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A regularity result for the p-laplacian near uniform ellipticity / Mercuri, C.; Riey, G.; Sciunzi, B.. - In: SIAM JOURNAL ON MATHEMATICAL ANALYSIS. - ISSN 0036-1410. - 48:3(2016), pp. 2059-2075.  
[10.1137/16M1058546]

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### **Paper:**

Mercuri, C., Riey, G. & Sciunzi, B. (in press). A regularity result for the  $\Delta_p$ -laplacian near uniform ellipticity. *SIAM Journal on Mathematical Analysis*

<http://dx.doi.org/10.1137/16M1058546>

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# A REGULARITY RESULT FOR THE P-LAPLACIAN NEAR UNIFORM ELLIPTICITY

CARLO MERCURI, GIUSEPPE RIEY, AND BERARDINO SCIUNZI

ABSTRACT. We consider weak solutions to a class of Dirichlet boundary value problems involving the  $p$ -Laplace operator, and prove that the second weak derivatives are in  $L^q$  with  $q$  as large as it is desirable, provided  $p$  is sufficiently close to  $p_0 = 2$ . We show that this phenomenon is driven by the classical Calderón-Zygmund constant. As a byproduct of our analysis we show that  $C^{1,\alpha}$  regularity improves up to  $C^{1,1^-}$ , when  $p$  is close enough to 2. This result we believe it is particularly interesting in higher dimensions  $n > 2$ , when optimal  $C^{1,\alpha}$  regularity is related to the optimal regularity of  $p$ -harmonic mappings, which is still open (see e.g. [27]).

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## 1. INTRODUCTION AND RESULTS

In this paper we deal with the  $W^{2,q}$  regularity of the weak solutions to

$$(1.1) \quad \begin{cases} -\Delta_p u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $p > 1$ ,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator,  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^n$ ,  $n \geq 2$ , and

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*2010 Mathematics Subject Classification.* 35J92, 35B33, 35B06.

$$f \in \begin{cases} W^{1,r}(\Omega), & r \in (n, \infty], & \text{if } p > 2, \\ C(\bar{\Omega}), & & \text{if } p \leq 2. \end{cases}$$

Namely, we consider possibly sign-changing functions  $u \in W_0^{1,p}(\Omega)$  such that

$$(1.2) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi \, dx = \int_{\Omega} f \psi \, dx,$$

for all  $\psi \in W_0^{1,p}(\Omega)$ .

It is well-known that under our assumptions  $u \in C^{1,\alpha}(\bar{\Omega})$ , for some  $\alpha < 1$ , as it follows by the classical results [8, 12, 28]. Furthermore, results on the optimal Hölder exponent  $\alpha$  are also known in the literature, see e.g. the recent paper [27] that is also based on previous results obtained in [16].

It is worth mentioning that, since by classical Morrey's embedding theorem our function  $f$  is (up to the boundary Hölder) continuous, the notion of weak and viscosity solutions (see e.g. [5]) coincide. This follows by the result of [15]; the recent paper [14] contains a new interesting proof of this known fact.

Here we address the study of the summability of the second derivatives of the solutions on the whole  $\Omega$ . From [7], [25, 26]) it is known under the above assumptions that  $u \in W_{\text{loc}}^{2,2}(\Omega)$  if  $1 < p < 3$ , and that if  $p \geq 3$  and the source term  $f$  is strictly positive then  $u \in W_{\text{loc}}^{2,q}(\Omega)$  for  $q < \frac{p-1}{p-2}$ . We observe that, in the case  $p \geq 3$ , it is possible to construct examples which show that such a regularity is optimal, see e.g. [26]. The above-mentioned regularity results are obtained exploiting improved weighted estimates on the summability of the second derivatives, see e.g. [28]. Note that the Calderón-Zygmund theory cannot be extended trivially to the context of quasilinear elliptic problems. The interested reader is referred to [20, 21] and the references therein.

As a preliminary observation we first would like to point out that the aforementioned weighted estimates on the second derivatives holds up to the boundary in those cases where one can handle problems caused by the intersection of the critical set  $Z_u = \{x \in \bar{\Omega} : \nabla u = \mathbf{0}\}$  with  $\partial\Omega$ . We have the following

**Proposition 1.1.** *Let  $p > 1$  and let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ , and  $u$  be weak solution of (1.1) with  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ . Assume that  $f \in W^{1,r}(\Omega)$  for some  $r \in (n, \infty]$  and for any  $p > 1$ . Then, for any  $\beta < 1$  there exists a constant  $C_\beta = C_\beta(n, p, f) > 0$  such that*

$$(1.3) \quad \int_{\Omega} |\nabla u|^{p-2-\beta} \|D^2 u\|^2 < C_\beta.$$

Here  $D^2 u$  denotes the Hessian matrix of  $u$ , and  $\|D^2 u\|$  is any equivalent norm of it. This implies in particular the global regularity  $u \in W^{2,2}(\Omega)$  if  $1 < p < 3$ . Note that no sign assumption on  $u$  nor on the source term  $f$  is required.

*Proof.* By a classical result of G. Stampacchia it is well-known that the second derivatives of  $u$  vanish almost everywhere on the critical set  $Z_u$ . From Corollary 2.1 [7] (see also [25, 26]) we have

$$(1.4) \quad \int_{\omega} |\nabla u|^{p-2-\beta} \|D^2 u\|^2 < \infty$$

on every open set  $\omega \subset \Omega$  which is strictly contained in  $\Omega$ . This estimate holds in fact up to the boundary. This can be proved word by word as in Corollary 2.1 in [7] simply replacing in

the proof Hopf's boundary point lemma by our boundary assumption on the gradient. And this concludes the proof.  $\square$

Note that, as we will discuss later on, the general assumption  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$  is fulfilled e.g. in all those cases when the Hopf boundary lemma applies (see [30]). We are now able to highlight a regularising effect that occurs when the quasilinear equation approaches the semilinear one, namely when  $p$  approaches  $p_0 = 2$ . This phenomenon is driven by the classical elliptic regularity theory, namely by the constant  $C(n, q)$  in the Calderón-Zygmund estimate

$$(1.5) \quad \|D^2 w\|_{L^q(\Omega)} \leq C(n, q) \|\Delta w\|_{L^q(\Omega)},$$

see e.g. Corollary 9.10 in [9]. We point out that it is because of this result that we restrict ourselves to solutions which vanish on  $\partial\Omega$ .

To describe this phenomenon let us start recalling that, formally we have

$$\Delta_p u = |\nabla u|^{p-2} \Delta u + (p-2) |\nabla u|^{p-4} \Delta_\infty u$$

where  $\Delta_\infty u = (D^2 u \nabla u, \nabla u)$ . Still formally, using the above decomposition, we can rewrite our equation as

$$-\Delta u = (p-2) \frac{\Delta_\infty u}{|\nabla u|^2} + \frac{f}{|\nabla u|^{p-2}}.$$

Then we show that the term  $(p-2) |\nabla u|^{-2} \Delta_\infty u$  is negligible when  $p$  is close to  $p_0 = 2$  and apply the standard Calderón-Zygmund theory. Note that, to do this, we also need information on the summability of the term  $\frac{f}{|\nabla u|^{p-2}}$  that we deduce, following [7], as a consequence of the aforementioned weighted estimate (1.3). Such information is provided by the following

**Proposition 1.2.** *Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ , and  $u$  be weak solution to (1.1) with  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ . Then, for  $p > 2$  and for any fixed  $1 \leq q < \frac{p-1}{p-2}$ , there exists a constant  $C = C(n, p, q, f) > 0$  such that*

$$(1.6) \quad \int_{\Omega} \frac{f^2}{\left[|\nabla u|^{p-2}\right]^q} \leq C.$$

The same bound holds for any  $1 \leq q < +\infty$  when  $1 < p \leq 2$ .

This proposition is proved in Section 2. As a consequence of the above proposition we have the following

**Theorem 1.3.** *Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ , and  $u$  be weak solution to (1.1) with  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ . Let  $q \geq 2$  and  $p$  be such that*

$$|p-2| < \frac{1}{C(n, q)},$$

with  $C(n, q)$  given by (1.5). Then, if  $1 < p \leq 2$ , there holds

$$u \in W^{2,q}(\Omega).$$

In the case  $p > 2$ , the same conclusion holds provided  $p$  is in addition such that  $q < \frac{p-1}{p-2}$ . As a consequence, for any  $\gamma \in (0, 1)$  there exist values  $1 < p_1 < 2 < p_2$  such that for all  $p \in (p_1, p_2)$  there holds  $u \in C^{1,\gamma}(\overline{\Omega})$ .

As an application of the above result we consider

$$(1.7) \quad \begin{cases} -\Delta_p u = u^{p^*} - \Lambda u^{p-1} & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^n$ ,  $1 < p < n$ , and  $p^* = np/(n-p)$  is the critical Sobolev exponent. The case  $p = 2$  and  $n \geq 3$  had been extensively studied since the pioneering papers of Brezis and Nirenberg [2], Coron [4], and Bahri and Coron [1], inspiring a very broad literature on related existence, non-existence, multiplicity, symmetry, and classification results. The case  $p \neq 2$  has several interesting features and related open problems, see e.g. [10, 11] and more recently [3], [17],[18], [19]. A solution  $u$  to problem (1.7) can be found for suitable  $\Lambda$  as nonnegative constrained minimiser, by overcoming well-known lack of compactness phenomena for the minimising (or Palais-Smale) sequences which typically occur in the presence of the critical Sobolev exponent. In particular  $u \in L^\infty(\Omega)$  by variant of Moser's iteration (see e.g. Appendix E in [23], and [29]), and hence  $u \in C^{1,\alpha}(\overline{\Omega})$ . The positivity is due to the maximum principle ([30], [24]). In particular Hopf's boundary point lemma implies that  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ , therefore  $u$  satisfies the hypotheses of Theorem 1.3.

We point out that the basic regularity estimate given by Proposition 1.1 on the second derivatives up the boundary, is known essentially only in those cases when Hopf's boundary point lemma (see [30]) can be applied, this to rule out the existence of critical points of the solutions along the boundary. For this reason Hopf's boundary point lemma allows to use standard elliptic regularity theory. However there are many cases of interest when the assumptions of Hopf's lemma are not satisfied. In fact, if for instance we consider a sign-changing solution whose nodal line touches the boundary, then necessarily, at the touching point, the gradient of  $u$  vanishes. This is the case when the second eigenfunction of the  $p$ -Laplacian with Dirichlet boundary condition is considered:

$$(1.8) \quad \begin{cases} -\Delta_p u_2 = \lambda_2 |u_2|^{p-2} u_2 & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case  $u_2$  has exactly two nodal regions, as it had been pointed out in [6], see also [13]. Motivated by the above problem and similar ones, it is natural to try to obtain regularity results of the same flavour of Theorem 1.3 which possibly hold for those equations involving a source term of the form  $f(u)$ , allowed to be strictly negative near the boundary.

To this aim, for  $p > 1$  and  $n \geq 2$  we consider the following boundary value problem. Let  $u$  be a weak solution of:

$$(1.9) \quad \begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and only when  $p > 2$  we assume in addition that  $f$  is locally Lipschitz satisfying

$$\begin{aligned} H1) \quad & \exists \gamma > 0, k > 0 : \lim_{t \rightarrow 0} \frac{f(t)}{|t|^{k-1}t} = \gamma \\ H2) \quad & f(0) = 0, \quad f(t) \cdot t > 0 \quad \forall t \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

**Theorem 1.4.** *Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ ,  $f$  continuous, and  $u \in C^{1,\alpha}(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$  be weak solution to (1.9). Let  $q \geq 2$  and  $p$  be such that*

$$|p - 2| < \frac{1}{C(n, q)},$$

with  $C(n, q)$  given by (1.5).

Then, if  $1 < p \leq 2$ , there holds

$$u \in W^{2,q}(\Omega).$$

In the case  $p > 2$ , the same conclusion holds provided  $f$  is locally Lipschitz and satisfies  $H1), H2)$ , and  $p, q$  and  $k$  are such that

$$(1.10) \quad \max\left(\frac{2(k+1)}{k}, q\right) < \frac{p-1}{p-2}.$$

As a consequence, for any  $\gamma \in (0, 1)$  there exist values  $1 < p_1 < 2 < p_2$  such that for all  $p \in (p_1, p_2)$  there holds  $u \in C^{1,\gamma}(\bar{\Omega})$ .

Going back to our model problem (1.8) we can see that Theorem 1.4 is applicable to  $u_2$ , where we set  $k := p - 1$ . In particular as far as the degenerate case  $p > 2$  is concerned, if  $q < \frac{2(k+1)}{k} = \frac{2p}{p-1}$ , then the condition  $\frac{2(k+1)}{k} < \frac{p-1}{p-2}$  holds if and only if  $p < 1 + \sqrt{2}$ . Then  $u \in W^{2,q}(\Omega)$ , provided  $p < \min\left(2 + \frac{1}{C(n,q)}, 1 + \sqrt{2}\right)$ .

In order to prove Theorem 1.4 we need a weaker form of equation (1.3) which holds in the case  $p > 2$  when we do not assume  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ . To this aim we perform a linearisation argument which is in the spirit of [7] and [25, 26].

**1.1. Related questions.** There are several related questions which are left open by the present paper to be considered in future projects. We believe that the major ones are the following.

A) Determining an optimal value for the constant  $C(n, q)$  involved in the Calderón-Zygmund estimate (1.5) would give sharper regularity results in Theorem 1.3 and Theorem 1.4.

B) We wonder whether results of similar flavour could be obtained for solutions  $u \in W_{\text{loc}}^{1,p}(\Omega)$  such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi \, dx = \int_{\Omega} f \psi \, dx,$$

for all  $\psi \in C_c^\infty(\Omega)$ . Even the case  $f \equiv 0$  of  $p$ -harmonic functions, especially in higher dimensions  $n > 2$ , would be significative. In this case one should take into account the effect of the boundary of  $\Omega$ , which we do not see in our context because of the Dirichlet boundary condition. This is the reason why the regularising phenomenon we highlight is simply driven by the constant  $C(n, q)$ .

C) To prove Theorem 1.4 a weaker form of the weighted Hessian estimate has been obtained and used. It would be interesting to check, under the same assumptions of Theorem 1.4, whether or not the same Hessian estimate (1.3) still holds. This could be perhaps accomplished by flattening the boundary and trying to obtain a local version of the same estimate in the spirit of [7, 25, 26], which is known to the best of our knowledge, only when Hopf's boundary point lemma is applicable. However, many nontrivial technical

difficulties arise with such approach when trying to obtain our summability results on the second derivatives of the solutions.

D) It would be interesting to weaken our assumptions on  $f$ , for instance by considering the non-autonomous case, i.e. the case with a nonlinearity either of the form  $f(x, u)$  or more in general  $f(x, u, \nabla u)$ .

**1.2. Organization of the paper.** The paper is organized as follows. In Section 2 we collect some preliminary results which are useful to prove Theorem 1.3, which we will prove in Section 3; in Section 2 in particular we prove Proposition 1.2, which follows from the more general Proposition 2.3. In Section 4, which looks slightly more technical, we derive a Hessian estimate which plays the same role of Proposition 2.3 in the new setting of Theorem 1.4. Finally, in Section 5 we prove Theorem 1.4.

**1.3. Notation.** We use the following standard notation:

- Limits for sequences of functions  $(u_\varepsilon)_{\varepsilon>0}$  as  $\varepsilon \rightarrow 0^+$  are meant to be performed for suitable sequences  $\varepsilon_i \rightarrow 0^+$ ,  $i \rightarrow \infty$ .
- $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the classical  $p$ -Laplacian operator.
- $|x|$  is the euclidean norm of  $x \in \mathbb{R}^n$ .
- $\chi_\Omega$  is the characteristic function of a measurable set  $\Omega$ .
- $C^k(\Omega)$  is the space of real valued functions  $k$  times continuously differentiable on  $\Omega$ .
- $C^\infty(\Omega)$  is the space of real valued functions which are infinitely times continuously differentiable on  $\Omega$ .
- $C_c^k(\Omega)$ ,  $C_c^\infty(\Omega)$  are the spaces made up of compactly supported functions of respectively  $C^k(\Omega)$ , and  $C^\infty(\Omega)$
- $C^{k,\alpha}(\Omega)$  and  $C_{\text{loc}}^{k,\alpha}(\Omega)$  are classical Hölder spaces.
- $L^q(\Omega)$  and  $L_{\text{loc}}^q(\Omega)$  with  $\Omega \subset \mathbb{R}^n$  measurable set and  $q \geq 1$ , are classical Lebesgue space.
- $W^{k,p}(\Omega)$  and  $W_{\text{loc}}^{k,p}(\Omega)$  are classical Sobolev spaces.
- $c_1, \dots, c_k, C_1, \dots, C_k, c, c', c'' \dots, C, C', \dots, C'' \dots$  are positive constants.
- $\int \dots$  denotes standard Lebesgue integration.

## 2. PRELIMINARIES

Throughout this section we assume that  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^n$ , and  $u$  is a weak solution to (1.1) with  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ .

We deal now with the proof of Proposition 1.2 which will follow from Proposition 2.3 as a particular case ( $\varepsilon = 0$ ).

**2.1. A classical approximation.** We start observing that in order to deal with the formal expression

$$\Delta_p u = |\nabla u|^{p-2} \Delta u + (p-2) |\nabla u|^{p-4} \Delta_\infty u$$



mentioned in the introduction, we consider for  $\varepsilon \in [0, 1]$  the parametric problem

$$(2.1) \quad \begin{cases} -\operatorname{div} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon = f, & \text{in } \Omega \\ u_\varepsilon = 0. & \text{on } \partial\Omega. \end{cases}$$

It is standard to see that, when  $\varepsilon \in (0, 1)$  the above problem regularises the solution to (1.1), as by standard regularity theory it follows that there exists a unique solution

$$u_\varepsilon \in C^{2,\alpha}(\bar{\Omega}) \cap W_0^{1,p}(\Omega);$$

$u_\varepsilon$  is therefore a classical solution of

$$(2.2) \quad -\Delta u_\varepsilon = (p-2) \frac{(D^2 u_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)}{(\varepsilon + |\nabla u_\varepsilon|^2)} + \frac{f}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}}$$

in  $\Omega$ .

Note that, by [8, 12, 28] we have that

$$\|u_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \leq C.$$

Therefore, passing if necessary to a subsequence, by classical Arzela-Ascoli compactness theorem we have

$$u_\varepsilon \xrightarrow{C^{1,\alpha'}(\bar{\Omega})} w$$

for some  $0 < \alpha' < \alpha$ . It follows easily that  $w$  is a weak solution to (1.1) and consequently, by uniqueness of the solution of (1.1), we get

$$(2.3) \quad u_\varepsilon \xrightarrow{C^{1,\alpha'}(\bar{\Omega})} u.$$

**2.2. On second derivatives.** We recall that by the classical Calderón-Zygmund theory for elliptic operators we have the following:

**Lemma 2.1.** *Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$  and let  $w \in W_0^{2,q}(\Omega)$ . Then there exists a positive constant  $C = C(n, q)$  such that*

$$(2.4) \quad \|D^2 w\|_{L^q(\Omega)} \leq C \|\Delta w\|_{L^q(\Omega)}.$$

See e.g. Corollary 9.10 in [9].

We will need the following weighted Hessian regularity result.

**Proposition 2.2.** *Let  $u_\varepsilon$  be solution to (2.1) and  $f$  as in Proposition 1.1. Then, for any  $\beta < 1$ , there exists a constant  $C_\beta = C_\beta(n, p, f) > 0$  such that*

$$\int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2-\beta}{2}} \|D^2 u_\varepsilon\|^2 < C_\beta$$

for all  $\varepsilon \in [0, 1]$ .

*Proof.* The case  $\varepsilon = 0$  which is the most delicate one, is given by Proposition 1.1. The case  $\varepsilon > 0$  follows by the same arguments. And this concludes the proof.  $\square$

**2.3. Proof of Proposition 1.2.** The main result of the present section is the following

**Proposition 2.3.** *Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ , and with  $\varepsilon \in [0, 1)$  let  $u_\varepsilon$  be defined by (2.1), and in particular for  $\varepsilon = 0$  let  $u_0 := u$  be the weak solution to (1.1) with  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ . Then, for  $p > 2$  and for any fixed  $1 \leq q < \frac{p-1}{p-2}$ , there exists a constant  $C = C(n, p, q, f) > 0$  which does not depend on  $\varepsilon$  such that*

$$(2.5) \quad \int_{\Omega} \frac{f^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} \leq C.$$

The same bound holds for any  $1 \leq q < +\infty$  when  $1 < p \leq 2$ .

We do not assume here  $q \geq 2$ .

*Proof.* For some small  $\delta > 0$  define  $\varphi_\delta = j_\delta * \chi_{K_\delta}$  where  $\{j_\delta\}_{\delta>0}$  are standard radial mollifiers,  $\{K_\delta\}_{\delta>0}$  is a monotone family of compact smooth subsets of  $\Omega$  such that the characteristic functions  $\chi_{K_\delta} \rightarrow \chi_\Omega$  as  $\delta \rightarrow 0$ , in  $L^1(\Omega)$ , and  $\Omega_\delta = \{x \in \Omega : |x - y| \leq \delta, \text{ for some } y \in K_\delta\} \subset \Omega$ . By construction  $0 \leq \varphi_\delta \leq 1$ ,  $\varphi_\delta \equiv 0$  outside  $\Omega_\delta$ ,  $\lim_{\delta \rightarrow 0} \varphi_\delta = 1$  for all  $x \in \Omega$ , and zero outside. Moreover by scaling  $|\nabla \varphi_\delta| < C/\delta$ .

We prove

$$(2.6) \quad \int_{\Omega} \frac{f^2 \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} \leq C$$

for some uniform constant  $C$ , the bound (2.5) will then follow by Fatou's lemma as  $\delta \rightarrow 0$ .

The bounds are deduced for  $\varepsilon \in (0, 1)$  as the case  $\varepsilon = 0$  is a consequence again of Fatou's lemma, by using (2.3) and the fact that the estimates are uniform in  $\varepsilon$ . We test equation (2.1) against

$$\psi := \frac{f \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q}.$$

In fact using (2.1) we estimate

$$(2.7) \quad \int_{\Omega} \frac{f^2 \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} \leq \left| \int (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} (\nabla u_\varepsilon, \nabla \psi) \right| \leq C_0(A_\varepsilon + B_\varepsilon + C_\varepsilon),$$

where we have set

$$\begin{aligned} A_\varepsilon &= \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}(1-q)} |\nabla u_\varepsilon| \cdot \varphi_\delta \cdot |\nabla \varphi_\delta| \cdot |f| \\ B_\varepsilon &= \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}(1-q)} |\nabla u_\varepsilon| \cdot |\varphi_\delta|^2 \cdot |\nabla f| \\ C_\varepsilon &= \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}(1-q)-1} |\nabla u_\varepsilon|^2 \cdot \|D^2 u_\varepsilon\| \cdot |f| \cdot |\varphi_\delta|^2. \end{aligned}$$

We handle each term separately.

1) Estimate on  $A_\varepsilon$ . The restriction of  $q$  and the fact that  $\|u_\varepsilon\|_{C^{1,\alpha}} \leq C_1$  uniformly with respect to  $\varepsilon$  yields  $\|(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}(1-q)} |\nabla u_\varepsilon|\|_\infty \leq C_2$ . Notice that, in the construction of  $\{K_\delta\}_{\delta>0}$  and  $\Omega_\delta$ , we can assume with no loss of generality that  $|\text{supp} \nabla \varphi_\delta| < C_3 \delta$ , for some constant  $C_3 > 0$ . Therefore it follows the uniform bound

$$A_\varepsilon \leq C_4 \int_{\text{supp} \nabla \varphi_\delta} |\nabla \varphi_\delta| \leq C_5 \frac{|\text{supp} \nabla \varphi_\delta|}{\delta} \leq C_6.$$

2) Estimate on  $B_\varepsilon$ . Again we use that  $\|(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}(1-q)} |\nabla u_\varepsilon|\|_\infty$  is uniformly bounded together with the fact that  $|\nabla f| \in L^1(\Omega)$ , to conclude that

$$B_\varepsilon \leq C_7.$$

3) Estimate on  $C_\varepsilon$ . We obviously have

$$C_\varepsilon \leq \int_\Omega \frac{|f| \varphi_\delta}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{4}} \right]^q} \cdot (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2} - \frac{(p-2)}{4}q} \|D^2 u_\varepsilon\|.$$

By the elementary inequality on positive numbers  $ab \leq \eta a^2 + \frac{b^2}{\eta}$  it follows that

$$C_\varepsilon \leq \eta \int_\Omega \frac{f^2 \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} + \frac{1}{\eta} \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{p-2 - \frac{(p-2)}{2}q} \|D^2 u_\varepsilon\|^2.$$

In view of the restriction on  $q$  we can apply Proposition 2.2, obtaining

$$C_\varepsilon \leq \eta \int_\Omega \frac{f^2 \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} + \frac{1}{\eta} C_8$$

Conclusion. By using the above estimates on  $A_\varepsilon, B_\varepsilon, C_\varepsilon$ , equation (2.7) yields for some  $\eta < \frac{1}{C_0}$

$$(1 - \eta C_0) \int_\Omega \frac{f^2 \varphi_\delta^2}{\left[ (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{(p-2)}{2}} \right]^q} \leq C(\eta)$$

where the constant  $C(\eta)$  does not depend on  $\varepsilon$ . This concludes the proof.  $\square$

### 3. PROOF OF THEOREM 1.3

With the results of the preceding sections at hand we are now in position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let us start considering the case  $p > 2$ . Note that, as already recalled in Section 2, passing if necessary to a subsequence we have that

$$(3.1) \quad u_\varepsilon \xrightarrow{C^{1,\alpha'}(\overline{\Omega})} u.$$

Since  $2 \leq q < \frac{p-1}{p-2}$ , by Proposition 2.3, Proposition 2.2, by equations (2.2) and (2.4) we deduce that

$$\begin{aligned} \|D^2 u_\varepsilon\|_{L^q(\Omega)} &\leq C(n, q) \left\| (p-2) \frac{(D^2 u_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)}{(\varepsilon + |\nabla u_\varepsilon|^2)} + \frac{f}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}} \right\|_{L^q(\Omega)} \\ &\leq C(n, q)(p-2) \|D^2 u_\varepsilon\|_{L^q(\Omega)} + C(n, q) \left\| \frac{f}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}} \right\|_{L^q(\Omega)} \\ &\leq C(n, q)(p-2) \|D^2 u_\varepsilon\|_{L^q(\Omega)} + \tilde{C}(n, p, q, f) \end{aligned}$$

where we used that  $2 \leq q < \frac{p-1}{p-2}$ . Here  $\tilde{C}(n, p, q, f) = C \cdot C(n, q)$  where  $C$  is given by Proposition 2.3 and we also have used that  $f$  is bounded by classical Morrey's embedding theorem.

It follows

$$(1 - C(n, q)(p-2)) \|D^2 u_\varepsilon\|_{L^q(\Omega)} \leq \tilde{C}(n, p, q, f).$$

Since  $p-2 < \frac{1}{C(n, q)}$  there holds

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{W^{2, q}(\Omega)} < \infty.$$

Classical Rellich's theorem implies now that passing if necessary to a subsequence

$$u_\varepsilon \rightharpoonup w \in W^{2, q}(\Omega), \quad \text{and almost everywhere in } \Omega.$$

Therefore we infer that

$$u \equiv w \in W^{2, q}(\Omega).$$

The proof in the case  $1 < p < 2$  can be carried out exactly in the same way and observing that (2.5) is obvious, being non-singular, therefore Proposition 2.3 is not needed in this case.

Let now  $\gamma \in (0, 1)$  be fixed. There exists  $q$  such that

$$\gamma = 1 - \frac{n}{q},$$

and by the preceding part of the proof  $u \in W^{2, q}(\Omega)$ , for all  $p$  in a suitable open interval containing  $p_0 = 2$ . It follows by classical Morrey's embedding that

$$\partial_i u \in C^{0, \gamma}(\bar{\Omega}), \quad i = 1, \dots, n.$$

This concludes the proof. □

#### 4. THE AUTONOMOUS EQUATION

**4.1. The approximated equation.** Let  $\Omega$  be a bounded smooth domain of  $\mathbb{R}^n$ , and  $p > 1$ . We consider now the autonomous equation

$$(4.1) \quad \begin{cases} -\Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and  $u \in C^{1,\alpha}(\overline{\Omega})$  is a weak solution. We adapt here the approximation argument which has been used in the preceding section. To this aim we consider the equation:

$$(4.2) \quad \begin{cases} -\operatorname{div} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon = f(u) & \text{in } \Omega, \quad \varepsilon \in (0, 1), \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously  $u_\varepsilon \in C^2(\overline{\Omega})$  defined by (4.2) is such that

$$(4.3) \quad \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon \nabla \psi \, dx = \int_{\Omega} f(u) \psi \, dx,$$

for all  $\psi \in C_c^\infty(\Omega)$ .

Moreover  $u_\varepsilon$  is a classical solution of

$$(4.4) \quad -\Delta u_\varepsilon = (p-2) \frac{(D^2 u_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)}{(\varepsilon + |\nabla u_\varepsilon|^2)} + \frac{f(u)}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}}$$

in  $\Omega$ .

Although (4.1) may certainly have multiple solutions, uniqueness holds for equation (4.2), as well as for

$$(4.5) \quad \begin{cases} -\Delta_p v = f(u) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By these observations, arguing exactly as in the preceding section again by the classical Arzela-Ascoli compactness theorem and up to a subsequence, there holds

$$u_\varepsilon \xrightarrow{C^{1,\alpha'}(\overline{\Omega})} u$$

for some  $0 < \alpha' < \alpha$ .

**4.2. Hessian estimates in the degenerate case  $p > 2$ .** Throughout the present section  $p > 2$ ; moreover the subscript  $i$  indicates the derivative with respect to  $x_i : u_i = \frac{\partial u}{\partial x_i}$ ,  $i = 1, \dots, n$ . We also assume  $f : \mathbb{R} \rightarrow \mathbb{R}$  to be a locally Lipschitz continuous function such that:

$$(4.6) \quad \exists \gamma > 0, k > 0 : \lim_{t \rightarrow 0} \frac{f(t)}{|t|^{k-1}t} = \gamma$$

and

$$(4.7) \quad f(0) = 0, \quad f(t) \cdot t > 0 \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

In the following we will use (4.3), and as in the preceding section, we will again make a suitable choice for a test function which after integrating by parts, with some abuse of language, linearises equation (4.3).

For any  $i = 1, \dots, n$ , plugging  $\varphi_i$  as test function into (4.3) and integrating by parts, we obtain

$$(4.8) \quad \begin{aligned} & \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_{\varepsilon,i}, \nabla \varphi \rangle + (p-2) \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u_\varepsilon \rangle \langle \nabla u_\varepsilon, \nabla \varphi \rangle = \\ & = \int_{\Omega} f'(u) u_i \varphi, \end{aligned}$$

for all  $\varphi \in C_c^1(\Omega)$  and by density for all  $\varphi \in W_0^{1,1}(\Omega)$ .

We have the following Hessian estimate which plays the same role of equation (1.3) given in Proposition 1.1, which is suitable in the present autonomous case, without assuming  $\nabla u \neq \mathbf{0}$  on  $\partial\Omega$ .

**Proposition 4.1** (Weighted Hessian estimate). *Let  $u_\varepsilon$  be as above. For all  $\beta \in [0, 1]$ , there holds:*

$$(4.9) \quad \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2-\beta}{2}} |D^2 u_\varepsilon|^2 u^2 \leq C,$$

where  $C = C(\beta, p, n, f) > 0$  is independent on  $\varepsilon$ . The same estimate holds for  $\varepsilon = 0$  and  $u_0 := u$ .

*Proof.* Let  $G_\xi : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$G_\xi(s) = \begin{cases} s & \text{if } |s| \geq 2\xi, \\ 2 \left[ s - \xi \frac{s}{|s|} \right] & \text{if } \xi < |s| < 2\xi, \\ 0 & \text{if } |s| \leq \xi. \end{cases}$$

Fix  $\beta \in [0, 1]$  and set

$$T_\xi(t) = \frac{G_\xi(t)}{|t|^\beta}$$

and

$$(4.10) \quad \varphi(x) = T_\xi(u_{\varepsilon,i}(x))u^2(x)$$

In the sequel we omit the dependence on  $x$ . Using  $\varphi$  as test function in (4.8), we have

$$(4.11) \quad \begin{aligned} & \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 T'_\xi(u_{\varepsilon,i})u^2 + \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u \rangle T_\xi(u_{\varepsilon,i})2u \\ & + (p-2) \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u_\varepsilon \rangle^2 T'_\xi(u_{\varepsilon,i})u^2 \\ & + (p-2) \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u_\varepsilon \rangle \langle \nabla u_\varepsilon, \nabla u \rangle T_\xi(u_{\varepsilon,i})2u \\ & = \int_{\Omega} f'(u)u_i |T_\xi(u_{\varepsilon,i})|u^2. \end{aligned}$$

In the sequel  $c$  and  $C$  will denote positive constants, possibly depending on  $\|u_\varepsilon\|_{W^{1,\infty}}$ , whose value can vary from line to line.

We set

$$(4.12) \quad \begin{aligned} I_1 &= \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 T'_\xi(u_{\varepsilon,i})u^2 \\ I_2 &= (p-2) \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u_\varepsilon \rangle^2 T'_\xi(u_{\varepsilon,i})u^2 \\ I_3 &= \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u \rangle T_\xi(u_{\varepsilon,i})2u \\ I_4 &= (p-2) \int_{\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla u_{\varepsilon,i}, \nabla u_\varepsilon \rangle \langle \nabla u_\varepsilon, \nabla u \rangle T_\xi(u_{\varepsilon,i})2u \end{aligned}$$

$$I_5 = \int_{\Omega} f'(u)u_i T_{\xi}(u_{\varepsilon,i})u^2.$$

If  $p \geq 2$ , then  $I_2$  is positive and hence we have

$$(4.13) \quad I_1 + I_2 \geq I_1.$$

By (4.11) and (4.13) we infer

$$(4.14) \quad I_1 \leq |I_3| + |I_4| + |I_5|$$

and hence, recalling that  $u \in C^1(\bar{\Omega})$ , we get

$$(4.15) \quad \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 T'_{\xi}(u_{\varepsilon,i})u^2 \leq c \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}| |T_{\xi}(u_{\varepsilon,i})| |u| + \int_{\Omega} |f'(u)| |u_i| |T_{\xi}(u_{\varepsilon,i})| u^2.$$

We recall that there exists  $M > 0$  such that

$$(4.16) \quad \sup_{\varepsilon} \|u_{\varepsilon}\|_{W^{1,\infty}} \leq M.$$

Therefore, recalling that  $f$  is locally Lipschitz continuous, by (4.16) we get

$$(4.17) \quad \int_{\Omega} |f'(u)| |u_i| |T_{\xi}(u_{\varepsilon,i})| u^2 \leq C.$$

Using (4.16) and the elementary inequality  $ab \leq \theta a^2 + \frac{1}{4\theta} b^2$  (for all  $a, b \in \mathbb{R}$  and  $\theta > 0$ ), we have

$$(4.18) \quad \begin{aligned} & \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}| |T_{\xi}(u_{\varepsilon,i})| |u| \\ &= \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{4}} |\nabla u_{\varepsilon,i}| G_{\xi}(u_{\varepsilon,i})^{\frac{1}{2}} |u|}{|u_{\varepsilon,i}|^{\frac{\beta}{2}} |u_{\varepsilon,i}|^{\frac{1}{2}}} \cdot \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{4}} G_{\xi}(u_{\varepsilon,i})^{\frac{1}{2}} |u_{\varepsilon,i}| |\nabla u_{\varepsilon}|}{|u_{\varepsilon,i}|^{\frac{\beta}{2}} |u_{\varepsilon,i}|^{\frac{1}{2}}} \\ &\leq \theta \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 G_{\xi}(u_{\varepsilon,i}) u^2}{|u_{\varepsilon,i}|^{\beta} u_{\varepsilon,i}} + \frac{1}{4\theta} \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-\beta+2}{2}} G_{\xi}(u_{\varepsilon,i}) \\ &\leq \theta \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 G_{\xi}(u_{\varepsilon,i}) u^2}{|u_{\varepsilon,i}|^{\beta} u_{\varepsilon,i}} + C. \end{aligned}$$

Since

$$T'_{\xi}(s) = \frac{1}{|s|^{\beta}} \left[ G'_{\xi}(s) - \beta \frac{G_{\xi}(s)}{s} \right],$$

after setting  $\vartheta = c\theta$ , by (4.15) we get

$$(4.19) \quad \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2}{|u_{\varepsilon,i}|^{\beta}} \left( G'_{\xi}(u_{\varepsilon,i}) - (\beta + \vartheta) \frac{G_{\xi}(u_{\varepsilon,i})}{u_{\varepsilon,i}} \right) u^2 \leq C.$$

Choosing  $\vartheta$  such that  $\beta + \vartheta < 1$ , we have that  $G'_{\xi}(w_j^{\varepsilon}) - (\beta + \vartheta) \frac{G_{\xi}(u_{\varepsilon,i})}{u_{\varepsilon,i}}$  is positive and by definition of  $G_{\xi}$  it follows that

$$G'_{\xi}(s) - (\beta + \vartheta) \frac{G_{\xi}(s)}{s} \rightarrow 1 - (\beta + \theta)$$

as  $\xi \rightarrow 0$  and hence by Fatou's Lemma

$$(4.20) \quad \int_{\Omega \setminus \{u_{\varepsilon,i}=0\}} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2}{|u_{\varepsilon,i}|^{\beta}} u^2 \leq C.$$

Moreover, since  $|u_{\varepsilon,i}| \leq |\nabla u_{\varepsilon}|$ , we have

$$\begin{aligned} \int_{\Omega \setminus \{u_{\varepsilon,i}=0\}} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2-\beta}{2}} |\nabla u_{\varepsilon,i}|^2 u^2 &= \int_{\Omega \setminus \{u_{\varepsilon,i}=0\}} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 u^2}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{\beta}{2}}} \\ &\leq \int_{\Omega \setminus \{u_{\varepsilon,i}=0\}} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon,i}|^2 u^2}{|u_{\varepsilon,i}|^{\beta}} \end{aligned}$$

and hence by (4.20) it follows

$$(4.21) \quad \int_{\Omega \setminus \{u_{\varepsilon,i}=0\}} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2-\beta}{2}} |\nabla u_{\varepsilon,i}|^2 u^2 \leq C$$

where  $C$  depends on  $n, p, \beta, f$ . Since  $\nabla u_{\varepsilon,i} = 0$  almost everywhere on  $\{u_{\varepsilon,i} = 0\}$  the statement is proved for all  $\varepsilon \in (0, 1)$ .

To prove the statement for  $\varepsilon = 0$  we argue as follows. Observe that since  $\nabla u_{\varepsilon} \rightarrow \nabla u$  uniformly and by elliptic regularity theory, we have

$$u_{\varepsilon} \rightarrow u$$

in some  $C^{2,\bar{\alpha}}(\bar{\omega})$  for all  $\omega$  strictly contained in  $\Omega \setminus Z_u$ , where  $Z_u = \{x \in \bar{\Omega} : \nabla u = \mathbf{0}\}$  is the critical set of  $u$ . Therefore, for some sequence,  $\partial_{i,j} u_{\varepsilon} \rightarrow \partial_{i,j} u$  almost everywhere on  $\Omega \setminus Z_u$ . Finally by Fatou's lemma we get from (4.21)

$$(4.22) \quad \int_{\Omega \setminus Z_u} |\nabla u|^{p-2-\beta} |\nabla u_i|^2 u^2 \leq C,$$

and the estimate holds on the whole  $\Omega$ , as by Stampacchia theorem  $\partial_{i,j} u$  vanish almost everywhere on  $Z_u$ . This concludes the proof.  $\square$

We now handle the summability of singularly weighted integrals involving  $f$ . The following statements are obvious for  $p \leq 2$ .

**Proposition 4.2** (Singularly weighted estimate). *For  $p > 2$  let  $u_{\varepsilon}$  be given by (4.2). Let  $s$  and  $p$  be such that  $1 \leq s < \frac{p-1}{p-2}$ . Then there exists a positive constant  $C = C(p, n, f)$ , independent on  $\varepsilon$ , such that:*

$$(4.23) \quad \int_{\Omega} \frac{|u|^{2(k+1)}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}} \leq C.$$

The same estimate holds for  $\varepsilon = 0$  and  $u_0 := u$ .

*Proof.* We use

$$\psi = \frac{|u|^{k+1} u}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}}$$

as test function in (4.3). After setting  $L = \|u\|_{\infty}$ , (4.6) and (4.7) imply that there exists  $\lambda' > 0$  such that

$$(4.24) \quad \frac{f(t)}{|t|^{k-1} t} \geq \lambda' \quad \text{for all } 0 < |t| \leq L.$$



Hence we have:

$$\begin{aligned}
\lambda' \int_{\Omega} \frac{u^{2(k+1)}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}} &\leq \int_{\Omega} f(u)\psi \\
&\leq c \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon}|^2 |D^2 u_{\varepsilon}| |u|^{k+2}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s+1}} \\
&\quad + c \int_{\Omega} \frac{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla u_{\varepsilon}| |\nabla u| |u|^{k+1}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}} \\
&\leq c \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}(1-s)-1} |\nabla u_{\varepsilon}|^2 |D^2 u_{\varepsilon}| |u|^{k+2} \\
&\quad + c \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-1-(p-2)s}{2}} |u|^{k+1} \\
&\leq c \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}(1-s)-1} |\nabla u_{\varepsilon}|^2 |D^2 u_{\varepsilon}| |u|^{k+2} \\
&\quad + C \\
&= c \int_{\Omega} \frac{|u|^{k+1}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{4}s}} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{4}s + \frac{p-2}{2}(1-s)-1} |\nabla u_{\varepsilon}|^2 |D^2 u_{\varepsilon}| |u| \\
&\quad + C \\
\text{(Young's inequality)} &\leq c\theta \int_{\Omega} \frac{|u|^{2(k+1)}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}} \\
&\quad + \frac{c}{4\theta} \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s + (p-2)(1-s)-2} |\nabla u_{\varepsilon}|^4 |D^2 u_{\varepsilon}|^2 |u|^2.
\end{aligned}$$

After setting  $\vartheta = c\theta$ , we have:

$$(4.25) \quad (\lambda' - \vartheta) \int_{\Omega} \frac{|u|^{2(k+1)}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}s}} \leq \int_{\Omega} (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2-(p-2)(s-1)}{2}} |D^2 u_{\varepsilon}|^2 |u|^2$$

and, recalling that  $s < \frac{p-1}{p-2}$ , we can apply Proposition 4.1 with  $\beta = (p-2)(s-1)$ . The statement for  $\varepsilon = 0$  follows by Fatou's lemma, and this conclude the proof.  $\square$

**Proposition 4.3** (Singularly weighted estimate for  $f$ ). *Let  $p > 2$  let  $u_{\varepsilon}$  be given by (4.2), and let  $r \geq 1, k > 0$  and  $p$  be such that*

$$(4.26) \quad \max\left(\frac{2(k+1)}{k}, r\right) < \frac{p-1}{p-2}.$$

*Then there exists a positive constant  $C = C(p, n, f)$ , independent on  $\varepsilon$ , such that:*

$$(4.27) \quad \int_{\Omega} \left( \frac{|f(u)|}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right)^r \leq C.$$

*The same estimate holds for  $\varepsilon = 0$  and  $u_0 := u$ .*

*Proof.* After setting  $L = \|u\|_{\infty}$ , (4.6) and (4.7) imply that there exists  $\lambda' > 0$  such that:

$$(4.28) \quad |f(t)| \leq \lambda' |t|^k \quad \text{for all } |t| \leq L.$$

Since (4.26) holds we have two cases:  $r \geq \frac{2(k+1)}{k}$  and  $r < \frac{2(k+1)}{k}$ .

If  $r \geq \frac{2(k+1)}{k}$  by (4.28) we have:

$$(4.29) \quad \int_{\Omega} \left( \frac{|f(u)|}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right)^r \leq c \int_{\Omega} \frac{|u|^{kr}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}r}} \leq c' \int_{\Omega} \frac{|u|^{2(k+1)}}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}r}}$$

and the conclusion follows by (4.23) taking  $s = r$ . If  $r < \frac{2(k+1)}{k}$ , we set  $s = \frac{2(k+1)}{k}$  and by Hölder inequality we estimate

$$(4.30) \quad \int_{\Omega} \left( \frac{|f(u)|}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right)^r \leq |\Omega|^{1-\frac{r}{s}} \left( \int_{\Omega} \left( \frac{|f(u)|}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right)^s \right)^{\frac{r}{s}}$$

and the right hand side is uniformly bounded because of the preceding case.

Finally the case  $\varepsilon = 0$  follows as in the preceding proof by Fatou's lemma, and this concludes the proof.  $\square$

## 5. PROOF OF THEOREM 1.4

In the present section we prove Theorem 1.4 following the same scheme of the proof of Theorem 1.3.

*Proof of Theorem 1.4.* Again we consider first the case  $p > 2$ . Note that, as already observed earlier in the preceding section,

$$(5.1) \quad u_{\varepsilon} \xrightarrow{C^{1,\alpha'}(\bar{\Omega})} u.$$

By Proposition 4.1, Proposition 4.3 with  $r = q$ , (4.4) and (2.4) we deduce that

$$\begin{aligned} \|D^2 u_{\varepsilon}\|_{L^q(\Omega)} &\leq C(n, q) \left\| (p-2) \frac{(D^2 u_{\varepsilon} \nabla u_{\varepsilon}, \nabla u_{\varepsilon})}{(\varepsilon + |\nabla u_{\varepsilon}|^2)} + \frac{f}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right\|_{L^q(\Omega)} \\ &\leq C(n, q)(p-2) \|D^2 u_{\varepsilon}\|_{L^q(\Omega)} + C(n, q) \left\| \frac{f}{(\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}}} \right\|_{L^q(\Omega)} \\ &\leq C(n, q)(p-2) \|D^2 u_{\varepsilon}\|_{L^q(\Omega)} + \tilde{C}. \end{aligned}$$

Here  $\tilde{C} = \tilde{C}(p, q, n, f) = C \cdot C(n, q)$  where  $C$  is given by Proposition 4.3. It follows

$$(1 - C(n, q)(p-2)) \|D^2 u_{\varepsilon}\|_{L^q(\Omega)} \leq \tilde{C}.$$

Since  $p-2 < \frac{1}{C(n, q)}$  we have that

$$\sup_{\varepsilon > 0} \|u_{\varepsilon}\|_{W^{2,q}(\Omega)} < \infty.$$

Classical Rellich's theorem implies now that up to subsequences

$$u_{\varepsilon} \rightharpoonup w \in W^{2,q}(\Omega), \quad \text{and almost everywhere in } \Omega.$$

Therefore we have that

$$u \equiv w \in W^{2,q}(\Omega).$$

The proof in the case  $1 < p < 2$  can be carried out exactly in the same way observing that Proposition 4.3 is not needed, as weighted integrals are non-singular in this case.

The statement on the  $C^{1,\gamma}$  regularity follows by the same argument used for Theorem 1.3. And this concludes the proof.  $\square$

## ACKNOWLEDGEMENTS

C.M. would like to thank the Department of Mathematics and Computer Science - Università della Calabria (Italy) for the warm hospitality when the present paper has been written.

## REFERENCES

- [1] A. Bahri, J.-M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. 41 (1988), 253–294.
- [2] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.
- [3] M. Clapp, S. Tiwari, *Solutions to a pure supercritical problem for the  $p$ -Laplacian equations*, preprint
- [4] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C.R. Acad. Sc. Paris 299 (1984), 209–212.
- [5] M.G. Crandall, H. Ishii, P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*. Bull. Amer. Math. Soc. (N.S.), 27(1): 1992,1–67.
- [6] M. Cuesta, D. de Figueiredo J. P. Gossez, *The beginning of the Fucik spectrum for the  $p$ -Laplacian*, J. Differential Equations 159 (1999), no. 1, 212–238.
- [7] L. Damascelli, B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of  $m$ -Laplace equations*. J. Differential Equations, 206(2), 2004, 483–515.
- [8] E. Di Benedetto,  *$C^{1+\alpha}$  local regularity of weak solutions of degenerate elliptic equations*. Nonlinear Anal., 7(8), 1983, 827–850.
- [9] D. Gilbarg, N.-S. Trudinger, *Elliptic Partial Differential Equations*. Springer, (1998).
- [10] M. Guedda, L. Veron, *Local and global properties of solutions of quasi-linear elliptic equations*, J. Differential Equations 76 (1988), 159–189.
- [11] M. Guedda, L. Veron *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. 13 (1989), 879–902.
- [12] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*. Nonlinear Anal., 12(11), 1988, 1203–1219.
- [13] P. Lindqvist, *On the equation  $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$* . Proc. Amer. Math. Soc., 109(1), 1990, 157–164.
- [14] V. Julin and P. Juutinen *A new proof for the equivalence of weak and viscosity solutions for the  $p$ -Laplace equation*. Comm. Partial Differential Equations, 37(5), 2012, 934–946.
- [15] P. Juutinen, P. Lindqvist and J. J. Manfredi, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*. SIAM J. Math. Anal., 33(3), 2001, 699–717.
- [16] T. Kuusi and G. Mingione, *Universal potential estimates*. J. Funct. Anal., 262(10), 2012, 4205–4269.
- [17] C. Mercuri, B. Sciunzi, and M. Squassina, *On Coron's problem for the  $p$ -laplacian*, *J. Math. Anal. Appl.* **421** (2015), n.11, 362–369.
- [18] C. Mercuri, F. Pacella, *On the pure critical exponent problem for the  $p$ -laplacian*, *Calc. Var. Partial Differential Equations*, 49 (2014), no. 3-4, 1075–1090.
- [19] C. Mercuri, M. Willem, *A global compactness result for the  $p$ -laplacian involving critical nonlinearities*, *Discrete Contin. Dyn. Syst.* **28** (2010), 469–493.
- [20] G. Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, *Applications of Mathematics*, 51 (4), (2006), 355–426.
- [21] G. Mingione, *The Calderon-Zygmund theory for elliptic problems with measure data*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 6, (2007), 195–261.
- [22] M.K.V. Murthy, G. Stampacchia, *Boundary value problems for some degenerate-elliptic operators*, Ann. Mat. Pura Appl. 80(4) (1968), 1-122

- [23] I. Peral, Multiplicity of solutions for the  $p$ -Laplacian, Lecture notes for the Second School of Nonlinear Functional Analysis and Applications to Differential Equations (1997). International Centre of Theoretical Physics - Trieste (Italy).
- [24] P. Pucci, J. Serrin, *The maximum principle*. Birkhauser, Boston (2007).
- [25] B. Sciunzi, *Some results on the qualitative properties of positive solutions of quasilinear elliptic equations*. NoDEA. Nonlinear Differential Equations and Applications, 14(3-4), 2007, 315–334.
- [26] B. Sciunzi, *Regularity and comparison principles for  $p$ -Laplace equations with vanishing source term*. Comm. Cont. Math., 16(6), 2014, 1450013, 20.
- [27] E. Teixeira, *Regularity for quasilinear equations on degenerate singular sets*. Math. Ann., 358(1-2), 2014, 241–256.
- [28] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*. J. Differential Equations, 51(1), 1984, 126 – 150.
- [29] N.S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa 27(3) (1973), 265-308
- [30] J. L. Vazquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.*, 12(3), pp. 191 – 202, 1984.

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