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UNIMORE
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Topology in colored tensor models

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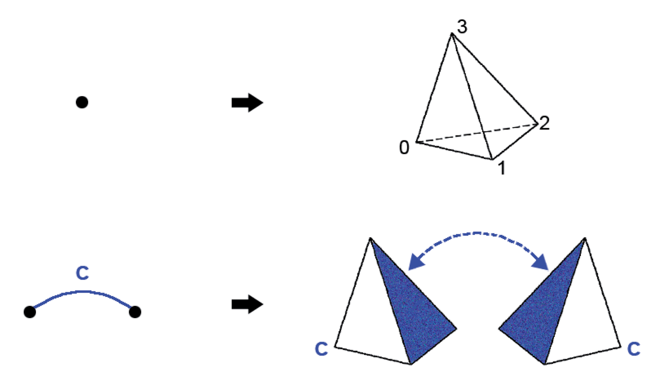
1. Edge-colored graphs

A ***$(d+1)$-colored graph*** is (Γ, γ), with:

- $\Gamma = (V(\Gamma), E(\Gamma))$ regular graph of degree $d+1$
- $\gamma : E(\Gamma) \to \Delta_d = \{0, \dots, d\}$ such that $\gamma(e) \neq \gamma(f)$ for adjacent edges $e, f \in E(\Gamma)$ (*edge-coloration*)

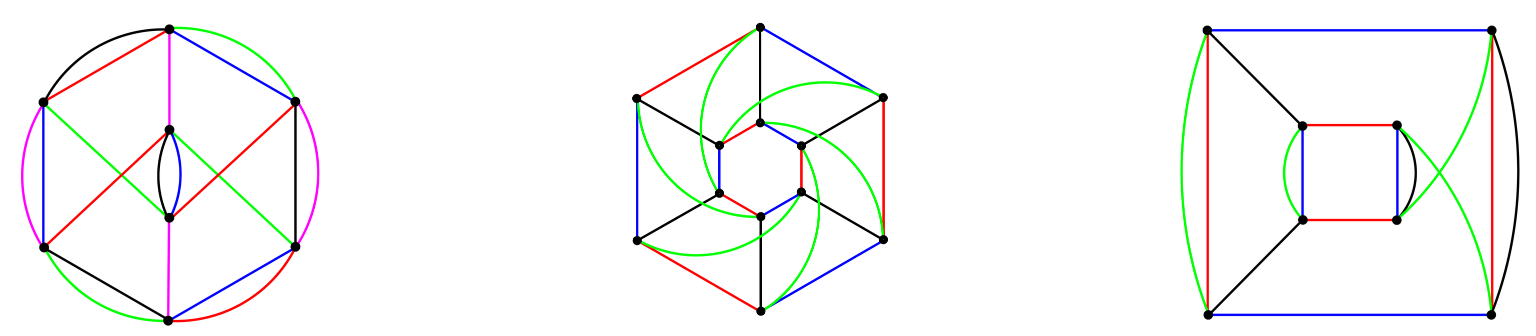
A ***colored pseudocomplex*** $K(\Gamma)$ is associated to (Γ, γ) :



from (Γ, γ) to $K(\Gamma)$, if $d = 3$

Existence Theorem: *Any orientable (resp. non-orientable) PL manifold M^d admits a bipartite (resp. non-bipartite) colored graph (Γ, γ)* ***representing*** *it, i.e. such that $M^d \cong_{PL} |K(\Gamma)|$.*

The same result holds for *singular d-manifolds*. As a consequence, $(d+1)$-colored graphs can be used to represent d-manifolds with boundary, too.

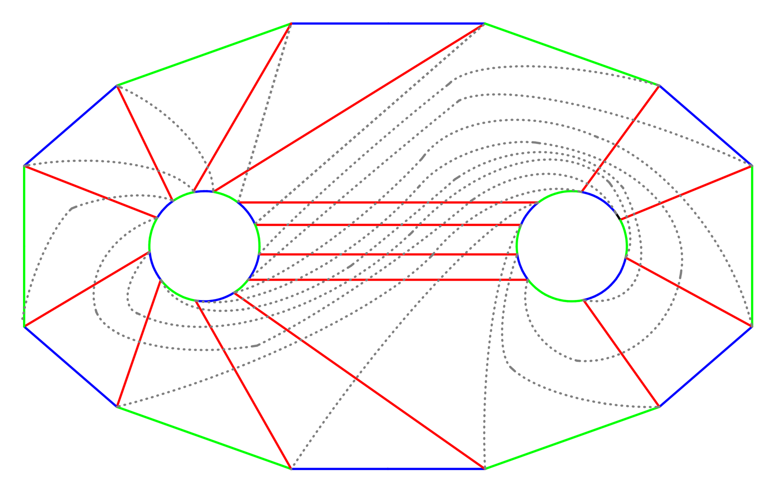


$\mathbb{CP}^2$ $\quad$ $L(3,1)$ $\quad$ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$

2. G-degree and gem-complexity

For each $(d+1)$-colored graph (Γ, γ) and for every cyclic permutation ε of Δ_d, there exists a ***regular embedding*** of Γ onto a suitable surface F_ε.

Example: Regular embedding corresponding to $\varepsilon =$ (green, red, blue, grey)



$\rho_\varepsilon(\Gamma) = 2$, with Γ representing the Poincaré homology sphere

- ***regular genus of Γ with respect to ε***: $\rho_\epsilon(\Gamma) = genus(F_\varepsilon)$ (or its half, if F_ε is non-orientable, i.e. Γ non-bipartite).
- ***G-degree of Γ***: $\boldsymbol{\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)}$
  where the $\varepsilon^{(i)}$'s are the cyclic permutations of Δ_d up to inverse.
- ***G-degree of a PL d-manifold M^d***:
$$\mathcal{D}(M^d) = \min\left\{\omega_G(\Gamma) \;/\; |K(\Gamma)| \cong_{PL} M^d\right\}$$
- ***gem-complexity of a PL d-manifold M^d***:
$$k(M^d) = \min\left\{\tfrac{1}{2}(\#V(\Gamma)) - 1 \;/\; |K(\Gamma)| \cong_{PL} M^d\right\}$$

3. Colored tensor models

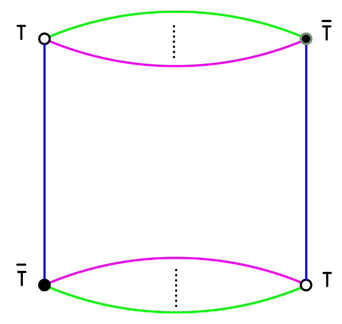
A ***$(d+1)$-dimensional colored tensor model*** is a formal partition function

$$\mathcal{Z}[N, \{\alpha_B\}] := \int_{\mathrm{f}} \frac{dT d\overline{T}}{(2\pi)^{N^d}} \exp(-N^{d-1}\overline{T} \cdot T + \sum_B \alpha_B B(T, \overline{T})),$$

where T belongs to $(\mathbb{C}^N)^{\otimes d}$, $\overline{T}$ to its dual and $B(T, \overline{T})$ are *trace invariants* obtained by contracting the indices of the components of T and $\overline{T}$.
In this framework, colored graphs naturally arise as Feynman graphs encoding tensor trace invariants:

Example (quartic invariant):

$$Q(T, \overline{T}) = \sum_{\substack{i_1, \dots, i_d = 1 \\ j_1, \dots, j_d = 1}}^{N} \overline{T}_{i_1, i_2, \dots, i_d} T_{j_1, i_2, \dots, i_d} \overline{T}_{j_1, j_2, \dots, j_d} T_{i_1, j_2, \dots, j_d}$$



- White (black) **vertices** for T ($\overline{T}$)
- **Edges colored** by the position of the index

$1/N$-expansion of the free energy:

$$\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}] = \sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!}\omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]],$$

where the coefficients $F_{\omega_G}[\{t_B\}]$ are generating functions of connected bipartite $(d+1)$-colored graphs with fixed G-degree ω_G.

Motivations and trends

From a "geometric topology" point of view, the theory of **manifold representation by means of edge-colored graphs** has been deeply studied since 1975 and many results have been achieved: its great advantage is the possibility of encoding, in any dimension, every PL d-manifold by means of a totally combinatorial tool.
Edge-colored graphs also play an important rôle within **colored tensor models theory, considered as a possible approach to the study of Quantum Gravity**: the key tool is the G-degree of the involved graphs, which drives the 1/N expansion in the higher dimensional tensor models context, exactly as it happens for the genus of surfaces in the two-dimensional matrix model setting.
Therefore, topological and geometrical properties of the represented PL manifolds, with respect to the G-degree, have specific relevance in the tensor models framework, showing **a direct fruitful interaction between tensor models and discrete geometry, via edge-colored graphs**.

In colored tensor models, manifolds and pseudomanifolds are (almost) on the same footing, since they constitute the class of polyhedra represented by edge-colored Feynman graphs arising in this context; thus, a promising research trend is to look for classification results concerning all pseudomanifolds represented by graphs of a given G-degree. In dimension 4, the goal has already been achieved - via singular 4-manifolds - for all compact PL 4-manifolds with connected boundary up to G-degree 24.

In the same dimension, the existence of colored graphs encoding different PL manifolds with the same underlying TOP manifold, suggests also to investigate the ability of tensor models to accurately reflect geometric degrees of freedom of Quantum Gravity.

4. Results in dimension d

- If Γ is bipartite, $\boldsymbol{\omega_G(\Gamma) < \frac{d!}{2} \implies |K(\Gamma)| \cong_{PL} \mathbb{S}^d}$
- The G-degree is finite-to-one within the class of PL manifolds. The same result holds for singular manifolds with a fixed number of singularities.
- Suppose d even and $d \geq 4$, then:
  Γ bipartite or Γ representing a singular d-manifold $\Longrightarrow \boldsymbol{\omega_G(\Gamma) \equiv 0 \mod (d-1)!}$

5. Results in dimension 3

- For any 3-manifold M^3:
$$\boldsymbol{\mathcal{D}_G(M^3) \;=\; k(M^3)}$$
- If Γ represents a prime, handle-free orientable (resp. non-orientable) 3-manifold M^3, the topological classification of M^3 is known up to $\omega_G(\Gamma) = 32$ (resp. $\omega_G(\Gamma) = 30$).
- If Γ represents an orientable 3-dimensional singular manifold M^3, the topological classification of M^3 is known up to $\omega_G(\Gamma) = 6$.

6. Results in dimension 4

- For each Γ and for each pair $(\varepsilon, \varepsilon')$ of associated permutations of Δ_4,
$$\omega_G(\Gamma) \;=\; 6\Big(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma)\Big)$$
- For any PL 4-manifold M^4:
$$\boldsymbol{\mathcal{D}_G(M^4) = 6(\underbrace{k(M^4)}_{\text{PL}} + \underbrace{\chi(M^4)}_{\text{TOP}} - 2)}$$
- If Γ represents an orientable PL 4-manifold M^4 and $\omega_G(\Gamma) < 48$, then M^4 is PL-homeomorphic to $\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1, \mathbb{CP}^2, \#_2(\mathbb{S}^3 \times \mathbb{S}^1), \#_3(\mathbb{S}^3 \times \mathbb{S}^1)$ or $(\mathbb{S}^3 \times \mathbb{S}^1)\#\mathbb{CP}^2$.
- If Γ represents a non-orientable PL 4-manifold M^4 and $\omega_G(\Gamma) < 36$, then M^4 is PL-homeomorphic to $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ or $\#_2(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$.
- PL 4-manifolds N and N' exist, so that $\mathcal{D}_G(N \# N') \neq \mathcal{D}_G(N) + \mathcal{D}_G(N')$
- If Γ represents a simply-connected PL 4-manifold M^4 and $\omega_G(\Gamma) \leq 527$, then M^4 is TOP-homeomorphic to
$$(\#_r \mathbb{CP}^2)\#(\#_{r'}(-\mathbb{CP}^2)) \quad \text{or} \quad \#_s(\mathbb{S}^2 \times \mathbb{S}^2),$$
  where $r + r' = \beta_2(M^4)$ and $s = \frac{1}{2}\beta_2(M^4)$, with $\beta_2(M^4) \leq \frac{1}{24} \cdot \omega_G(\Gamma)$.

References

- C. Gagliardi: *Regular imbeddings of edge-coloured graphs*, Geom. Dedicata, 11, 397–414 (1981).
- R. Gurau, V. Rivasseau: *The 1/N expansion of colored tensor models in arbitrary dimension*, Europhys. Lett., 95(5), 50004 (2011).
- R. Gurau: *Random Tensors*, Oxford University Press, 2016.
- M.R. Casali, P. Cristofori, C. Gagliardi: *Classifying PL 4-manifolds via crystallizations: results and open problems*, in: "A Mathematical Tribute to Professor José María Montesinos Amilibia", Universidad Complutense Madrid (2016). [ISBN: 978-84-608-1684-3]
- M.R. Casali, P. Cristofori, S. Dartois, L. Grasselli: *Topology in colored tensor models via crystallization theory*, J. Geom. Phys., 129, 142–167 (2018). https://doi.org/10.1016/j.geomphys.2018.01.001
- M.R. Casali, P. Cristofori, L. Grasselli: *G-degree for singular manifolds*, RACSAM 112(3), 693-704 (2018). https://doi.org/10.1007/s13398-017-0456-x
- M.R. Casali, L. Grasselli: *Combinatorial properties of the G-degree*, Revista Matemática Complutense 32, 239-254 (2019). https://doi.org/10.1007/s13163-018-0279-0
- M.R. Casali, P. Cristofori: *Classifying compact 4-manifolds via generalized regular genus and G-degree*, Ann. Inst. Henri Poincarè D (Combinatorics, Physics and their interactions) 10 (1), 121-158 (2023). https://doi.org/10.4171/aihpd/128