34) W should be written in the place of U).

325 Similarly, the refined Love assumption (32) becomes

$$326 \quad u_{,\zeta} = -\nu_0 w_{,\zeta\zeta} - \frac{1}{2(\kappa^2 - 2)} (\hat{c}_B^2 w_{,\zeta\zeta} - w_{,tt}) - \delta \frac{\nu_0 (k_3 \nu_0 - 2k_5) + 3k_6}{\kappa^2 - 2} w_{,\zeta} w_{,\zeta\zeta} + O(\delta^2), \quad (38)$$

327 that, to first correction terms, gives

$$328 \quad U_{,Z} = -\nu_0 W_{,ZZ} - \frac{\hat{c}_B^2}{2(\kappa^2 - 2)} (W_{,ZZ} - c_B^{-2} W_{,TT}) - \frac{\nu_0 (k_3 \nu_0 - 2k_5) + 3k_6}{\kappa^2 - 2} W_{,Z} W_{,ZZ}. \quad (39)$$

329 The refined Love hypothesis (28), in the original variables, reads

$$330 \quad U = -\nu_0 W_{,Z} + K^2 \nu_0 \frac{\hat{c}_B^2 - 1}{4(\kappa^2 - 1)} W_{,ZZZ} - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} W_{,Z}^2, \quad (40)$$

331 and it may be compared with Eq.(2.52) in Samsonov (2001), which, however,
 332 emerges from a completely different setting, by enforcing the free boundary
 333 conditions. The two equations reveal a similar structure, although coefficients
 334 are very different for each correction term (dispersive and nonlinear). In fact,
 335 according to Samsonov, the linear correction term sign is opposite and exhibits
 336 a quadratic dependence on the radius R (whence it disappears on the rod axis).
 337 Nonetheless, this structural correspondence is somewhat remarkable, consider-
 338 ing that equations emerge from very different assumptions.

339 4. Numerical results

340 We now show how the model equation (33) compares to the nonlinear cou-
 341 pled system (17) and to the solution of the wave equation. Specifically, we are
 342 interested in the accuracy of the transversal motion, with special regard to the
 343 nonlinear contribution. It is pointed out that this is not the same comparison
 344 that is presented in Garbuzov et al. (2020), which instead focuses on propa-
 345 gating solitons. We consider the dimensionless variables $\delta u(\zeta, t)$ and $\delta w(\zeta, t)$,
 346 where δ is introduced to properly scale quantities as in (21). We choose the
 347 parameter set

$$348 \quad \kappa = 3, k_1 = 1.1, k_2 = 1, k_3 = 5, k_4 = 2, k_5 = 0.5, k_6 = 0.3,$$

349 with the aim to produce a significant nonlinear contribution. A few numerical
 350 experiments show that this particular choice of parameters is of little importance

351 for the overall picture. In contrast, it is crucial to take for δ a small value:
 352 hereinafter we begin with $\delta = 0.1$, which warrants equally small initial conditions
 353 to remain within the weakly nonlinear hypothesis. In the following, we assume
 354 a periodic system, because our approximation relies on the far-field concept
 355 which is at odd with a finite domain. Therefore, we chose a periodic boundary
 356 conditions, namely

$$357 \quad w(-\pi, t) = w(\pi, t), \quad u(-\pi, t) = u(\pi, t). \quad (41)$$

358 in the period range $\zeta \in [-\pi, \pi]$. For the initial condition on w , we assume

$$359 \quad w(\zeta, 0) = \sin(\zeta), \quad (42)$$

360 and, for the sake of definiteness, we assume zero initial velocity, i.e.

$$361 \quad w_{,t}(\zeta, 0) = u_{,t}(\zeta, 0) \equiv 0. \quad (43)$$

362 For $u(\zeta, 0)$, we adopt the linear Love assumption, whereby

$$363 \quad u(\zeta, 0) = -\nu_0 w_{,\zeta}(\zeta, 0) = -\nu_0 \cos \zeta, \quad (44)$$

364 with the understanding that this choice works in favor of the accuracy of Love's
 365 linear hypothesis (25).

366 To determine the approximating solution, we integrate the Boussinesq-type
 367 Eq.(33) by standard methods. Here, it is important to point out that a fourth
 368 space derivative appears, which calls for two sets of boundary conditions, pre-
 369 cisely as in (41). To bring about periodicity on u , we call upon (28) and thereby
 370 impose

$$371 \quad -\nu_0 u_{,\zeta}(-\pi, t) + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} u_{,\zeta\zeta\zeta}(-\pi, t) - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} u_{,\zeta}^2(-\pi, t) \\
 372 \quad = -\nu_0 u_{,\zeta}(\pi, t) + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} u_{,\zeta\zeta\zeta}(\pi, t) - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} u_{,\zeta}^2(\pi, t).$$

373 Finally, for the wave equation, we have the analytic solution

$$374 \quad w(\zeta, t) = \frac{1}{2} (\sin(\zeta + c_B t) - \sin(\zeta - c_B t)), \quad (45)$$

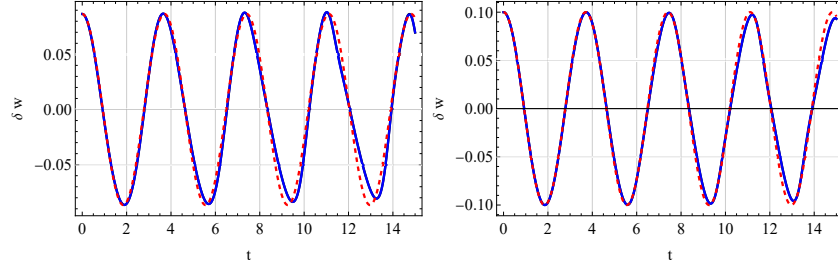


Figure 1: $\delta w(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

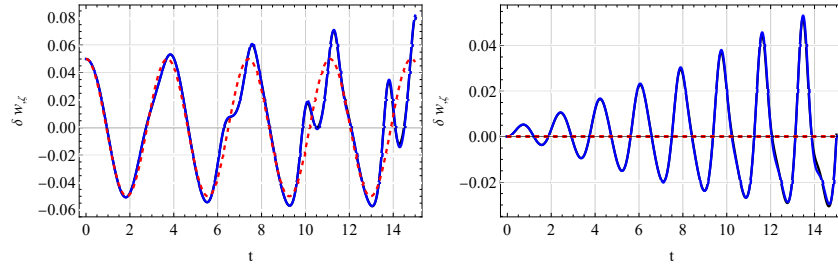


Figure 2: $\delta w_{,\zeta}(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

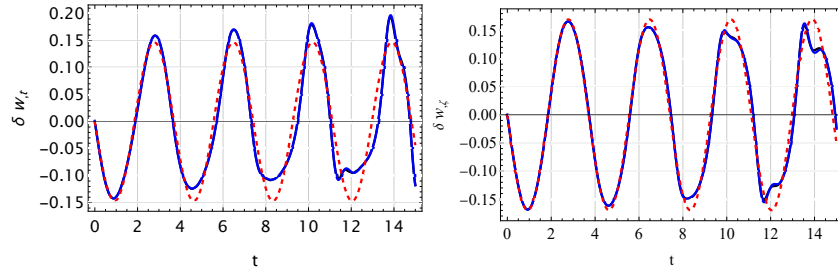


Figure 3: $\delta w_{,t}(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

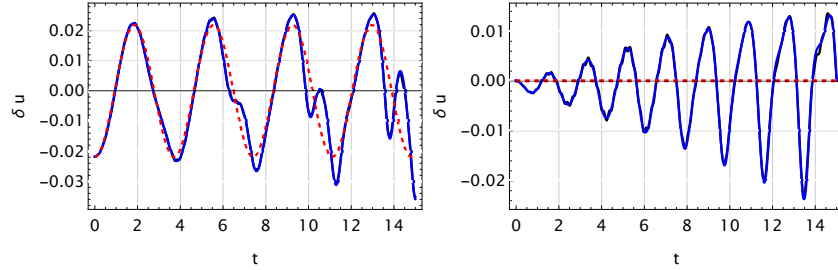


Figure 4: $\delta u(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the refined nonlinear Love hypothesis (blue, solid) and the linear Love hypothesis applied to the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

375 which disposes of the initial velocity, according to the first of Eqs.(43).

376 Figure 1 plots $\delta w(\zeta, t)$ as it emerges from the numerical solution of the non-
 377 linear Mindlin-Herrmann system (17), from the numerical solution of the KdV
 378 (33) and finally from solving the wave equation. Clearly, all three solutions ap-
 379 pear very close, at least initially, the deviation from the wave equations building
 380 up slowly in time. Yet, it is interesting to look at corresponding plots for the
 381 space derivative of w , that are shown in Fig.2 where it clearly appears that
 382 the wave equation is unable to reproduce the features of the solution. This is
 383 especially true for the position $\zeta = \pi/2$ because there the wave equation solu-
 384 tion vanishes, thus leaving only the nonlinear term as the leading source for the
 385 solution. In contrast, the numerical solution of the KdV offers an outstanding
 386 approximation, which can be hardly resolved from the Mindlin-Herrmann sys-
 387 tem. The same comparison is given in Fig.3 with respect to the time derivative
 388 of w and similar conclusions may be drawn.

389 We now compare the numerical solution for δu obtained from solving the
 390 nonlinear Mindlin-Herrmann system (17) with the refined Love hypothesis (28)
 391 and with the linear Love hypothesis (25) applied to the solution of the wave
 392 equation (namely a $O(1)$ solution). Fig.4 presents this comparison for $\zeta =$
 393 $\pi/3$ and for $\zeta = \pi/2$ and confirms the excellent approximation offered by the
 394 refined Love hypothesis, that is indeed capable of reproducing the nonlinear
 395 contributions over time. Once again, the location $\zeta = \pi/2$ is expedient to rule

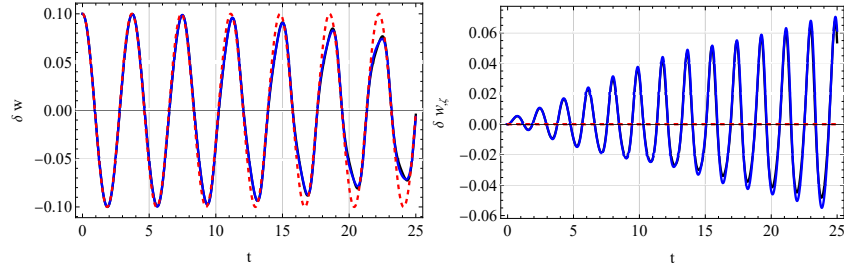


Figure 5: $\delta w(\pi/2, t)$ (left) and $\delta w_\zeta(\pi/2, t)$ (right) as a function of t for $\delta = 0.2$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV (blue, solid) and the solution of the wave equation (red, dashed). The solid curves begin to resolve for $t > 10$ in the derivative only.

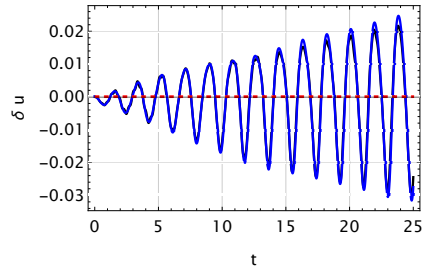


Figure 6: $\delta u(\pi/2, t)$ as a function of t for $\delta = 0.2$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the refined nonlinear Love hypothesis (blue, solid) and the linear Love hypothesis applied to the solution of the wave equation (red, dashed). The latter is zero throughout. The solid curves exhibit some small differences at large times.

396 out any contribution from the linearized solution and single out the outcomes
 397 of the nonlinear terms. It should be emphasized that obtaining the numerical
 398 solution of the Mindlin-Herrmann system is rather delicate and time consuming,
 399 especially for long time frames.

400 We now investigate the robustness of the approximation in dependence of
 401 the parameter δ . Fig.5 shows the displacement δw and its space derivative δw_ζ
 402 at $\zeta = \pi/2$, having let $\delta = 0.2$. Some little deviations of the KdV approxima-
 403 tion may be appreciated for long times in the space derivative only. A similar
 404 comparison, this time for u and the refined Love hypothesis, is plotted in Fig.6
 405 where again little deviations begin to appear at large times. This trend becomes
 406 more evident for $\delta = 0.25$ and Fig.7 reveals that the nonlinear approximation
 407 deteriorates substantially over time, although it still fares a lot better than the

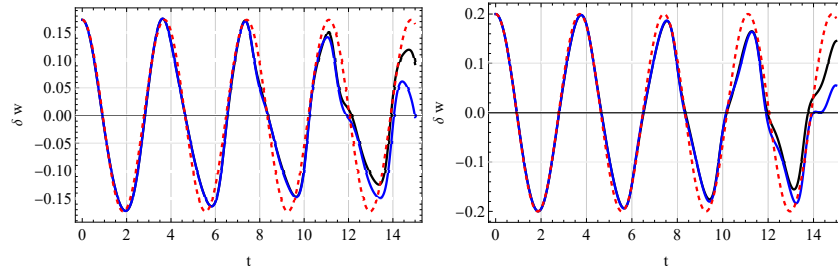


Figure 7: $\delta w(\pi/3, t)$ (left) and $\delta w(\pi/2, t)$ (right) as a function of t for $\delta = 0.25$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV (blue, solid) and the solution of the wave equation (red, dashed). The solid curves begin to resolve for $t > 10$ in both locations.

408 wave equation.

409 5. Unimodal Lagrangian

410 Plugging the refined Love hypothesis (40) into the Lagrangian (16) lends a
 411 unimodal system whose only variable is W . As already observed in Nobili and
 412 Saccomandi (2024), the corresponding Euler-Lagrange equation corresponds to
 413 the Love equation (Love, 1927, §278)

$$414 \quad W_{,ZZ} + \frac{\nu_0^2 K^2}{c_B^2} W_{,ZZTT} = \frac{W_{,TT}}{c_B^2},$$

415 only to leading order, while different coefficients are obtained in the first cor-
 416 rection. This outcome, that is a result of the fact that the Love Lagrangian
 417 accommodates for transversal motion only in the inertia term and not in the
 418 elastic potential, occurs regardless of the correction terms to the linear Love
 419 hypothesis (20). Accordingly, the first correction plays no role in affecting the
 420 E-L equation of the unimodal Lagrangian up to first order. Besides, comparing
 421 the model equation against the Pochhammer-Chree solution of linear elasticity
 422 as in Dai and Fan (2004) to assess its validity brings little value, because we
 423 already know that the bimodal kinematics is doomed to fail (and equally so
 424 in the case of richer models, provided that they are power series corrections of
 425 the bimodal system). Yet, it is even more remarkable that the E-L equation
 426 of the unimodal cubic Lagrangian corresponds to the BE (34) *regardless* of the

427 correction terms. In other words, plugging the linear Love hypothesis (20) into
 428 the Murnaghan Lagrangian (16) yields the correct model equation, be it (34)
 429 or (36). This time, this outcome is a consequence of the first constraint in the
 430 Murnaghan material parameters (12). It is concluded that, as it was the case
 431 in linear elasticity, plugging a family of Love assumptions, which all differ from
 432 the linear Love hypothesis (20) by first correction terms into the nonlinear La-
 433 grangian (16) (and regularizing) always leads the model equations (34) or (36).
 434 Indeed, the unimodal Lagrangian corresponding to (34) is given by

$$435 \quad \mathcal{L}_1 = -\frac{1}{2}W_{,Z}^2 + \frac{1}{2}c_B^{-2}W_{,T}^2 + \frac{1}{2}\nu_0^2K^2(1 - \hat{c}_B^{-2})W_{,ZZ}^2 - \hat{c}_B^{-2}\beta_1W_{,Z}^3. \quad (46)$$

436 Similarly, letting the Love Lagrangian Graff (2012)

$$437 \quad \mathcal{L}^* = -\frac{1}{2}W_{,Z}^2 + \frac{1}{2}c_B^{-2}W_{,T}^2 + \frac{1}{2}\nu_0^2K^2c_B^{-2}W_{,ZT}^2, \quad (47)$$

438 we see that the corrected Lagrangian

$$439 \quad \mathcal{L}_2 = \mathcal{L}^* - \frac{1}{2}\nu_0^2K^2c_B^{-2}\hat{c}_B^{-2}W_{,ZT}^2 - \hat{c}_B^{-2}\beta_1W_{,Z}^3, \quad (48)$$

440 yields the E-L equation (36).

441 6. Conclusions

442 **In this paper, slow time perturbation of the motion equations is proposed to**
 443 **systematically derive the (weakly) nonlinear counterpart of the Love hypothe-**
 444 **sis of linear elasticity. For the sake of illustration, the method is shown in the**
 445 **simplest possible setting.** Indeed, a bimodal kinematics is assumed to study
 446 longitudinal waves propagating in Murnaghan hyperelastic straight rods and,
 447 as a result, a complicated pair of coupled nonlinear PDEs arises, which is the
 448 nonlinear generalization of the Mindlin-Herrmann system of linear elasticity.
 449 This nonlinear system is difficult to analyze and calls for numerical investiga-
 450 tion. Alternatively, a refined weakly nonlinear Love hypothesis is derived by
 451 perturbation of the motion equation in slow time. This relation may be seen as
 452 a generalization of the linear hypothesis originally proposed by Love (1927) in
 453 the context of linear elasticity. **In the process, a unidimensional model equation**

454 is obtained for longitudinal strain, that, in the moving frame, is the celebrated
455 KdV. In the stationary frame, the KdV turns into the Boussinesq and into the
456 improved Boussinesq equations, which are asymptotically equivalent. In fact, as
457 noted by Garbuzov et al. (2019), the asymptotic approach justifies the presence
458 of many different model equations in the literature. In terms of longitudinal
459 displacement, the Boussinesq-type model of Ostrovskii and Sutin (1977) is re-
460 trieved, which was obtained by the linear Love hypothesis, although coefficients
461 in the dispersive term are different (the nonlinear term is also different by a 1/2
462 factor which, however, may be a typo).

463 The solution of either Boussinesq equation is compared against the numerical
464 solution of the nonlinear Mindlin-Herrmann system and remarkable accuracy is
465 found, both in terms of longitudinal as well as transversal displacement (and
466 their derivatives alike). In fact, assessment of the accuracy of the transversal
467 displacement, as provided by the refined nonlinear Love equation, is especially
468 interesting and seems not yet explored in the literature. Besides, the accuracy
469 of the approximation seems unexpectedly robust for not-so-small values of the
470 small parameter δ . This procedure may be easily generalized to more involved
471 scenarios, such as the presence of reinforcing fibres or exotic constitutive models,
472 see Amendola et al. (2024). Finally, a unimodal Lagrangian is derived which
473 proves capable of producing (either of) the Boussinesq equations *regardless* of
474 the first correction terms. This surprising outcome emerges in light of the
475 restriction that exists on the Murnaghan material parameters and, possibly,
476 it may be a by-product of the bimodal kinematics.

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489 **Conflict of Interest statement**

490 The author has no conflict of interest to declare.

491 **Data Availability**

492 This paper makes use of no data.

493 **Appendix A. Murnaghan strain energy density**

494 The Murnaghan strain energy density, within the restricted kinematics (1),
495 reads

$$\begin{aligned}
\mathcal{W} = & \mu (2U^2 + W_{,Z}^2 + \frac{1}{2}R^2U_{,Z}^2) + \frac{1}{2}\lambda \left[\frac{1}{2} (R^2U_{,Z}^2 + (W_{,Z} + 1)^2 - 1) + U(U + 2) \right]^2 \\
& + \frac{1}{24}\mu \left\{ \kappa_2 (R^2U_{,Z}^2 + 2U^2 + 4U + W_{,Z} (W_{,Z} + 2))^3 \right. \\
& + 3\kappa_1 (2U^2 (R^2U_{,Z}^2 + 4) + 2R^2U_{,Z}^2 (W_{,Z} + 1)^2 + 4R^2UU_{,Z}^2 + R^4U_{,Z}^4 + 2U^4 + 8U^3 + W_{,Z}^2 (W_{,Z} + 2)^2) \\
& \quad \times (R^2U_{,Z}^2 + 2U^2 + 4U + W_{,Z} (W_{,Z} + 2)) \\
& + \kappa_4 \left[3R^2U^2U_{,Z}^2 (R^2U_{,Z}^2 + W_{,Z}^2 + 2W_{,Z} + 5) + 3U^4 (R^2U_{,Z}^2 + 8) + 4U^3 (3R^2U_{,Z}^2 + 4) \right. \\
& + 6R^2UU_{,Z}^2 (R^2U_{,Z}^2 + (W_{,Z} + 1)^2) + 3R^4U_{,Z}^4 (W_{,Z} + 1)^2 + 3R^2U_{,Z}^2W_{,Z} (W_{,Z} + 1)^2 (W_{,Z} + 2) \\
& \quad \left. + R^6U_{,Z}^6 + 2U^6 + 12U^5 + W_{,Z}^3 (W_{,Z} + 2)^3 \right] \\
& \left. + 6 (2R^2U_{,Z}^2 (W_{,Z} + 1)^2 + 4R^2UU_{,Z}^2 + R^4U_{,Z}^4 + 2U^4 + 8U^3 + W_{,Z}^4 + 4W_{,Z}^3) \right\}.
\end{aligned}
\tag{A.1}$$

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: