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A GLOBAL COMPACTNESS RESULT FOR THE P-LAPLACIAN INVOLVING CRITICAL NONLINEARITIES

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Dedicated to Louis Nirenberg, a Master in Analysis, with admiration.

Abstract. We prove a representation theorem for Palais-Smale sequences involving the p-Laplacian and critical nonlinearities. Applications are given to a critical problem.

1. Introduction.

The following global compactness result was proved by M. Struwe in 1984 (see [25]). We assume that Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $\mu > 0$ and $a \in L^{N/2}(\Omega)$ is such that

$$\inf_{\substack{u \in W_0^{1,2}(\Omega) \\ \|\nabla u\|_{L^2}=1}} \int_\Omega |\nabla u|^2 + a|u|^2 dx > 0.$$

We define the following functionals

$$\phi_2(u) = \int_\Omega \frac{|\nabla u|^2}{2} + a\frac{|u|^2}{2} - \mu\frac{|u|^{2^*}}{2^*}dx, \quad u \in W_0^{1,2}(\Omega),$$

$$\psi_2(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} - \mu\frac{|u|^{2^*}}{2^*}dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $2^* := 2N/(N-2)$ and $\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) \,:\, \nabla u \in L^2(\mathbb{R}^N)\}$ with the norm

$$\|u\| := \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

Theorem A *Under the above assumptions, let $\{u_n\}_n \subset W_0^{1,2}(\Omega)$ be such that*

$$\phi_2(u_n) \to c \qquad \phi_2'(u_n) \to 0 \quad \text{in } W^{-1,2}(\Omega).$$

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Then, passing if necessary to a subsequence, there exists a solution $v_0 \in W_0^{1,2}(\Omega)$ *of*

$$-\Delta u + au = \mu |u|^{2^*-2} u,$$

a finite sequence $\{v_1, ..., v_k\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ *of solutions of*

$$-\Delta u = \mu |u|^{2^*-2} u \quad on\ \mathbb{R}^N$$

and k *sequences* $\{y_n^i\}_n \subset \Omega$ *and* $\{\lambda_n^i\}_n \subset \mathbb{R}_+$*, satisfying*

$$\frac{1}{\lambda_n^i} \mathrm{dist}(y_n^i, \partial\Omega) \to \infty, \quad n \to \infty,$$

$$\|u_n - v_0 - \sum_{i=1}^{k} (\lambda_n^i)^{(2-N)/2} v_i((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^2 \to \sum_{i=0}^{k} \|v_i\|^2, \quad n \to \infty,$$

$$\phi_2(v_0) + \sum_{i=1}^{k} \psi_2(v_i) = c.$$

Motivated by a problem with lack of compactness, C.O. Alves proved, in 2002, an extension to the p-Laplacian when $p > 2$ and $\Omega = \mathbb{R}^N$. However the convergence a.e. of the gradients in the proof of Theorem 2 in [1] needs a justification. The case of a smooth Riemannian manifold without boundary was considered by N. Saintier [23] in 2006, mainly for a sequence of positive critical points.
In this paper we consider the case of a smooth bounded domain Ω in $\mathbb{R}^N$. New difficulties arise from the fact that $\partial\Omega \neq \emptyset$.

We are motivated by an extension to the p-Laplacian of the existence result of [7]. This extension is contained in Section 6.

We use the following notation:

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty,$$

$$p^* := Np/(N-p), \quad 1 < p < N,$$

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$$

$$||u|| = ||\nabla u||_{L^p(\mathbb{R}^N)}$$

$$u_+ = \max(u, 0), u_- = \max(-u, 0)$$

$$\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}.$$

We denote by $\mathcal{D}_0^{1,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

We need the following nonexistence result.

Theorem 1.1. *Let* $1 < p < N$ *and let* $u \in \mathcal{D}_0^{1,p}(\mathbb{R}_+^N)$ *be a nonnegative solution of the equation*

$$-\Delta_p u = \mu u^{p^*-1} \quad in \quad \mathbb{R}_+^N.$$

Then $u \equiv 0$.

Our main result is the following theorem. We assume that

(A) Ω is a smooth bounded domain of $\mathbb{R}^N$, $1 < p < N$, $a \in L^{N/p}(\Omega)$ and $\mu > 0$,

(B)
$$\inf_{\substack{u \in W_0^{1,p}(\Omega) \\ \|\nabla u\|_{L^p}=1}} \int_\Omega |\nabla u|^p + a|u|^p dx > 0.$$

We define on $W_0^{1,p}(\Omega)$
$$\phi(u) = \int_\Omega \frac{|\nabla u|^p}{p} + a\frac{|u|^p}{p} - \mu\frac{|u|^{p^*}}{p^*}dx,$$
and on $\mathcal{D}^{1,p}(\mathbb{R}^N)$
$$\phi_\infty(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} - \mu\frac{u_+^{p^*}}{p^*}dx.$$
Let us recall that
$$< \phi'(u), h >= \int_\Omega [|\nabla u|^{p-2}\nabla u \cdot \nabla h + a|u|^{p-2}u\, h - \mu|u|^{p^*-2}u\, h]dx,$$
$$< \phi'_\infty(u), h >= \int_{\mathbb{R}^N} [|\nabla u|^{p-2}\nabla u \cdot \nabla h - \mu u_+^{p^*-1}\, h]dx.$$

Theorem 1.2. *Let* $1 < p < N$. *Under assumptions* ***(A)*** *and* ***(B),*** *let* $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ *such that*
$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega)$$
and
$$\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0, \quad n \to \infty.$$
Then, passing if necessary to a subsequence, there exists a solution $v_0 \in W_0^{1,p}(\Omega)$ *of*
$$-\Delta_p u + au^{p-1} = \mu u^{p^*-1} \quad \text{in} \quad \Omega,$$
$$u \geq 0 \quad \text{in} \quad \Omega,$$
a finite sequence $\{v_1, ..., v_k\} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *of solutions of*
$$-\Delta_p u = \mu u^{p^*-1} \quad \text{in} \quad \mathbb{R}^N,$$
$$u \geq 0 \quad \text{in} \quad \mathbb{R}^N$$
and k *sequences* $\{y_n^i\}_n \subset \Omega$ *and* $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, *satisfying*
$$\frac{1}{\lambda_n^i} \operatorname{dist}(y_n^i, \partial\Omega) \to \infty, \quad n \to \infty,$$
$$\|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{(p-N)/p} v_i((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,$$
$$\|u_n\|^p \to \sum_{i=0}^k \|v_i\|^p, \quad n \to \infty,$$

$$\phi(v_0)+\sum_{i=1}^{k}\phi_\infty(v_i)=c.$$

If $k\geq 1$, then $v_i>0$, for $i=1,...,k$, by the strong maximum principle.

The organization of the paper is the following:

1 Introduction

2 Proof of Theorem 1.1

3 Preliminary results

4 Proof of Theorem 1.2

5 Variants and open problems

6 A critical problem for the p-Laplacian

2. **Proof of Theorem 1.1.** We need the following version of the divergence theorem.

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ with outer normal unit vector $n(\cdot)$ and $v\in C(\mathbb{R}^N,\mathbb{R}^N)$ be such that $div\, v\in L^1_{loc}(\mathbb{R}^N)$. Then*

$$\int_\Omega \operatorname{div} v\, dx=\int_{\partial\Omega} v(\sigma)\cdot n(\sigma)d\sigma.$$

Proof. Let ρ_n be a sequence of mollifiers and let us define $v_n:=(\rho_n * v_1,...,\rho_n * v_N)$. By the classical divergence theorem we have

$$\int_\Omega \operatorname{div} v_n\, dx=\int_{\partial\Omega} v_n(\sigma)\cdot n(\sigma)d\sigma.$$

Since $v_n\to v$ uniformly on compact sets of $\mathbb{R}^N$ and since $\operatorname{div} v_n\to \operatorname{div} v$ in $L^1_{loc}(\mathbb{R}^N)$, the proof is complete. □

Remark 2.2. When $v\in C(\bar\Omega,\mathbb{R}^N)$ and $\operatorname{div} v\in L^1(\Omega)$, the divergence theorem holds for C^2-domains, see [[9] , p. 742].

Hereafter we define $H:=\mathbb{R}^N_+$. The following lemma is in the spirit of [21].

Lemma 2.3. *Let $1<p<N$ and $u\in\mathcal{D}^{1,p}_0(H)$ be a weak solution of the equation*

$$\Delta_p u+|u|^{p^*-2}u=0. \tag{1}$$

For $p>2$ we assume in addition $u\geq 0$. Then $\partial_N u=0$ everywhere on ∂H.

Proof. Let $u\in\mathcal{D}^{1,p}_0(H)$ be a solution of (1). By anti-reflection with respect to ∂H we can extend u in $\mathbb{R}^N\setminus H$ obtaining a function $\bar u$, which satisfies

$$\Delta_p\bar u+|\bar u|^{p^*-2}\bar u=0 \quad \text{in} \quad \mathbb{R}^N$$

weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, see [[8] p. 446]. From [Theorem 2.2 in [22]] and [Corollary, p. 830 in [12]] we have

$$\bar u\in C^{1,\alpha}_{loc}(\mathbb{R}^N)$$

and by [Theorem 2.5 in [22]], (see also [28] and [24]) we have

$$\bar{u} \in W^{2,p}_{loc}(\mathbb{R}^N), \quad p \leq 2.$$

Moreover

$$\bar{u} \in W^{2,2}_{loc}(\mathbb{R}^N), \quad 2 \leq p < 3 \tag{2}$$

and

$$\bar{u} \in W^{2,q}_{loc}(\mathbb{R}^N), \quad q < \frac{p-1}{p-2}, \quad p \geq 3. \tag{3}$$

For (2) and (3) we argue as follows. Since $p > 2$, the nonlinearity $f(s) := |s|^{p^*-2}s$ is locally lipschitz and $f(s) > 0$ for $s > 0$. We observe that $H = \bigcup_i O_i$, where $O_i \subset H$ are smooth bounded domains, possibly such that $\partial O_i \cap \partial H \neq \emptyset$. By Vázquez strong maximum principle [30], both $\bar{u}\big|_{H\cap O_i}$ and $-\bar{u}\big|_{-(H\cap O_i)}$ are positive solutions in $H \cap O_i$ and (respectively) in $-(H \cap O_i)$ of the equation

$$-\Delta_p u = f(u).$$

Statements (2) and (3) follow then by [[10], Proposition 2.2].
Observe, by regularization of $|\nabla \bar{u}|^{p-2}\nabla \bar{u}$, that

$$\text{div}\,[\partial_N \bar{u}|\nabla \bar{u}|^{p-2}\nabla \bar{u}] = \partial_N \bar{u}[\text{div}\ |\nabla \bar{u}|^{p-2}\nabla \bar{u}] + |\nabla \bar{u}|^{p-2}\nabla \bar{u}\cdot\nabla\partial_N \bar{u} \in L^1_{loc}(\mathbb{R}^N).$$

Using Lemma 2.1, we have

$$\begin{aligned}
&\int_{H\cap B_R} \partial_N u[\text{div}\ |\nabla u|^{p-2}\nabla u]dx \\
&\quad = \int_{\partial(H\cap B_R)} \partial_N u|\nabla u|^{p-2}\nabla u\cdot n(\sigma)d\sigma - \int_{H\cap B_R} |\nabla u|^{p-2}\nabla u\cdot\nabla\partial_N u\ dx \\
&\quad = \int_{\partial(H\cap B_R)} \partial_N u|\nabla u|^{p-2}\nabla u\cdot n(\sigma)d\sigma - \int_{\partial(H\cap B_R)} \frac{|\nabla u|^p}{p} n_N(\sigma)d\sigma
\end{aligned}$$

and

$$\int_{H\cap B_R} \partial_N u|u|^{p^*-2}u\ dx = \int_{\partial(H\cap B_R)} \frac{|u|^{p^*}}{p^*} n_N(\sigma)d\sigma = \int_{H\cap\partial B_R} \frac{|u|^{p^*}}{p^*} n_N(\sigma)d\sigma.$$

It follows from (1) that

$$\begin{aligned}
&\left(1-\frac{1}{p}\right)\int_{B_R\cap\partial H} |\partial_N u|^p d\sigma \\
&\quad = \int_{H\cap\partial B_R} [\partial_N u\ |\nabla u|^{p-2}\nabla u\cdot n(\sigma) - \frac{|\nabla u|^p}{p} n_N(\sigma)]d\sigma + \int_{H\cap\partial B_R} \frac{|u|^{p^*}}{p^*} n_N(\sigma)d\sigma.
\end{aligned}$$

The right hand side number is bounded by

$$M(R) = \int_{H\cap\partial B_R} \left(1+\frac{1}{p}\right)|\nabla u|^p + \frac{|u|^{p^*}}{p^*} d\sigma.$$

Since $\nabla u \in L^p(H)$ and $u \in L^{p^*}(H)$, there exists a sequence $R_n \to \infty$ such that $M(R_n) \to 0$. The monotone convergence theorem implies that $\int_{\partial H} |\partial_N u|^p d\sigma = 0$. □

Remark 2.4. Arguing as in [10], it is possible to prove statement (2), removing the assumption $u \geq 0$, for $2 < p < 3$.

Proof of Theorem 1.1. Assume that u is nontrivial. Since $u \geq 0$, we have $\Delta_p u \leq 0$. By Lemma 2.3 we have $\partial_N u(x_0) = 0$ for every $x_0 \in \partial H$. On the other hand, by the $C^{1,\alpha}_{loc}(\bar{H})$ regularity we can apply the strong maximum principle by Vázquez [30], obtaining $\partial_N u(x_0) > 0$. This yields a contradiction and the theorem follows. □

Remark 2.5. We extend to the p-Laplacian Proposition I.2 in [13].

3. **Preliminary results.** In sections 3 and 4, we follow the scheme of the proof for the case $p = 2$ of Theorem A given in [31]. This proof is in the spirit of [4]. We need some preliminary lemmas.

Lemma 3.1. *Let $1 < p < 2$. Then*

$$c := \sup_{\substack{h \neq 0 \\ x \in \mathbb{R}^N}} \Big| \frac{|x+h|^{p-2}(x+h) - |x|^{p-2}x}{|h|^{p-1}} \Big| < \infty.$$

Proof. Define

$$F(x,h) := \Big| \frac{|x+h|^{p-2}(x+h) - |x|^{p-2}x}{|h|^{p-1}} \Big|$$

and notice that, by homogeneity,

$$F(x, th) = F(\frac{x}{t}, h).$$

Hence

$$c \equiv \sup_{\substack{|h|=1 \\ x \in \mathbb{R}^N}} F(x,h).$$

Since, by continuity,

$$c_1 := \sup_{\substack{|h|=1 \\ |x| \leq 2}} F(x,h) < \infty,$$

it suffices to prove that

$$c_2 := \sup_{\substack{|h|=1 \\ |x| > 2}} F(x,h) < \infty. \tag{4}$$

Assume that $|h| = 1$, $|x| > 2$ and $t \in [0,1]$. Then

$$|x + th| \geq |x| - |h| > 1,$$

so we have, for any $k = 1, ..., N$

$$\begin{aligned} &||x+h|^{p-2}(x_k + h_k) - |x|^{p-2}x_k| = \\ = \;& \Big| \int_0^1 |x+th|^{p-2}h_k + (p-2)(x_k + th_k)|x+th|^{p-4}(x+th)\cdot h \, dt \Big| \\ \leq \;& \int_0^1 |x+th|^{p-2}|h| dt + (2-p) \int_0^1 |x+th|^{p-2}|h| \, dt \\ \leq \;& 3 - p. \end{aligned}$$

This yields (4) and the lemma follows. □

The following result is a variant of the Brezis-Lieb Lemma [5]. The case $p \geq 2$ is due to Alves [1].

Lemma 3.2. *Let $p > 1$ and define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(y) := |y|^{p-2}y$. Let $\{u_n\}_n \in L^p(\Omega, \mu; \mathbb{R}^N)$ be such that $u_n \to u$ $\mu-$a.e. on Ω and $\sup_n \|u_n\|_p < \infty$. Then,*

$$\lim_{n\to\infty} \int_\Omega |A(u_n) - A(u_n - u) - A(u)|^{\frac{p}{p-1}} d\mu = 0$$

Proof. The case $p \geq 2$ is treated in [1]. We assume that $1 < p < 2$. Since

$$c := \sup_{\substack{h\neq 0 \\ x\in\mathbb{R}^N}} \left| \frac{|x+h|^{p-2}(x+h) - |x|^{p-2}x}{|h|^{p-1}} \right| < \infty$$

by the previous lemma, the statement follows from the dominated convergence theorem. □

Define

$$T := \begin{cases} s & \text{if } |s| \leq 1 \\ \frac{s}{|s|} & \text{if } |s| > 1. \end{cases}$$

Theorem 3.3. *Let (Ω_k) be a sequence of open subsets of Ω such that $\Omega_k \subset \Omega_{k+1}$ and $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$. Let $p > 1$, $\{v_n\}_n \subset W^{1,p}(\Omega)$ be such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and, for every k,*

$$\lim_{n\to\infty} \int_{\Omega_k} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v) \cdot \nabla T(v_n - v) dx = 0.$$

Then

(a) $\nabla v_n \to \nabla v$ a.e. on Ω, *passing if necessary to a subsequence,*

(b) $\lim_{n\to\infty} (\|\nabla v_n\|^p_{L^p} - \|\nabla(v_n - v)\|^p_{L^p}) = \|\nabla v\|^p_{L^p}$

(c) $|\nabla v_n|^{p-2}\nabla v_n - |\nabla(v_n - v)|^{p-2}\nabla(v_n - v) \to |\nabla v|^{p-2}\nabla v$ in $L^{p/(p-1)}(\Omega)$.

If $\Omega = \mathbb{R}^N$, the statements continue to hold replacing $W^{1,p}(\Omega)$ by $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proof. By [11], $\nabla v_n \to \nabla v$ a.e. on Ω_k passing to a subsequence. It suffices then to use the Cantor diagonal argument. This proves (a). Statement (b) follows from the Brezis-Lieb Lemma ([5]). Statement (c) follows from Lemma 3.2 with $u_n := \nabla v_n$. □

We need the following lemma, which is a remark on the Brezis-Lieb Lemma.

Lemma 3.4. *Let $p > 0$ and $\{u_n\}_n \subset L^p(\Omega, \mu)$ be such that*

(a) $\lim_{n\to\infty} u_n \to u$ $\mu - a.e.$ *on* Ω,

(b) $\int_\Omega |(u_n)_-|^p d\mu \to 0, \quad n \to \infty,$

(c) $\sup_n \int_\Omega |u_n|^p d\mu < \infty.$

Then $u \geq 0$,

$$\int_\Omega |u|^p d\mu < \infty, \tag{5}$$

$$\int_\Omega |(u_n - u)_-|^p d\mu \to 0 \tag{6}$$

and

$$\lim_{n\to\infty} (\int_\Omega |u_n|^p d\mu - \int_\Omega |(u_n - u)_+|^p d\mu) = \int_\Omega (u)^p d\mu. \tag{7}$$

Proof. Fatou's Lemma yields (5), while (7) follows by (6), (a) and (c) using the Brezis-Lieb Lemma. To prove (6) observe that, by (b) there exists a function $g \in L^p(\Omega, \mu)$ such that, passing if necessary to a subsequence,

$$|(u_n)_-(x)| \leq g(x), \quad \mu - \text{a.e. on } \Omega.$$

We have

$$|(u_n - u)_- - (u_n)_-| \leq |u|.$$

It follows that

$$0 \leq (u_n - u)_- \leq u + g \quad \mu - \text{a.e. on } \Omega.$$

Hence (6) follows from the dominated convergence Theorem, using (a) and (5). Since the subsequence is arbitrary, the proof is complete. □

Lemma 3.5. *Let us assume* **(A)**, **(B)** *and*

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ &u_n \to u \quad \text{a.e. on } \Omega, \\ &\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0 \\ &\phi(u_n) \to c \\ &\phi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega). \end{aligned}$$

Then, passing if necessary to a subsequence, $\nabla u_n \to \nabla u$ *a.e. on* Ω *and* $\phi'(u) = 0$. *Moreover* $v_n := u_n - u$ *is such that*

$$\begin{aligned} i) &\quad \lim_{n\to\infty} (||u_n||^p - ||v_n||^p) = ||u||^p, \\ ii) &\quad \phi_\infty(v_n) \to c - \phi(u), \\ iii) &\quad \phi'_\infty(v_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega). \end{aligned}$$

Proof. Since T is bounded, $\int_\Omega |T(u_n - u)|^q dx \to 0$, for all $q > 1$. Moreover $T(u_n - u) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$. Hence we have

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla T(u_n - u) dx$$

$$= < \phi'(u_n), T(u_n - u) > - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla T(u_n - u) dx$$

$$- \int_\Omega (a(x)|u_n|^{p-2}u_n - \mu|u_n|^{p^*-2}u_n) T(u_n - u) dx \longrightarrow 0.$$

Statement $i)$ follows from Theorem 3.2 (b) and, passing if necessary to a subsequence, $\nabla u_n \to \nabla u$ a.e. on Ω. Now notice that $a(\cdot) \in (L^{N/(N-p)}(\Omega))'$, $\{|v_n|^p\}_n$ is bounded in $L^{N/(N-p)}(\Omega)$ and $v_n \to 0$ a.e. on Ω. It follows from Proposition 20.6 in [32] that

$$\int_\Omega a(x)|v_n|^p = o(1).$$

Therefore, by Lemma 3.4 we have $ii)$:

$$\begin{aligned}\phi_\infty(v_n) &= \phi(v_n) + o(1)\\ &= \phi(u_n) - \phi(u) + o(1)\\ &= c - \phi(u) + o(1).\end{aligned}$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\phi'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, it is easy to verify that $\phi'(u) = 0$. Finally, Lemma 3.2 yields $iii)$:

$$\begin{aligned}\phi'_\infty(v_n) &= \phi'(v_n) + o(1)\\ &= \phi'(u_n) - \phi'(u) + o(1)\\ &= o(1).\end{aligned}$$

□

Lemma 3.6. *Under assumptions* **(A)** *and* **(B)**, *let* $\{y_n\}_n \subset \Omega$ *and* $\{\lambda_n\}_n \subset]0,\infty[$ *be such that*

$$\frac{1}{\lambda_n}\operatorname{dist}(y_n, \partial\Omega) \to \infty. \tag{8}$$

Assume that the sequence $\{u_n\}_n \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *and the sequence*

$$v_n(x) := \lambda_n^{(N-p)/p} u_n(\lambda_n x + y_n)$$

are such that

$$\begin{aligned}&v_n \rightharpoonup v \quad \text{in } \mathcal{D}^{1,p}(\mathbb{R}^N),\\ &v_n \to v \quad \text{a.e. on } \mathbb{R}^N,\\ &\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0,\\ &\phi_\infty(u_n) \to c,\\ &\phi'_\infty(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega).\end{aligned}$$

Then, passing to a subsequence, $\nabla v_n \to \nabla v$ *a.e. on* $\mathbb{R}^N$ *and* $\phi'_\infty(v) = 0$. *Moreover, the sequence*

$$w_n(z) := u_n(z) - (\lambda_n)^{(p-N)/p} v((z - y_n)/\lambda_n)$$

satisfies

$$\begin{aligned}i)&\quad \lim_{n\to\infty}(||u_n||^p - ||w_n||^p) = ||v||^p,\\ ii)&\quad \phi_\infty(w_n) \to c - \phi_\infty(v),\\ iii)&\quad \phi'_\infty(w_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega).\end{aligned}$$

Proof. 1) We first prove the existence of a subsequence on ∇v_n converging a.e. on $\mathbb{R}^N$.

For $h \in \mathcal{D}(\mathbb{R}^N)$, we define

$$h_n(z) = \lambda_n^{(p-N)/p} h((z - y_n)/\lambda_n).$$

Let us also define $B_k := B(0, k)$. For every n large enough, if $h \in \mathcal{D}(B_k)$ then, by assumption (8), $h_n \in \mathcal{D}(\Omega)$ and

$$|\langle\phi'_\infty(v_n), h\rangle| = |\langle\phi'_\infty(u_n), h_n\rangle| \le ||\phi'_\infty(u_n)||\ ||h_n|| = ||\phi'_\infty(u_n)||\ ||h||.$$

Hence $\phi'_\infty(v_n) \to 0$ in $W^{-1,p'}(B_k)$. Let $\rho \in \mathcal{D}(\mathbb{R}^N)$ be such that $0 \le \rho \le 1$ and

$$\begin{aligned} \rho(x) &= 1, \quad |x| \le k, \\ &= 0, \quad |x| \ge k+1. \end{aligned}$$

Consider the vector field

$$f_n = |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v.$$

We have

$$|\int_{B_k} f_n \cdot \nabla T(v_n - v)dx| \le |\int_{\mathbb{R}^N} f_n \cdot \nabla[\rho T(v_n - v)]dx| + |\int_{\mathbb{R}^N} T(v_n - v) f_n \cdot \nabla\rho \; dx|.$$

Since T is bounded, it is clear that

$$\int_{\mathbb{R}^N} T(v_n - v) f_n \cdot \nabla\rho \; dx \to 0, \quad n \to \infty.$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^N} f_n \cdot \nabla[\rho T(v_n - v)]dx &= \langle \phi'_\infty(v_n), \rho T(v_n - v)\rangle \\ &+ \mu \int_{\mathbb{R}^N} |v_n|^{p^*-2} v_n \rho T(v_n - v)dx \\ &+ \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla[\rho T(v_n - v)]dx \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} f_n \cdot \nabla[\rho T(v_n - v)]dx \to 0, \quad n \to \infty.$$

Finally,

$$\int_{B_k} f_n \cdot \nabla T(v_n - v)dx \to 0, \quad n \to \infty$$

and it suffices to use Theorem 3.2.

2) Using also Theorem 3.2, we obtain

$$\lim_{n\to\infty}(||v_n||^p - ||v_n - v||^p) = ||v||^p$$

or

$$\lim_{n\to\infty}(||u_n||^p - ||w_n||^p) = ||v||^p.$$

It follows then from the Brezis-Lieb Lemma that

$$\begin{aligned} \phi_\infty(w_n) &= \phi_\infty(v_n - v) \\ &= \phi_\infty(v_n) - \phi_\infty(v) + o(1) \\ &= \phi_\infty(u_n) - \phi_\infty(v) + o(1) \\ &= c - \phi_\infty(v) + o(1). \end{aligned}$$

3) Since, for every $h \in \mathcal{D}(\mathbb{R}^N)$,

$$\langle \phi'_\infty(v_n), h\rangle \to 0, \langle \phi'_\infty(v_n), h\rangle \to \langle \phi'_\infty(v), h\rangle,$$

it is clear that $\phi'_\infty(v) = 0$.

4) For $g \in \mathcal{D}(\Omega)$, we define

$$g_n(x) = \lambda_n^{(N-p)/p} g(\lambda_n x + y_n).$$

By combining Lemma 3.1 and Theorem 3.2, we obtain, uniformly for $g \in \mathcal{D}(\Omega)$, $||g|| = 1$,

$$\begin{aligned}\langle \phi'_\infty(w_n), g\rangle &= \langle \phi'_\infty(v_n - v), g_n\rangle \\ &= \langle \phi'_\infty(v_n), g_n\rangle - \langle \phi'_\infty(v), g_n\rangle + o(1) \\ &= \langle \phi'_\infty(u_n), g\rangle + o(1) = o(1).\end{aligned}$$

□

4. **Proof of Theorem 1.2.** For sake of clarity we divide the proof in 5 steps.

1) The sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$ (see [31] p. 15). Hence, passing if necessary to a subsequence, we can assume that $u_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $u_n \to v_0$ a.e. on Ω. By Lemma 3.5, it follows that $\phi'(v_0) = 0$ and $u_n^1 := u_n - v_0$ is such that

$$\begin{aligned}&i) \quad \|u_n^1\|^p = \|u_n\|^p - \|v_0\|^p + o(1),\\ &ii) \quad \phi_\infty(u_n^1) \to c - \phi(v_0),\\ &iii) \quad \phi'_\infty(u_n^1) \to 0 \quad \text{in } W^{-1,p'}(\Omega).\end{aligned}$$

2) If $u_n^1 \to 0$ in $L^{p^*}(\Omega)$, since $\phi'_\infty(u_n^1) \to 0$ in $W_0^{1,p}(\Omega)$, we have that $u_n^1 \to 0$ in $W_0^{1,p}(\Omega)$ and the proof is complete. Otherwise we can assume that

$$\int_\Omega |u_n^1|^{p^*} dx > \delta$$

for some $0 < \delta < (S_p/2\mu)^{N/p}$. Introducing the Levy concentration function

$$Q_n(r) := \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n^1|^{p^*} dx,$$

since $Q_n(0) = 0$ and $Q_n(\infty) > \delta$, there exists a sequence $\{\lambda_n^1\}_n \subset]0, \infty[$ and a sequence $\{y_n^1\}_n \subset \Omega$ such that

$$\delta = \sup_{y \in \mathbb{R}^N} \int_{B(y,\lambda_n^1)} |u_n^1|^{p^*} dx = \int_{B(y_n^1,\lambda_n^1)} |u_n^1|^{p^*} dx.$$

We define on

$$\Omega_n := \frac{1}{\lambda_n^1}(\Omega - y_n^1)$$

the sequence $v_n^1(x) := (\lambda_n^1)^{(N-p)/p} u_n^1(\lambda_n^1 x + y_n^1)$. We can assume that $v_n^1 \rightharpoonup v_1$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $v_n^1 \to v_1$ a.e. on $\mathbb{R}^N$. Observe also that

$$\delta = \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n^1|^{p^*} dx = \int_{B(0,1)} |v_n^1|^{p^*} dx. \tag{9}$$

3) We claim that $v_1 \neq 0$. Indeed let $f_n := (f_n^1, ..., f_n^N) \in (L^{p'}(\Omega))^N$ defined by the representation

$$(\phi'_\infty(u_n^1), h) = \sum_{i=1}^N \int_\Omega f_n^i \partial_i h dx, \qquad \forall h \in W_0^{1,p}(\Omega).$$

Define $g_n := (\lambda_n^1)^{(N-p)/p} f_n(\lambda_n^1 x + y_n^1)$. It is clear that

$$(\phi'_\infty(v_n^1), h) = \sum_{i=1}^N \int_{\Omega_n} g_n^i \partial_i h dx, \qquad \forall h \in W_0^{1,p}(\Omega)$$

and, since $\phi'_\infty(u_n^1) \to 0$,

$$\sum_{i=1}^N \int_{\Omega_n} |g_n^i|^{p'} dx = \sum_{i=1}^N \int_\Omega |f_n^i|^{p'} dx = o(1).$$

Suppose, by contradiction, that $v_1 = 0$. Then we can assume that $v_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^N)$. Take $h \in \mathcal{D}(\mathbb{R}^N)$ such that $\operatorname{supp} h \subset B(y, 1)$ for some $y \in \mathbb{R}^N$. From the Hölder inequality, it follows that

$$\int |h|^p |v_n^1|^{p^*} \le S_p^{-1} \Big(\int_{\operatorname{supp} h} |v_n^1|^{p^*} \Big)^{p/N} \Big(\int |h v_n^1|^{p^*} \Big)^{(N-p)/N}$$

and from the Sobolev inequality

$$\int |h|^p |v_n^1|^{p^*} \le S_p^{-1} \Big(\int_{\operatorname{supp} h} |v_n^1|^{p^*} \Big)^{p/N} \int |\nabla (h v_n^1)|^p.$$

Hence, since $v_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\int_{\Omega_n} |\nabla (h v_n^1)|^p &= \int |h|^p |\nabla v_n^1|^p + o(1) \\
&= \int |\nabla v_n^1|^{p-2} \nabla v_n^1 \nabla (|h|^p v_n^1) + o(1) \\
&= \mu \int |h|^p |v_n^1|^{p^*} + \sum_{i=1}^N \int_{\Omega_n} g_n^i \partial_i (|h|^p v_n^1) + o(1) \\
&\le \mu S_p^{-1} \delta^{p/N} \int |\nabla (h v_n^1)|^p + o(1) \\
&\le \frac{1}{2} \int |\nabla (h v_n^1)|^p + o(1)
\end{aligned}$$

As a consequence we have that $\nabla v_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ and by the Sobolev inequality we have that $v_n^1 \to 0$ in $L^{p^*}_{loc}(\mathbb{R}^N)$. Because of (9), this is a contradiction. Hence $v_1 \neq 0$.

4) Since Ω is bounded we may assume $y_n^1 \to y_0^1 \in \bar{\Omega}$ and $\lambda_n^1 \to \lambda_0^1 \ge 0$. If $\lambda_0^1 > 0$ then, as a consequence of the fact that $u_n^1 \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, we have $v_n^1 \rightharpoonup 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and this is impossible. If $\lambda_n^1 \to 0$ and

$$\liminf_{n \to \infty} \frac{1}{\lambda_n^1} \operatorname{dist}(y_n^1, \partial\Omega) < \infty,$$

then we also get a contradiction. Indeed, we have necessarily $y_0^1 \in \partial\Omega$. As in Lemma 3.6 it is easy to see that v_1, which is nonnegative, satisfies

$$\begin{aligned}
-\Delta_p u = \mu u^{p^*-1} \quad &\text{in} \quad H, \\
u = 0 \quad &\text{on} \quad \partial H,
\end{aligned}$$

where H is a halfspace. By Theorem 1.1 we have that $v_1 \equiv 0$.
It follows that, for some subsequence,

$$\frac{1}{\lambda_n^1}\text{dist}\,(y_n^1, \partial\Omega) \to \infty, \quad \lambda_n^1 \to 0.$$

By Step 1 and Lemma 3.6 we have that $\phi'_\infty(v_1) = 0$ and by the strong maximum principle $v_1 > 0$. The sequence

$$u_n^2(x) := u_n^1(x) - (\lambda_n^1)^{(p-N)/p} v_1((x - y_n^1)/\lambda_n^1)$$

satisfies

$$\begin{aligned} &\|u_n^2\|^p = \|u_n\|^p - \|v_0\|^p - \|v_1\|^p + o(1), \\ &\phi_\infty(u_n^2) \to c - \phi(v_0) - \phi_\infty(v_1), \\ &\phi'_\infty(u_n^2) \to 0 \quad \text{in } W^{-1,p'}(\Omega). \end{aligned}$$

5) Since for any nontrivial critical point u of ϕ_∞ we have

$$S_p\|u\|^p_{L^{p^*}(\mathbb{R}^N)} \le \|u\|^p = \mu\|u\|^{p^*}_{L^{p^*}(\mathbb{R}^N)},$$

we get

$$\phi_\infty(u) \ge c^* := \frac{\mu}{N}\Big(\frac{S_p}{\mu}\Big)^{N/p}.$$

As a consequence the above procedure iterates constructing sequences $\{v^i\}$ $\{\lambda_n^i\}$ $\{v_n^i\}$. But, since $\phi_\infty(v_i) \ge c^*$, only a finite number of iterations is allowed and this concludes the proof.

5. **Variants and open problems.** The result obtained in the previous sections is an extension to the p-Laplacian operator of the Struwe result, for PS sequences with negative part such that $\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0$. This hypothesis is satisfied for a large class of problems

$$\begin{cases} -\Delta_p u + a u^{p-1} = & \mu u^{p^*-1} \quad \text{in } \Omega, \\ & u \ge 0 \quad \text{in } \Omega, \\ & u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Indeed, one can construct PS sequences which are nearby the positive cone P in $W_0^{1,p}(\Omega)$. More precisely, define

$$P := \{u \in W_0^{1,p}(\Omega) \ : \ u \ge 0\}$$

and let

$$P_n := \{u \in W_0^{1,p}(\Omega) \ : \ \text{dist}(u, P) < 1/n\}$$

be a $1/n-$ neighbourhood of P in $W_0^{1,p}(\Omega)$. By Lemma 2.3 in [31], under suitable assumptions on a, it is possible to select a PS sequence for ϕ such that $\{u_n\}_n \subset P_n$. As a consequence there exists a sequence $\{y_n\}_n \subset P$ such that

$$\|y_n - u_n\| \to 0$$

and, by the Sobolev inequality

$$\|y_n - u_n\|_{L^{p^*}(\Omega)} \to 0.$$

Hence,

$$\begin{aligned} \|(u_n)_-\|_{L^{p^*}(\Omega)} &= ||(u_n)_- - (y_n)_-||_{L^{p^*}(\Omega)} \\ &\le ||u_n - y_n||_{L^{p^*}(\Omega)} \to 0, \quad n \to \infty. \end{aligned}$$

For the study of sign changing solutions for the problem

$$-\Delta_p u + a|u|^{p-2}u = \mu|u|^{p^*-2}u \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

the above argument is not suitable anymore. The main problem is to generalize Theorem 1.1 to sign changing solutions. Let $1 < p < N$. The existence of a non-zero solution of

$$\begin{cases} -\Delta_p u = \mu|u|^{p^*-2}u & \text{in } \mathbb{R}^N_+ \\ u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N_+), \end{cases}$$

seems an open problem when $p \neq 2$. When $p = 2$, 0 is the only solution because of the unique continuation principle and Lemma 2.3.
Similarly, the existence of a non-radial solution of

$$\begin{cases} -\Delta_p u = \mu|u|^{p^*-2}u & \text{in } B(0,1) \subset \mathbb{R}^N, \\ u \in W^{1,p}_0(B(0,1)), \end{cases}$$

seems also an open problem when $p \neq 2$. For $p = 2$, 0 is the only solution by the unique continuation principle. When $\frac{2N}{N+2} \leq p \leq 2$, 0 is the only radial solution. Indeed, define $u(x) := v(|x|)$, where v satisfies on $]0,1]$,

$$-(r^{N-1}|v'|^{p-2}v')' = \mu r^{N-1}|v|^{p^*-2}v.$$

Let us define

$$w := r^{N-1}|v'|^{p-2}v'$$

so that we obtain the system

$$\begin{cases} w' &= -\mu r^{N-1}|v|^{p^*-2}v \\ v' &= r^{(1-N)/(p'-1)}|w|^{p'-2}w, \end{cases}$$

where p' is the usual conjugate exponent of p. Since $v(1) = 0$ and, by the Pohozaev identity [[14], Theorem 1.1],

$$w(1) = |v'(1)|^{p-2}v'(1) = 0,$$

we obtain $v \equiv 0$. We use the fact that

$$\begin{cases} p' &\geq 2, \quad \text{since} \quad p \leq 2 \\ p^* &\geq 2, \quad \text{since} \quad p \geq \frac{2N}{N+2}. \end{cases}$$

The above argument was suggested to us by J. Mawhin [17].

5.1. **The radial case.** We assume now that

(C) Ω is the unit ball in $\mathbb{R}^N$ and a is a radial function.

We denote by $W^{1,p}_{0,\text{rad}}(\Omega)$ (resp. $\mathcal{D}^{1,p}_{\text{rad}}(\mathbb{R}^N)$) the space of radial functions in $W^{1,p}_0(\Omega)$ (resp. $\mathcal{D}^{1,p}(\mathbb{R}^N)$).
We define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\tilde{\phi}_\infty(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} - \mu\frac{|u|^{p^*}}{p^*}dx.$$

Theorem 5.1. *Under assumptions* **(A)**, **(B)** *and* **(C)**, *let* $\{u_n\}_n$ *be a sequence in* $W^{1,p}_{0,\mathrm{rad}}(\Omega)$ *such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_{0,rad}(\Omega))'.$$

Then, passing if necessary to a subsequence, there exists a solution $v_0 \in W^{1,p}_{0,\mathrm{rad}}(\Omega)$ *of*

$$-\Delta_p u + a|u|^{p-2}u = \mu|u|^{p^*-2}u,$$

a finite sequence $\{v_1, ..., v_k\} \subset \mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ *of solutions of*

$$-\Delta_p u = \mu|u|^{p^*-2}u \quad \text{on } \mathbb{R}^N$$

and k *sequences* $\{\lambda^i_n\}_n \subset \mathbb{R}_+$, *satisfying* $\lambda^i_n \to 0$, $n \to \infty$, *and*

$$\|u_n - v_0 - \sum_{i=1}^{k} (\lambda^i_n)^{(p-N)/p} v_i(\cdot/\lambda^i_n)\| \to 0,$$

$$\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p,$$

$$\phi(v_0) + \sum_{i=1}^{k} \tilde{\phi}_\infty(v_i) = c.$$

Proof. Dropping the hypothesis $\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0$, and the subscript $+$ to ϕ_∞, Lemma 3.5 and Lemma 3.6 continue to hold. The proof is the same, simply replacing Lemma 3.4 by the Brezis-Lieb Lemma.

We follow step 1) of the proof of Theorem 1.2 and we assume that

$$\int_\Omega |u^1_n|^{p^*} dx > \delta.$$

There exists a sequence $\{\lambda^1_n\}_n \subset]0, \infty[$ such that

$$\delta = \int_{B(0,\lambda^1_n)} |u^1_n|^{p^*} dx.$$

We assume that $v^1_n(x) := (\lambda^1_n)^{(N-p)/p} u^1_n(\lambda^1_n x)$ converges weakly to v_1 in $\mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and a.e. on $\mathbb{R}^N$. Using the fact that $v^1_n \to v_1$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\})$ (see e.g. [16]), it is easy to verify, as in step 3) of the proof of Theorem 1.2, that $v_1 \neq 0$. It is then easy to adapt the end of the proof of Theorem 1.2. ☐

5.2. **The case $\Omega = \mathbb{R}^N$.** We assume that

(D) $1 < p < N$, $\mu > 0$ and $a \in L^{N/p}(\mathbb{R}^N)$ is such that

$$\inf_{\substack{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \\ ||\nabla u||_{L^p} = 1}} \int_{\mathbb{R}^N} |\nabla u|^p + a|u|^p dx > 0.$$

We define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\phi(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} + a\frac{|u|^p}{p} - \mu\frac{|u|^{p^*}}{p^*} dx$$

and

$$\tilde{\phi}_\infty(u) := \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} - \mu \frac{|u|^{p^*}}{p^*} dx$$

The next result is due to Benci and Cerami [3] when $p = 2$ and to Alves [1] when $p > 2$.

Theorem 5.2. *Under assumption* **(D)**, *let* $\{u_n\}_n$ *be a sequence in* $\mathcal{D}^{1,p}(\mathbb{R}^N)$ *such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (\mathcal{D}^{1,p}(\mathbb{R}^N))'.$$

Then, passing if necessary to a subsequence, there exists a solution $v_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *of*

$$-\Delta_p u + a|u|^{p-2}u = \mu|u|^{p^*-2}u$$

and a finite sequence $\{v_1, ..., v_k\} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *of nontrivial solutions of*

$$-\Delta_p u = \mu|u|^{p^*-2}u \quad \text{on } \mathbb{R}^N$$

and k *sequences* $\{y_n^i\}_n \subset \mathbb{R}^N$ *and* $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, *satisfying*

$$\|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{(p-N)/p} v_i((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^p \to \sum_{i=0}^k \|v_i\|^p, \quad n \to \infty,$$

$$\phi(v_0) + \sum_{i=1}^k \tilde{\phi}_\infty(v_i) = c.$$

If $y_n^i \to y^i$, *then* $\lambda_n^i \to 0$ *or* $\lambda_n^i \to \infty$, *as* $n \to \infty$.

The proof is an easy adaptation of the proof of Theorem 1.2.

Remark 5.3. *If* $k \geq 1$ *and* $(u_n)_- \to 0$ *a.e., by the strong maximum principle* $v_i > 0$ *on* $\mathbb{R}^N$, *for* $1 \leq i \leq k$. This is the case when $\{u_n\}_n$ is nearby the positive cone in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Now we assume that

(E) $1 < p < N$, $\mu > 0$ and $a \in L^{N/p}(\mathbb{R}^N)$ radial is such that

$$\inf_{\substack{u \in \mathcal{D}^{1,p}_{rad}(\mathbb{R}^N) \\ ||\nabla u||_{L^p} = 1}} \int_{\mathbb{R}^N} |\nabla u|^p + a|u|^p dx > 0.$$

Let us define ϕ and $\tilde{\phi}_\infty$ on $\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ as in Theorem 5.2.

Theorem 5.4. *Under assumption* **(E)**, *let* $\{u_n\}_n$ *be a sequence in* $\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ *such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad \text{in } (\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N))'.$$

Then, passing if necessary to a subsequence, there exists a solution $v_0 \in \mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ *of*

$$-\Delta_p u + a|u|^{p-2}u = \mu|u|^{p^*-2}u$$

and a finite sequence $\{v_1, ..., v_k\} \subset \mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ *of nontrivial solutions of*

$$-\Delta_p u = \mu |u|^{p^*-2} u \quad on\ \mathbb{R}^N$$

and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, such that $\lambda_n^i \to 0$ or $\lambda_n^i \to \infty$ satisfying

$$\|u_n - v_0 - \sum_{i=1}^{k} (\lambda_n^i)^{(p-N)/p} v_i(\cdot/\lambda_n^i)\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p, \quad n \to \infty,$$

$$\phi(v_0) + \sum_{i=1}^{k} \tilde{\phi}_\infty(v_i) = c.$$

The proof is an easy adaptation of the proof of Theorem 5.1.

5.3. **Exterior domains.** In this section we extend to the p-Laplacian a result obtained in 1987 by Benci and Cerami in the case of the Laplacian (see [2]).

We assume that

(F) Ω is a smooth domain of $\mathbb{R}^N$ with bounded complement, $\mu > 0$, $1 < p < N$, $p < q < p^*$, $a \in \mathcal{C}(\Omega)$, $\lim_{|x| \to \infty} a(x) = 1$, $\inf_{\Omega} a > 0$.

The norm on $W_0^{1,p}(\Omega)$ is given by

$$||u||_{W^{1,p}} = (||u||_{L^p}^p + ||\nabla u||_{L^p}^p)^{1/p}.$$

We define ψ on $W_0^{1,p}(\Omega)$ by

$$\psi(u) = \int_\Omega \frac{|\nabla u|^p}{p} + a\frac{|u|^p}{p} - \mu \frac{|u|^q}{q} dx.$$

We define also ψ_∞ on $W^{1,p}(\mathbb{R}^N)$ by

$$\psi_\infty(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} + \frac{|u|^p}{p} - \mu \frac{|u|^q}{q} dx.$$

Theorem 5.5. *Under assumption **(F)**, let $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ such that*

$$\psi(u_n) \to c \qquad \psi'(u_n) \to 0 \quad in\ W^{-1,p'}(\Omega).$$

Then, passing if necessary to a subsequence, there exists a solution $v_0 \in W_0^{1,p}(\Omega)$ of

$$-\Delta_p u + a|u|^{p-2} u = \mu |u|^{q-2} u \quad in \quad \Omega,$$

a finite sequence $\{v_1, ..., v_k\} \subset W^{1,p}(\mathbb{R}^N)$ of solutions of

$$-\Delta_p u + |u|^{p-2} u = \mu |u|^{q-2} u \quad in \quad \mathbb{R}^N,$$

and k sequences $\{y_n^i\}_n$ satisfying

$$|y_n^i| \to \infty, \quad |y_n^i - y_n^{i'}| \to \infty, \quad \forall i \neq i', \quad n \to \infty,$$

$$\|u_n - v_0 - \sum_{i=1}^{k} v_i(\cdot - y_n^i)\|_{W^{1,p}} \to 0, \quad n \to \infty,$$

$$\|u_n\|_{W^{1,p}}^p \to \sum_{i=0}^{k} \|v_i\|_{W^{1,p}}^p, \quad n \to \infty,$$

$$\psi(v_0) + \sum_{i=1}^{k} \psi_\infty(v_i) = c.$$

The proof is an adaptation using Lemma 3.1 and Theorem 3.2 of the proof given in [[31], p. 120] for the case $p = 2$.

Remark 5.6. *As observed before, if* $k \geq 1$ *and* $(u_n)_- \to 0$ *a.e., then* $v_i > 0$ *on* $\mathbb{R}^N$, *for* $1 \leq i \leq k$. This is the case when $\{u_n\}_n$ is nearby the positive cone in $W^{1,p}(\Omega)$.

6. **A critical problem for the p-Laplacian.** In this section we extend to the p-Laplacian operator the existence results of [7], where the case $p = 2$ has been treated. More precisely, we consider the problem

$$\begin{cases} -\Delta_p u + \frac{f(|x|/\lambda)}{\lambda^p}|u|^{p-2}u = |u|^{p^*-2}u & \text{in} \quad B \\ u \geq 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases} \tag{10}$$

where B is the unit ball in $\mathbb{R}^N$ and $1 < p < N$.
We assume, similarly as in [7], that

$$f : [0, \infty) \to [0, \infty) \quad \text{is such that}$$

$$f \not\equiv 0 \quad \text{on a set of positive measure}, \quad f \in L_{loc}^{N/p}([0, \infty), s^{N-1}ds) \tag{11}$$

and

$$\lim_{\lambda \downarrow 0} \lambda^{N-p} \int_0^{1/\lambda} f(s)s^{N-1}ds = 0. \tag{12}$$

Our main result is

Theorem 6.1. *Assume* f *satisfies (11),(12). Then there exists* λ_0 *such that the problem (10) admits a nontrivial radial solution in* $W_0^{1,p}(B)$, *for* $0 < \lambda < \lambda_0$.

The problem

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in} \quad B \\ u > 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases}$$

was solved in the classical paper [6] by H. Brezis and L. Nirenberg. When $p = 2$, Theorem 6.1 is due to H. Brezis and the second author. When $p = 2$ and $f \in L^{N/2}([0, \infty), s^{N-1}ds)$, the existence of a positive solution was proved by D. Passaseo in [19].

We define the manifolds

$$V(B) = \{u \in W_0^{1,p}(B) : \ u \text{ radial}, \ \|u\|_{p^*} = 1\}$$

$$V(\mathbb{R}^N) = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \ u \text{ radial}, \ \|u\|_{p^*} = 1\},$$

and for any $\lambda > 0$, the functionals

$$\varphi_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^p + \frac{f(|x|/\lambda)}{\lambda^p}|u|^p dx, \qquad \psi_\lambda(u) = \int_{\mathbb{R}^N} \frac{|x|}{\lambda + |x|}|u|^{p^*} dx.$$

It is worth pointing out that φ_λ can be unbounded although it is defined on $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

We also define the levels

$$c(\lambda) = \inf\{\varphi_\lambda(u) : u \in V(B), \psi_\lambda(u) \geq 1/2\},$$
$$d(\lambda) = \inf\{\varphi_\lambda(u) : u \in V(B), \psi_\lambda(u) = 1/2\},$$

and

$$d = \inf\{\varphi_1(u) : u \in V(\mathbb{R}^N), \psi_1(u) = 1/2\}.$$

As in [7] we shall prove $c(\lambda)$ to be a critical value for $\varphi_\lambda\big|_{V(B)}$, by estimating $d(\lambda)$ and d.

The following lemmas extend Lemma 4.1 and Lemma 4.2 in [7] to the p-Laplacian operator. We define S_p to be the best Sobolev constant (see e.g. [26]):

$$S_p := \inf_{u\in V(\mathbb{R}^N)} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}.$$

Lemma 6.2. *Under the assumptions (11), for every $\lambda > 0$, we have $S_p < d \leq d(\lambda.)$*

Proof. Observe that $S_p \leq d$. Suppose by contradiction that $S_p = d$. Then there exists a sequence $\{u_n\}_n \subset V(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^p + f(|x|)|u_n|^p dx \to S_p \qquad \int_{\mathbb{R}^N} \frac{|x|}{1+|x|}|u_n|^{p^*} dx = \frac{1}{2}, \qquad \|u_n\|_{p^*} = 1.$$

Since f is nonnegative we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \to S_p, \qquad \|u_n\|_{p^*} = 1.$$

Define

$$S(u) := \int_{\mathbb{R}^N} |\nabla u_n|^p dx$$

and

$$\phi(u) := \int_{\mathbb{R}^N} (\frac{|\nabla u_n|^p}{p} - \frac{|u|^{p^*}}{p^*}) dx.$$

By the Ekeland variational principle there exists a Palais-Smale sequence for $S\big|_{V(\mathbb{R}^N)}$ at the level S_p, namely there exist $\{\beta_n\}_n \subset \mathbb{R}_+$ and $\{u_n\}_n \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that

$$\|u_n - \tilde{u}_n\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \longrightarrow 0$$

and

$$S(\tilde{u}_n) \longrightarrow S_p \qquad -\Delta_p \tilde{u}_n - \beta_n |\tilde{u}_n|^{p^*-2}\tilde{u}_n \longrightarrow 0 \quad \text{in} \, (\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N))'.$$

Put $v_n := \beta_n^{1/(p^*-p)}\tilde{u}_n$, so we have

$$\beta_n \longrightarrow S_p$$

and

$$\phi(v_n) \longrightarrow \frac{S_p^{N/p}}{N}, \qquad \phi'(v_n) \longrightarrow 0.$$

We can assume, passing to a subsequence, that $v_n \rightharpoonup v_0$ in $\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$. By Theorem 5.4 (for $a \equiv 0$) there exist k functions $v_1, ..., v_k \in \mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ such that

$$-\Delta_p v_i = |v_i|^{p^*-2} v_i \quad \text{in } \mathbb{R}^N$$

for $i = 0, 1, ..., k$ and

$$\phi(v_n) = \phi(v_0) + \sum_{i=1}^{k} \phi(v_i) + o(1) = \frac{1}{N} \sum_{i=1}^{k} \int_{\mathbb{R}^N} |v_i|^{p^*} + o(1). \tag{13}$$

Multiplying the equation by $(v_i)_+$ and $(v_i)_-$, and using the Sobolev inequality, for each $i = 0, 1, ..., k$ one of the following cases hold:

$$\int_{\mathbb{R}^N} |v_i|^{p^*} dx = 0,$$

$$\int_{\mathbb{R}^N} |v_i|^{p^*} dx \geq S_p^{N/p},$$

$$\int_{\mathbb{R}^N} |v_i|^{p^*} dx \geq 2S_p^{N/p},$$

according to the cases v_i is (respectively) zero, has constant sign or changes sign.

If $v_0 \neq 0$, from (13) we have

$$\frac{S_p^{N/p}}{N} \geq (k+1)\frac{S_p^{N/p}}{N},$$

hence $k = 0$. This implies that $\{v_n\}_n$ is relatively compact in $\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ and $u := S_p^{1/(p-p^*)} v_0$ is optimal for S_p. This occurs only if u is a Talenti function $u(x) = [a + b|x|^{p/(p-1)}]^{1-N/p}$, (see [26],[6]). By using the Fatou lemma together with the weakly lower semicontinuity of $S(\cdot)$ we get a contradiction:

$$S_p < \int_{\mathbb{R}^N} |\nabla u|^p + f(|x|)|u|^p dx \leq S_p.$$

If $v_0 = 0$, the above argument yields

$$\frac{S_p^{N/p}}{N} \geq k\frac{S_p^{N/p}}{N},$$

hence $k = 0, 1$.

If $k = 0$, we have $u_n \to 0$ strongly in $\mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ and this is impossible, since $\|u_n\|_{p^*} = 1$.

If $k = 1$ we distinguish the cases $\lambda_n^1 \to 0, \infty$. Define

$$w_n := \lambda_n^{(p-N)/p} v_1(\cdot/\lambda_n).$$

If $\lambda_n^1 \to 0$ we have

$$\begin{aligned}
\beta_n^{N/p}(\frac{1}{2} + o(1)) &= \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |v_n|^{p^*} dx \\
&= \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |w_n|^{p^*} dx + o(1) \\
&= \int_{\mathbb{R}^N} \frac{|x|}{\lambda_n^{-1}+|x|} |v_1|^{p^*} dx + o(1) \longrightarrow 0,
\end{aligned}$$

where, in the last step the following invariance property has been used:

$$\psi_1(v_\lambda) = \psi_\lambda(u), \qquad \text{where} \quad v_\lambda(x) := \lambda^{\frac{N-p}{p}} u(\lambda x). \tag{14}$$

Hence we get a contradiction.
In the case $\lambda_n^1 \to \infty$, we have

$$\begin{aligned}
& \beta_n^{N/p}(\frac{1}{2}+o(1)) \\
= & \int_{\mathbb{R}^N} \frac{|x|}{1+|x|}|v_n|^{p^*}dx \quad \text{(by the Ekeland variational principle)} \\
= & \int_{\mathbb{R}^N} \frac{|x|}{1+|x|}|w_n|^{p^*}dx + o(1) \quad \text{(by Theorem 5.4)} \\
= & \int_{\mathbb{R}^N} \frac{|x|}{\lambda_n^{-1}+|x|}|v_1|^{p^*}dx + o(1) \quad \text{(by (14))} \\
= & \int_{\mathbb{R}^N} |v_1|^{p^*}dx + o(1) \quad \text{(by the dominated convergence theorem)} \\
= & \int_{\mathbb{R}^N} |w_n|^{p^*}dx + o(1) \quad \text{(by invariance)} \\
= & \int_{\mathbb{R}^N} |v_n|^{p^*}dx + o(1) \quad \text{(by Theorem 5.4)} \\
= & \beta_n^{N/p}(1+o(1)) \quad \text{(by the Ekeland variational principle).}
\end{aligned}$$

Since this is also a contradiction, this proves that $S_p < d$.

Finally, take $u \in V(B)$ such that $\psi_\lambda(u) = 1/2$ and define $v_\lambda(x) = \lambda^{\frac{N-p}{p}} u(\lambda x)$ if $|x| \leq 1/\lambda$ and $v_\lambda(x) = 0$ otherwise. Since

$$\psi_1(v_\lambda) = \psi_\lambda(u) = 1/2, \qquad \varphi_1(v_\lambda) = \varphi_\lambda(u), \qquad \|v_\lambda\|_{p^*} = \|u\|_{p^*} = 1$$

we get $d \leq d(\lambda)$. □

Lemma 6.3. *Under the assumptions (11),(12), for every $\lambda > 0$, we have $S_p < c(\lambda)$ and*

$$\lim_{\lambda \downarrow 0} c(\lambda) = S_p.$$

Proof. In order to prove $S_p < c(\lambda)$ we argue as in Lemma 6.2. Indeed, assume by contradiction that $S_p = c(\lambda)$, then there exist a sequence $\{u_n\}_n \subset V(B)$ such that

$$\int_B |\nabla u_n|^p + \frac{f(|x|/\lambda)}{\lambda^p}|u_n|^p dx \to S_p \qquad \int_B \frac{|x|}{\lambda+|x|}|u_n|^{p^*}dx \geq \frac{1}{2}, \qquad \|u_n\|_{p^*} = 1.$$

Arguing as in the previous lemma we can consider Palais-Smale sequences. We repeat the above argument, by using Theorem 5.1. In the case $v_0 = 0$ and $k = 1$, only the case $\lambda_n \to 0$ can occur. In this case we have

$$\begin{aligned}
& \beta_n^{N/p}(\frac{1}{2}+o(1)) \\
\leq & \int_B \frac{|x|}{1+|x|}|v_n|^{p^*}dx \\
= & \int_{|x|\leq\lambda_n} \frac{|x|}{1+|x|}|v_n|^{p^*}dx + o(1) \quad (\text{since } v_n \to 0 \text{ in } L^{p^*}_{loc}(\mathbb{R}^N \setminus \{0\})) \\
\leq & C\lambda_n + o(1) \longrightarrow 0,
\end{aligned}$$

for some positive constant C. This is a contradiction.

On the other hand, if $v_0 \neq 0$, then $u := S_p^{1/(p-p^*)} v_0 \in W_0^{1,p}(B)$ is optimal for S_p. This is again a contradiction, hence $S_p < c(\lambda)$.

The rest of the proof is the same as in Lemma 4.2 in [7] and requires condition (12). ◻

Remark 6.4. Lemma 6.2 and Lemma 6.3 can be proved, as in [7], without passing to Palais-Smale sequences, by using a decomposition lemma instead of Theorem 5.4.

Proof of Theorem 6.1. By Lemma 6.2 and Lemma 6.3, there exists δ such that, for $0 < \lambda < \delta$,

$$S_p < c(\lambda) < \min\{d, 2^{p/N} S_p\} \leq d(\lambda). \tag{15}$$

In particular, since $c(\lambda) < d(\lambda)$, by the Ekeland variational principle there exists a Palais-Smale sequence for $\varphi_\lambda\big|_{V(B)}$ at the level $c(\lambda)$. Namely, there exist a sequence $\{u_n\}_n \subset V(B)$ and a sequence $\{\alpha_n\}_n \subset \mathbb{R}$ such that

$$\varphi_\lambda(u_n) \longrightarrow c(\lambda) \quad -\Delta_p u_n + \frac{f(|x|/\lambda)}{\lambda^p}|u_n|^{p-2}u_n - \alpha_n |u_n|^{p^*-2}u_n \longrightarrow 0 \text{in}\,(W^{1,p}_{0,rad}(B))'.$$

Notice that, since $u_n \in V(B)$, then $\varphi_\lambda(u_n) - \alpha_n \to 0$ and $\alpha_n \to c(\lambda)$.
Now define $v_n = \alpha_n^{1/(p^*-p)} u_n$, and notice that

$$\Phi(v_n) = \int_B \frac{|\nabla v_n|^p}{p} + \frac{f(|x|/\lambda)}{\lambda^p}\frac{|v_n|^p}{p} - \frac{|v_n|^{p^*}}{p^*} dx \longrightarrow \frac{c(\lambda)^{N/p}}{N}, \tag{16}$$

$$-\Delta_p v_n + \frac{f(|x|/\lambda)}{\lambda^p}|v_n|^{p-2}v_n - |v_n|^{p^*-2}v_n \longrightarrow 0 \quad \text{in}\,(W^{1,p}_{0,rad}(B))'. \tag{17}$$

By (15) we have

$$\frac{S_p^{N/p}}{N} < \frac{c(\lambda)^{N/p}}{N} < 2\frac{S_p^{N/p}}{N}. \tag{18}$$

Theorem 5.1 yields the following decomposition:

$$\Phi(v_n) = \Phi(v) + \frac{1}{N}\sum_{i=1}^{k}\int_{\mathbb{R}^N}|w_i|^{p^*}dx + o(1),$$

$$\|v_n - v - \sum_{i=1}^{k}(\lambda_n^i)^{(p-N)/p} w_i(\cdot/\lambda_n^i)\| \to 0, \tag{19}$$

$w_i \in \mathcal{D}^{1,p}_{rad}(\mathbb{R}^N)$ being solutions of

$$-\Delta_p u = |u|^{p^*-2}u \quad \text{in}\,\mathbb{R}^N \tag{20}$$

and $v \in W^{1,p}_{0,rad}(B)$ satisfying

$$-\Delta_p v + \frac{f(|x|/\lambda)}{\lambda^p}|v|^{p-2}v = |v|^{p^*-2}v. \tag{21}$$

Hence

$$\Phi(v) + \frac{1}{N}\sum_{i=1}^{k}\int_{\mathbb{R}^N}|w_i|^{p^*}dx = \frac{c(\lambda)^{N/p}}{N}. \tag{22}$$

We can use the argument in [[7] p. 12-13]. Multiplying (20) by w_i^+ and w_i^-, for each i, one of the following cases hold:

$$\int_{\mathbb{R}^N}|w_i|^{p^*}dx = 0, \tag{23}$$

$$\int_{\mathbb{R}^N} |w_i|^{p^*} dx \geq S_p^{N/p},$$

$$\int_{\mathbb{R}^N} |w_i|^{p^*} dx \geq 2S_p^{N/p},$$

according to the cases w_i is (respectively) zero, has constant sign or changes sign. Similarly, we can have only one of the following possibilities:

$$v = 0 \quad \Rightarrow \quad \Phi(v) = 0,$$
$$\Phi(v) \geq \frac{S_p^{N/p}}{N}, \tag{24}$$
$$\Phi(v) \geq 2\frac{S_p^{N/p}}{N}.$$

Finally, as a consequence of (18) and (22), the only possible case is (23) together with (24).
Hence, by (19), $v_n \to v$ in $W_0^{1,p}(B)$ and we can choose $v \geq 0$. From (16) and (17), v is a solution of (21) and $\Phi(v) = \frac{c(\lambda)^{N/p}}{N}$, and this concludes the proof of Theorem 6.1. □

Theorem 6.5. *Let u be the solution found in Theorem 6.1 and $f \in L^\infty_{loc}(B)$, then u is positive. Furthermore, if $f \in L^{N/(p-\varepsilon)}(B)$, for some $\varepsilon \in (0,1]$, then $u > 0$ in B and $u \in W_0^{1,p}(B) \cap L^\infty(B)$.*

In order to prove the positivity of the solution we use the strong maximum principle (see e.g. [18],[30]). Indeed, since we have

$$-\Delta_p u + \frac{f(|x|/\lambda)}{\lambda^p} u^{p-1} \geq 0$$

and $f \in L^\infty_{loc}(B)$, we can apply Theorem 2, p. 434 in [18].
In order to prove the L^∞ regularity for u, we argue as in [20], Appendix E. By a slight modification of the method in [29] p. 272–273, it is possible to prove that

$$u \in L^{\beta p^*}(B),$$

for some $\beta > 1$. Writing the equation (21) as

$$-\Delta_p u = a(x) u^{p-1},$$

where $a(x) := u^{p^*-p} - \frac{f(|x|/\lambda)}{\lambda^p}$, since $f \in L^{N/(p-\varepsilon)}(B)$, the conclusion follows from the lemma below, which can be proved adapting, to the p-Laplacian operator, Lemma p. 268 in [29]). We refer to [20], Appendix E.

Lemma 6.6. *Let $u \in W_0^{1,p}(B)$ be a weak solution of the equation*

$$-\Delta_p u = a(x)|u|^{p-2}u \quad \text{in } B,$$

where $a \in L^r(B)$, for some $r > N/p$. Then u is bounded.

Remark 6.7. We point out that, under suitable assumptions on f, it is possible to check the $C^{1,\alpha}_{loc}$ regularity up to the boundary. See [12], [15], [24],[27] and [28] .

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