Bound sets approach to impulsive Floquet problems for vector second-order differential inclusions

Martina Pavlačková* **and Valentina Taddei**[**1]
(*) Dept. of Computer Science and Appl. Mathematics, Moravian Business College Olomouc, tř. Kosmonautů 1288/1, 77900 Olomouc, Czech Republic,
e-mail: pavlackovam@centrum.cz
ORCID ID 0000-0002-7376-5755
(**) Dept. of Sciences and Methods for Engineering,
University of Modena and Reggio Emilia,
Via G. Amendola, 2 - pad. Morselli, I-42122 Reggio Emilia, Italy,
e-mail: valentina.taddei@unimore.it
ORCID ID 0000-0001-6171-2499

Abstract

In this paper, the existence and the localization of a solution of an impulsive vector multivalued second-order Floquet boundary value problem are investigated. The method used in the paper is based on the combination of a fixed point index technique with bound sets approach. At first, problems with upper-Carathéodory right-hand sides are investigated and it is shown afterwards how can the conditions be simplified in more regular case of upper semi-continuous right hand side. In this more regular case, the conditions ensuring the existence and the localization of a solution are put directly on the boundary of the considered bound set. This strict localization of the sufficient conditions is very significant since it allows some solutions to escape from the set of candidate solutions. In both cases, the C^1-bounding functions with locally Lipschitzian gradients are considered at first and it is shown afterwards how the conditions change in case of C^2-bounding functions. The paper concludes with an application of obtained results to Liénard-type equations and inclusions and the comparisons of our conclusions with the few results related to impulsive periodic and antiperiodic Liénard equations are obtained.

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1 Introduction

Boundary value problems with impulses have been widely studied because of their applications in areas, where the parameters are subject to sudden perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment or in environmental sciences, they can describe the seasonal changes or harvesting. Standardly, the right-hand sides of studied impulsive problems have been single-valued. However, it is worth to study also the multivalued case since it comes from single-valued problems with discontinuous right-hand sides, from control theory, or from practical applications concerning population genetics, power law fluids, and many other branches.

[1]corresponding author

In this paper, the following second-order multivalued vector Floquet problem will be studied

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \tag{1}$$

$$x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0), \tag{2}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory (or upper semi-continuous) multivalued mapping and M and N are real $n \times n$ matrices.

In the paper, the solvability of the Floquet b.v.p. (1), (2) will be investigated in the presence of the following linear impulse conditions

$$x(t_i^+) = A_i x(t_i), \quad i = 1, \ldots, p, \tag{3}$$

$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \ldots, p, \tag{4}$$

where a finite number of impulse points $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$ and real $n \times n$ matrices A_i, B_i, $i = 1, \ldots, p$, are given and the notation $\lim_{t \to a^+} x(t) = x(a^+)$ is used. The presence of the impulses take into account the possibility that the natural phenomena described by the model doesn't propagate continuously, but it is subject to short term perturbations in time. For instance, in the periodic treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics, impulses may correspond to abrupt changes of prices. We consider the case of linear impulses described by means of matrices A_i and B_i.

By a *solution* of problem (1) – (4) we shall mean a function $x \in PAC^1([0, T], \mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1) – (4).

If we would focus the attention to the literature overview dealing with impulsive boundary value problems, we would find out that the theory of single-valued impulsive problems has been deeply examined and presents in many cases direct analogies with the results for problems without impulses (see, e.g., [8, 9, 20]). On the other hand, the theory dealing with multivalued impulsive problems has not been so deeply studied and the results have been obtained in particular for the first-order problems and using fixed point theorems or upper and lower-solutions methods; for the overview of known-results, see, e.g., the monographs [12, 18] and the references therein. Besides these techniques, also topological and variational approaches have been used for Dirichlet impulsive problems with right-hand sides not depending on the first derivatives or with impulses depending only on the first derivatives (see, e.g., [1, 14, 15, 16, 21]).

In this paper, we consider the second order inclusion (1) with the right-hand side depending also on the first derivative, together with impulses depending both on the solution and its first derivative, obtaining the existence and the localization results for the associated Floquet problem (1)-(4). The results will be proven by the combination of a continuation principle with bound sets technique. Bound sets approach which is used in Sections 3 and 4 of our paper was initiated by Gaines and Mawhin in [17] for proving the existence of periodic solutions of first-order as well as second-order systems of differential equations (see also the references therein). Bound sets theory for multivalued Dirichlet or Floquet problems without impulses was developed in [2]-[7],[24], [28]. Recently, bound sets approach has been applied also for multivalued impulsive Dirichlet problem in [25] and [26]. In this paper, it will be shown how changing the boundary conditions from Dirichlet to Floquet will affect the assumptions guaranteeing the existence of a bound set for the considered impulsive boundary value problem. Furthermore, it will be illustrated in the final part of the paper how the existence and localization results can be applied to Liénard type equation which is a generalization of the Duffing equation, the Josephson equation, the Van der Pol equation, or of the pendulum equation. Also some comparisons with the few results related to impulsive periodic and antiperiodic Liénard equations will be obtained, showing not only that we are able to threat a quite general equation, but also that the regularity assumptions that we need are weaker than those assumed in the literature.

The paper is organized as follows. In the second section, suitable definitions and statements which will be used in the sequel are recalled. Sections 3 and 4 are devoted to studying of bound sets and Liapunov-like bounding functions for impulsive Floquet problems with upper-Carathéodory or upper semi-continuous right-hand sides. In both sections, the C^1-bounding functions with locally Lipschitzian gradients are considered at first. Consequently, it is shown how conditions ensuring the existence of a bound set change in case of C^2-bounding functions. In Section 5, the bound sets approach is combined with the continuation principle and the existence and localization results are obtained in this way for the impulsive Floquet problem (1)-(4). Final section 6 deals with an application to the Liénard type equation.

2 Preliminaries

Let us start with notations we use in the paper. If (X, d) is a metric space and $A \subset X$, by $\overline{A}$, $Int\ A$ and ∂A, we mean the *closure*, the *interior* and the *boundary* of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

Consider a function $V : \mathbb{R}^n \to \mathbb{R}$. If V is of class C^1 we will denote by ∇V its gradient, i.e.

$$\nabla V = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n} \right).$$

If V is of class C^2 we will denote by HV its Hessian matrix, i.e.

$$HV = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} & \cdots & \frac{\partial^2 V}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{pmatrix}.$$

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J. In the sequel, the norm of a real $n \times n$ matrix will be denoted by $||\cdot||$ and the norm in $L^1(J, \mathbb{R})$ by the symbol $||\cdot||_1$.

Let $PAC^1([0, T], \mathbb{R}^n)$ be the space of all functions $x : [0, T] \to \mathbb{R}^n$ such that

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ . & \\ . & \\ . & \\ x_{[p]}(t), & \text{for } t \in (t_p, T], \end{cases}$$

where $x_{[0]} \in AC^1([0, t_1], \mathbb{R}^n), x_{[i]} \in AC^1((t_i, t_{i+1}], \mathbb{R}^n), x(t_i^+) = \lim\limits_{t \to t_i^+} x(t) \in \mathbb{R}$ and $\dot{x}(t_i^+) = \lim\limits_{t \to t_i^+} \dot{x}(t) \in \mathbb{R}$, for every $i = 1, ..., p$. The space $PAC^1([0, T], \mathbb{R}^n)$ is a normed space with the norm

$$||x||_E := \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |\dot{x}(t)|. \tag{5}$$

In the sequel, it will be denoted by $(E, ||\cdot||_E)$. In a similar way, we can define the spaces $PC([0,T],\mathbb{R}^n)$ and $PC^1([0,T],\mathbb{R}^n)$ as the spaces of functions $x : [0,T] \to \mathbb{R}^n$ satisfying the previous definition with $x_{[0]} \in C([0,t_1],\mathbb{R}^n)$ and $x_{[i]} \in C((t_i,t_{i+1}],\mathbb{R}^n)$ for every $i = 1,..,p$, or with $x_{[0]} \in C^1([0,t_1],\mathbb{R}^n)$ and $x_{[i]} \in C^1((t_i,t_{i+1}],\mathbb{R}^n)$, for every $i = 1,..,p$, respectively. The space $PC^1([0,T],\mathbb{R}^n)$ with the norm defined by (5) is a Banach space (see [22, page 128]).

We also need the following definitions and notions from multivalued theory in the sequel. We say that F is a *multivalued mapping* from X to Y (written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, i.e. the subset of $X \times Y$ defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

The single-valued function $f : X \to Y$ is called a *selection* of F if $\Gamma_f \subset \Gamma_F$, i.e. if $f(x) \in F(x)$, for every $x \in X$.

A multivalued mapping $F : X \multimap Y$ is called *upper semi-continuous* (shortly, u.s.c.) if, for each open set $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega,\mathcal{U},\mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that the mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\cdot,x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $F(t,\cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for a.a. $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times \mathbb{R}^m$.

We employ the following selection result in the sequel, which was proved in [13, Proposition 6] in a quite general setting for continuous function q. Its proof can be easily extended to the piecewise continuous functions, so we omit it here.

Proposition 2.1 *Let $J \subset \mathbb{R}$ be a compact interval and $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that for every $r > 0$ there exists an integrable function $\mu_r : J \to [0,\infty)$ satisfying $|y| \le \mu_r(t)$, for every $(t,x) \in J \times \mathbb{R}^m$, with $|x| \le r$, and every $y \in F(t,x)$. Then the composition $F(\cdot,q(\cdot))$ admits, for every $q \in PC(J,\mathbb{R}^m)$, a measurable selection.*

The following continuation principle, that was proven in [25, Proposition 2.4] for impulsive boundary value problems, is the crucial tool that will be used in the paper for obtaining the existence and localization result.

Proposition 2.2 *Let us consider the b.v.p.*

$$\left.\begin{array}{c} \ddot{x}(t) \in F(t,x(t),\dot{x}(t)),\ \textit{for a.a. } t \in [0,T], \\ x \in S, \end{array}\right\} \tag{6}$$

where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of E. Let $H : [0,T] \times \mathbb{R}^{4n} \times [0,1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset F(t,c,d), \quad \textit{for all } (t,c,d) \in [0,T] \times \mathbb{R}^{2n}.$$

Assume that

(i) *there exists a retract Q of $PC^1([0,T],\mathbb{R}^n)$, with $Q\setminus\partial Q\neq\emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\left.\begin{array}{c}\ddot{x}(t)\in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda),\quad \text{for a.a. } t\in[0,T],\\ x\in S_1\end{array}\right\}\tag{7}$$

has, for each $(q,\lambda)\in Q\times[0,1]$, a non-empty and convex set of solutions $\mathfrak{T}(q,\lambda)$;

(ii) *there exists a nonnegative, integrable function $\alpha:[0,T]\to\mathbb{R}$ such that*

$$|H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda)|\leq\alpha(t)(1+|x(t)|+|\dot{x}(t)|),\ \text{for a.a. } t\in[0,T],$$

for any $(q,\lambda,x)\in\Gamma_{\mathfrak{T}}$;

(iii) $\mathfrak{T}(Q\times\{0\})\subset Q$;

(iv) *there exist constants $M_0\geq 0,\ M_1\geq 0$ such that $|x(0)|\leq M_0$ and $|\dot{x}(0)|\leq M_1$, for all $x\in\mathfrak{T}(Q\times[0,1])$;*

(v) *the solution map $\mathfrak{T}(\cdot,\lambda)$ has no fixed points on the boundary ∂Q of Q, for every $\lambda\in[0,1)$.*

Then the b.v.p. (6) *has a solution in $S_1\cap Q$.*

The continuation principle described in Proposition 2.2 requires in particular that any of corresponding problems given in (7) does not have solutions tangent to the boundary of a given set Q of candidate solutions. This will be guaranteed in the paper using the bound sets approach studied in next two sections and by means of the following result based on Nagumo conditions (see [27, Lemma 2.1] and [19, Lemma 5.1]).

Proposition 2.3 *Let $\phi:[0,+\infty)\to[0,+\infty)$ be a continuous and non-decreasing function, with*

$$\lim_{s\to\infty}\frac{s^2}{\phi(s)}ds=\infty,\tag{8}$$

and let R be a positive constant. Then there exists a positive constant

$$B=\phi^{-1}(\phi(2R)+2R)\tag{9}$$

such that if $x\in PC^1([0,T],\mathbb{R}^n)$ is such that $|\ddot{x}(t)|\leq\phi(|\dot{x}(t)|)$, for a.a. $t\in[0,T]$, and $|x(t)|\leq R$, for every $t\in[0,T]$, then it holds that $|\dot{x}(t)|\leq B$, for every $t\in[0,T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [5]) that the statement holds also for piecewise continuously differentiable functions.

3 Bound sets theory for impulsive Floquet problem with upper-Carathéodory r.h.s.

The direct verification of transversality condition (v) in Proposition 2.2 is quite complicated. Therefore, a Liapunov-like function V, usually called a *bounding function*, which can guarantee this condition will be introduced now.

Hence, let $K\subset\mathbb{R}^n$ be a nonempty, open set and let $V:\mathbb{R}^n\to\mathbb{R}$ be a continuous function satisfying

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$.

Definition 3.1 The set K is called a *bound set for the impulsive Floquet problem* $(1)-(4)$ if every solution x of problem $(1)-(4)$ such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0,T]$.

Remark 3.1 Let us note that the existence of a bound set K for the Floquet problem $(1)-(4)$ does not guarantee the existence of a solution of problem $(1)-(4)$. It only ensures that if there would exist a solution laying in $\overline{K}$, then this solution would not touch the boundary of K at any point, i.e. it would lay in $Int\ K$.

At first, sufficient conditions for the existence of a bound set for the impulsive Floquet problem $(1)-(4)$ in the general case will be shown in Proposition 3.1 below. Afterwards, the regularity assumptions on the bounding function V will be made more strict and the practically applicable version of Proposition 3.1 will be obtained (see Corollary 3.1 below).

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be a nonempty open set, M and N be real $n \times n$ matrices with M invertible satisfying*

$$M\partial K = \partial K \tag{10}$$

and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1,\ldots,p$, be real $n \times n$ matrices, A_i invertible and such that $A_i\partial K = \partial K$, for all $i = 1,\ldots,p$.
Assume that there exists a function $V \in C^1(\mathbb{R}^n,\mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions $(H1)$ and $(H2)$. Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, the following condition

$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{11}$$

holds, for all $w \in F(t,x,v)$, and that

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0, \tag{12}$$

for all $i = 1,\ldots,p$, $x \in \partial K$ and $v \in \mathbb{R}^n$ with $\langle \nabla V(x), v\rangle \neq 0$.
Moreover, let

$$\langle \nabla V(Mx), Nv\rangle \cdot \langle \nabla V(x), v\rangle \geq 0, \tag{13}$$

for all $x \in \partial K$ and $v \in \mathbb{R}^n$.
Then K is a bound set for the impulsive Floquet problem (1)-(4).

Proof. We assume, by a contradiction, that K is not a bound set for the Floquet problem (1)-(4), i.e. that there exist a solution $x : [0,T] \to \overline{K}$ of problem (1)-(4) and $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. If the point t^* lays in $\{0,T\}$, we can take without the loss of generality, $t^* = T$, according to (10). Let us define the function $g : [0,T] \to \mathbb{R}$ by the formula $g(t) := V(x(t))$. According to the properties of x and V, $g \in PC^1([0,T],\mathbb{R})$ and $g(t) \leq 0$, for all $t \in [0,T]$. Since $g(T) = 0$, the point T is a local maximum point for g. Therefore, $\dot{g}(T) \geq 0$. According to the boundary conditions, also $x(0) \in \partial K$, i.e. also 0 is a local maximum point for g, hence

$$0 \geq \dot{g}(0) = \langle \nabla V(x(0)), \dot{x}(0)\rangle .$$

Moreover, since $x(T) = Mx(0)$ and $\dot{x}(T) = N\dot{x}(0)$, we have

$$0 \le \dot{g}(T) = \langle \nabla V(x(T)), \dot{x}(T) \rangle = \langle \nabla V(Mx(0)), N\dot{x}(0) \rangle .$$

Condition (13) then implies

$$\langle \nabla V(x(0)), \dot{x}(0) \rangle = \langle \nabla V(Mx(0)), N\dot{x}(0) \rangle = 0.$$

This is equivalent to $\dot{g}(0) = \dot{g}(T) = 0$. Therefore, whatever is $t^*, \dot{g}(t^*) = 0$ and we can subsequently proceed like in the proof of Proposition 3.3 in [25]. □

Definition 3.2 A function $V : \mathbb{R}^n \to \mathbb{R}$ from Proposition 3.1 satisfying $(H1)$, $(H2)$ and conditions (11), (12) and (13) is called a *bounding function* for the set K relative to $(1) - (4)$.

When the bounding function V is of class C^2, the condition (11) can be rewritten in terms of gradients and Hessian matrices and the following corollary immediately follows.

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be a nonempty open set, M and N be real $n \times n$ matrices with M invertible satisfying* (10) *and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T,\ p \in \mathbb{N}$, be given and let $A_i,\ B_i,\ i = 1, \ldots, p$, be real $n \times n$ matrices, A_i invertible and such that $A_i \partial K = \partial K$, for all $i = 1, \ldots, p$.*
Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions $(H1), (H2)$ and (12) *and* (13). *Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$ and $v \in \mathbb{R}^n$, condition*

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0 \tag{14}$$

holds, for all $w \in F(t, x, v)$.
Then K is a bound set for the impulsive Floquet problem $(1) - (4)$.

Proof. The statement of Corollary 3.1 follows immediately from the fact that if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T),\ v \in \mathbb{R}^n$ and $w \in F(t, x, v)$, there exists

$$\lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} = \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle.$$

□

Remark 3.2 In conditions (11)-(14), the element v plays the role of the first derivative of the solution x. If x is a solution of $(1) - (4)$ such that $x(t) \in \overline{K}$, for every $t \in [0,T]$, and if there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying condition (8) and such that

$$|F(t, c, d)| \le \phi(|d|),$$

for a.a. $t \in [0,T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \le R := \max\{|x| : x \in \overline{K}\}$, then, according to Proposition 2.3, it holds that $|\dot{x}(t)| \le B$, for every $t \in [0,T]$, where B is defined by (9). Hence, it is sufficient to require conditions (11)-(14) only for all $v \in \mathbb{R}^n$ with $|v| \le B$ and not for all $v \in \mathbb{R}^n$.

4 Bound sets theory for Floquet problem with upper semi-continuous r.h.s.

In this section, we will show how conditions ensuring the existence of a bound set for the impulsive Floquet problem $(1)-(4)$ change in case of upper semi-continuous r.h.s. Firstly, we will consider a smooth bounding function V with a locally Lipschitzian gradient.

Proposition 4.1 *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semi-continuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n,\mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Furthermore, assume that M and N are $n\times n$ matrices with M invertible and satisfying (10). Let a finite number of points $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1,\ldots,p$, be real $n\times n$ matrices, A_i invertible and such that $A_i\partial K = \partial K$, for all $i = 1,\ldots,p$.*
Suppose moreover that, for all $x \in \partial K$, $t \in (0,T)\setminus\{t_1,\ldots t_p\}$ and $v \in \mathbb{R}^n$ with

$$\langle \nabla V(x), v\rangle = 0,$$

the following condition holds

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} > 0, \tag{15}$$

for all $w \in F(t,x,v)$.
Furthermore, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ with

$$\langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle, \text{ for some } i = 1,\ldots,p,$$

the following condition

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} > 0 \tag{16}$$

holds, for all $w \in F(t_i,x,v)$.
At last, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ with

$$\langle \nabla V(x), v\rangle \le 0 \le \langle \nabla V(Mx), Nv\rangle, \tag{17}$$

the following condition

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(Mx+hNv), Nv+hw\rangle}{h} > 0 \tag{18}$$

holds, for all $w \in F(T,Mx,Nv)$.
Then K is a bound set for the impulsive Floquet problem $(1)-(4)$.

Proof. We assume, by a contradiction, that K is not a bound set for the Floquet problem (1)-(4), i.e. that there exist a solution $x : [0,T] \to \overline{K}$ of problem (1)-(4) and $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. If the point t^* lays in $(0,T)$, then we proceed like in the proof of Theorem 3.4 in [26]. Therefore, it is only necessary to solve the cases when $t^* = 0$ and $t^* = T$. According to condition (10), we can take, without any loss of generality, $t^* = T$.

Following the same reasoning as in the proof of Proposition 3.1, we obtain

$$\langle \nabla V(x(0)), \dot{x}(0)\rangle \le 0$$

and

$$0 \le \langle \nabla V(x(T)), \dot{x}(T)\rangle = \langle \nabla V(Mx(0)), N\dot{x}(0)\rangle.$$

Therefore the couple $(x, v) := (x(0), \dot{x}(0))$ satisfies condition (17).
Using the same procedure as in the proof of Proposition 3.2 in [26], for $\bar{t} = T$, we obtain the existence of a sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$ and of a point $w \in F(T, x(T), \dot{x}(T))$ such that

$$\frac{\dot{x}(T + h_k) - \dot{x}(T)}{h_k} \to w \quad \text{as } k \to \infty.$$

Finally, by similar arguments as in the proof of Proposition 3.2 in [26], we get

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(T) + h\dot{x}(T)), \dot{x}(T) + hw \rangle}{h} = \liminf_{h \to 0^-} \frac{\langle \nabla V(Mx(0) + hN\dot{x}(0)), N\dot{x}(0) + hw \rangle}{h} \leq 0. \tag{19}$$

Inequality (19) is in contradiction with condition (18), which completes the proof. □

Remark 4.1 Let us note that condition (18) can be replaced by the following assumption: For all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (17), it holds that

$$\liminf_{h \to 0^+} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0, \tag{20}$$

for all $w \in F(0, x, v)$.

Remark 4.2 If the bounding function V is of class C^2, then conditions (15), (16) and (18) can be rewritten in terms of gradients and Hessian matrices. More concretely, if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \partial K$, $t \in (0, T)$, $v \in \mathbb{R}^n$ and $w \in F(t, x, v)$, there exists

$$\begin{aligned} \lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} &= \lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} \\ &= \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle. \end{aligned}$$

Therefore, conditions (15), (16) and (18) take in this more regular case the following form: Suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ the following holds:

- if $\langle \nabla V(x), v \rangle = 0$, then $\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0$, for all $t \in (0, T) \setminus \{t_1, \ldots t_p\}$ and $w \in F(t, x, v)$,
- if $\langle \nabla V(A_i x), B_i v \rangle \leq 0 \leq \langle \nabla V(x), v \rangle$, for some $i = 1, ..., p$, then $\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0$, for all $w \in F(t_i, x, v)$,
- if $\langle \nabla V(x), v \rangle \leq 0 \leq \langle \nabla V(Mx), Nv \rangle$, then $\langle HV(Mx)Nv, Nv \rangle + \langle \nabla V(Mx), w \rangle > 0$, for all $w \in F(T, Mx, Nv)$.

Remark 4.3 In Proposition 4.1, the element v plays again the role of the first derivative of the solution x. Therefore, if all assumptions from Remark 3.2 hold, then it is possible to require all conditions in Proposition 4.1 only for all $v \in \mathbb{R}^n$ with $|v| \leq B$, where B is defined by (9), and not for all $v \in \mathbb{R}^n$.

5 The existence and localization result for Floquet problem

In this section, the impulsive Floquet problem (1) − (4) will be investigated by combining the continuation principle from Proposition 2.2 with bound sets results developed in previous two sections. After rewriting the impulsive Floquet problem (1) − (4) in the abstract form (6), we will be able to clearly verify all conditions in Proposition 2.2. Firstly, the case of upper-Carathéodory r.h.s. will be investigated and then also the case of more regular r.h.s. will be studied.

Theorem 5.1 *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider the impulsive Floquet problem $(1)-(4)$, where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, M and N are real $n \times n$ matrices with M invertible satisfying (10), $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, and A_i, B_i, $i = 1, \ldots, p$, are real $n \times n$ matrices, A_i invertible and such that $A_i \partial K = \partial K$, for all $i = 1, \ldots, p$. Moreover, assume that*

(*i*) *there exists a function $\phi : [0,\infty) \to [0,\infty)$ continuous and non-decreasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\phi(s)} ds = \infty, \tag{21}$$

such that

$$|F(t,c,d)| \le \phi(|d|), \tag{22}$$

for a.a. $t \in [0,T]$ and every $c, d \in \mathbb{R}^n$, with $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(*ii*) *matrices $(N - \prod_{i=1}^p B_i)$ and $(M - \prod_{i=1}^p A_i)$ are both invertible;*

(*iii*) *there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions $(H1)$ and $(H2)$;*

(*iv*) *there exists $\varepsilon > 0$ such that, for all $\lambda \in (0,1), x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$, the following condition*

$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0$$

holds, for all $w \in \lambda F(t,x,v)$;

(*v*) *for all $i = 1, \ldots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R)+2R)$ and $\langle \nabla V(x), v\rangle \ne 0$, it holds that*

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0;$$

(*vi*) *for all $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$*

$$\langle \nabla V(Mx), Nv\rangle \cdot \langle \nabla V(x), v\rangle \ge 0.$$

Then the Floquet problem $(1)-(4)$ has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0,T]$.

Proof. For every $c \in \overline{K}$, it holds that $|c| \le R$. According to Proposition 2.3, for every $x \in PC^1([0,T], \mathbb{R}^n)$ with $|\ddot{x}(t)| \le \phi(|\dot{x}(t)|)$, for a.a. $t \in [0,T]$, and $x(t) \in \overline{K}$, for every $t \in [0,T]$, it holds $|\dot{x}(t)| \le B$, for every $t \in [0,T]$, with B defined by

$$B = \phi^{-1}(\phi(2R) + 2R).$$

Define

$$\bar{F}(t,c,d) = \begin{cases} F(t,c,d) & \text{if } |c| \le R \\ F(t, R\frac{c}{|c|}, d) & \text{if } |c| > R. \end{cases}$$

Since F is upper-Carathéodory, $\bar{F}$ is upper Carathéodory as well. Moreover, since $|R\frac{c}{|c|}| = R$, it holds according to (22) that, for every $t \in [0,T]$ and $c, d \in \mathbb{R}^n$,

$$|\bar{F}(t,c,d)| \le \phi(|d|). \tag{23}$$

In order to apply the continuation principle from Proposition 2.2, put

$$Q := \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K},\ |\dot{q}(t)| \le 2B, \ \text{ for all } t \in [0,T]\},$$

$S = S_1 = Q$ and $H(t,c,d,e,f,\lambda) = \lambda \bar{F}(t,e,f)$. Thus the associated problem (7) is the fully linearized problem

$$\left.\begin{array}{c} \ddot{x}(t) \in \lambda \bar{F}(t,q(t),\dot{q}(t)), \text{ for a.a. } t \in [0,T], \\ x(T) = Mx(0), \\ \dot{x}(T) = N\dot{x}(0), \\ x(t_i^+) = A_i x(t_i),\ i = 1,\ldots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\ldots,p. \end{array}\right\} \tag{24}$$

Moreover, let us denote, for each $(q,\lambda) \in Q \times [0,1]$, by $\mathfrak{T}(q,\lambda)$ the solution set of (24).

We will check now that (24) satisfies all the assumptions of Proposition 2.2.

ad $(i)-(ii)$ Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0,T],\mathbb{R}^n)$.

For every $(t,c,d,e,f,\lambda) \in [0,T] \times \mathbb{R}^{4n} \times [0,1]$, it follows from (23) that

$$|H(t,c,d,e,f,\lambda)| = \lambda |\bar{F}(t,e,f)| \le \phi(|f|)$$

yielding that

$$|H(t,c,d,e,f,\lambda)| \le \phi(2B) \le \phi(2B)(1+|c|+|d|) \tag{25}$$

when $|f| \le 2B$ and that

$$|\bar{F}(t,e,f)| \le \phi(r) \tag{26}$$

when $|f| \le r$.

Let $q \in Q$ and let f_q be a measurable selection of $\bar{F}(\cdot,q(\cdot),\dot{q}(\cdot))$, whose existence is guaranteed applying Proposition 2.1 with $\mu_r(t) \equiv \phi(r)$. Then, for any $\lambda \in [0,1]$, λf_q is a measurable selection of $\lambda \bar{F}(\cdot,q(\cdot),\dot{q}(\cdot))$. Let us consider the corresponding single-valued linear problem with linear impulses

$$\left.\begin{array}{c} \ddot{x}(t) = \lambda f_q(t), \text{ for a.a. } t \in [0,T], \\ x(T) = Mx(0), \\ \dot{x}(T) = N\dot{x}(0), \\ x(t_i^+) = A_i x(t_i),\ i = 1,\ldots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\ldots,p. \end{array}\right\} \tag{27}$$

Since the matrices $(N - \prod_{i=1}^p B_i)$ and $(M - \prod_{i=1}^p A_i)$ are invertible, the problem (27) has a unique solution $x_{\lambda f_q}$ given by

$$x_{\lambda f_q}(t) = \begin{cases} x_{[0]}(t) := & x_{\lambda f_q}(0) + \dot{x}_{\lambda f_q}(0)t + \lambda \int_0^t (t-\tau) f_q(\tau)d\tau \\ & \text{if } t \in [0, t_1], \\ x_{[1]}(t) := & A_1[x_{\lambda f_q}(0) + x\dot{_{\lambda f_q}}(0)t_1 + \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau)d\tau] \\ & + B_1[\dot{x}_{\lambda f_q}(0)(t-t_1) + (t-t_1)\lambda \int_0^{t_1} f_q(\tau)d\tau] \\ & + \lambda \int_{t_1}^t (t-\tau) f_q(\tau)d\tau \\ & \text{if } t \in (t_1, t_2] \\ x_{[i]}(t) := & \prod_{k=1}^{i} A_k \left[x_{\lambda f_q}(0) + \dot{x}_{\lambda f_q}(0)t_1 + \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau)d\tau \right] \\ & + \prod_{l=1}^{i} B_l \dot{x}_{\lambda f_q}(0)(t - t_i) + \sum_{r=1}^{i} \prod_{l=r}^{i} (t-t_i)\lambda B_l \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \\ & + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \left[\prod_{l=1}^{j-1} B_l \dot{x}_{\lambda f_q}(0)(t_j - t_{j-1}) + \lambda \int_{t_{j-1}}^{t_j} (t_j - \tau) f_q(\tau)d\tau \right. \\ & \left. + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} (t_j - t_{j-1})\lambda B_l \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \right] + \lambda \int_{t_i}^{t} (t-\tau) f_q(\tau)d\tau \\ & \text{if } t \in (t_i, t_{i+1}],\ 2 \le i \le p \end{cases}$$

with

$$\dot{x}_{\lambda f_q}(0) = \left(N - \prod_{i=1}^{p} B_i \right)^{-1} \left[\sum_{m=1}^{p} \prod_{i=m}^{p} B_i \lambda \int_{t_{m-1}}^{t_m} f_q(\tau)d\tau + \lambda \int_{t_p}^{T} f_q(\tau)d\tau \right], \tag{28}$$

$$\begin{aligned} x_{\lambda f_q}(0) = \quad & \left(M - \prod_{k=1}^{p} A_k \right)^{-1} \left[\prod_{k=1}^{p} A_k \left(\dot{x}_{\lambda f_q}(0)t_1 + \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau)d\tau \right) \right. \\ & + (T - t_p) \prod_{l=1}^{p} B_l \dot{x}_{\lambda f_q}(0) + (T - t_p)\lambda \sum_{r=1}^{p} \prod_{l=r}^{p} B_l \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \\ & + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \left((t_j - t_{j-1}) \prod_{l=1}^{j-1} B_l \dot{x}_{\lambda f_q}(0) + \lambda \int_{t_{j-1}}^{t_j} (t_j - \tau) f_q(\tau)d\tau \right. \\ & \left. \left. + (t_j - t_{j-1}) \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} \lambda B_l \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \right) + \lambda \int_{t_p}^{T} (T-\tau) f_q(\tau)d\tau \right]. \end{aligned} \tag{29}$$

Therefore

$$\mathfrak{T}(q,\lambda) = \{x_{\lambda f_q} : f_q \text{ is a selection of } \bar{F}(\cdot, q(\cdot), \dot{q}(\cdot))\} \neq \emptyset.$$

Moreover, given $x_1, x_2 \in \mathfrak{T}(q,\lambda)$, there exist two measurable selections f_q^1 and f_q^2 of $\bar{F}(\cdot, q(\cdot), \dot{q}(\cdot))$ such that $x_1 = x_{\lambda f_q^1}$ and $x_2 = x_{\lambda f_q^2}$. Since the right-hand side $\bar{F}$ has convex values, it holds that, for any $c \in [0,1]$, $cf_q^1 + (1-c)f_q^2$ is a measurable selection of $\bar{F}(\cdot, q(\cdot), \dot{q}(\cdot))$ as well. The linearity of both the equation and of the impulses yields that $cx_1 + (1-c)x_2 = x_{cf_q^1 + (1-c)f_q^2}$, i.e. that the set of solutions of problem (24) is, for each $(q,\lambda) \in Q \times [0,1]$, convex.

Condition *(ii)* in Proposition 2.2 follows from (25) when replacing (t, c, d, e, f, λ) by $(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)$ with $q \in Q, x \in \mathfrak{T}(q,\lambda)$.

ad (iii) For $\lambda = 0$, the associated problem takes the form

$$\left.\begin{aligned}\ddot{x}(t) &= 0, \text{ for a.a. } t \in [0,T],\\ x(T) &= Mx(0),\\ \dot{x}(T) &= N\dot{x}(0),\\ x(t_i^+) &= A_i x(t_i),\ i = 1,\dots,p,\\ \dot{x}(t_i^+) &= B_i \dot{x}(t_i),\ i = 1,\dots,p.\end{aligned}\right\} \tag{30}$$

Assumption (ii) implies that the problem (30) has only one solution, i.e. only the trivial solution. The fulfillment of condition (iii) in Proposition 2.2 then follows immediately from the fact that $0 \in K$.

ad (iv) Let $x_{\lambda f_q}$ be the solution of the b.v.p. (27). Then, it is possible to show that according to formulas (26), (28) and (29), there exist constants

$$\begin{aligned} M_1 \le\ & \left\| \left(N - \prod_{i=1}^{p} B_i \right)^{-1} \right\| \left[\sum_{m=1}^{p} \prod_{i=m}^{p} \|B_i\| \int_{t_{m-1}}^{t_m} |f_q(\tau)| d\tau + \int_{t_p}^{T} |f_q(\tau)| d\tau \right] \\ \le\ & \left\| \left(N - \prod_{i=1}^{p} B_i \right)^{-1} \right\| \left[\prod_{i=m}^{p} \|B_i\| \phi(2B) t_p + \phi(2B)(T - t_p) \right] \\ =\ & \left\| \left(N - \prod_{i=1}^{p} B_i \right)^{-1} \right\| \max\left\{ \prod_{i=m}^{p} \|B_i\|, 1 \right\} \phi(2B) T \end{aligned}$$

and

$$
\begin{aligned}
M_0 \le\ & \left\| \left(M - \prod_{k=1}^{p} A_k \right)^{-1} \right\| \Bigg[\prod_{k=1}^{p} \|A_k\| \left(M_1 t_1 + t_1 \int_0^{t_1} |f_q(\tau)| d\tau \right) \\
& + (T - t_p) \prod_{l=1}^{p} \|B_l\| M_1 + (T - t_p) \prod_{l=1}^{p} \|B_l\| \int_0^{t_p} |f_q(\tau)| d\tau \\
& + \sum_{j=2}^{p} \prod_{k=j}^{p} \|A_k\| \Bigg((t_j - t_{j-1}) \prod_{l=1}^{j-1} \|B_l\| M_1 + (t_j - t_{j-1}) \int_{t_{j-1}}^{t_j} |f_q(\tau)| d\tau \\
& + (t_j - t_{j-1}) \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} \|B_l\| \int_{t_{r-1}}^{t_r} |f_q(\tau)| d\tau \Bigg) + (T - t_p) \int_{t_p}^{T} |f_q(\tau)| d\tau \Bigg] \\
\le\ & \left\| \left(M - \prod_{k=1}^{p} A_k \right)^{-1} \right\| \Bigg[\prod_{k=1}^{p} \|A_k\| t_1 \left(M_1 + t_1 \phi(2B) \right) \\
& + (T - t_p) \prod_{l=1}^{p} \|B_l\| \left(M_1 + t_p \phi(2B) \right) \\
& + \sum_{j=1}^{p} \prod_{k=1}^{p} \|A_k\| \Bigg((t_j - t_{j-1}) \prod_{l=1}^{p} \|B_l\| M_1 + (t_j - t_{j-1}) T \phi(2B) \\
& + (t_j - t_{j-1}) \sum_{r=1}^{p} \prod_{l=1}^{p} \|B_l\| (t_r - t_{r-1}) \phi(2B) \Bigg) + (T - t_p) T \phi(2B) \Bigg] \\
\le\ & \left\| \left(M - \prod_{k=1}^{p} A_k \right)^{-1} \right\| \Bigg[\prod_{k=1}^{p} \|A_k\| t_1 \left(M_1 + t_1 \phi(2B) \right) \\
& + (T - t_p) \prod_{l=1}^{p} \|B_l\| \left(M_1 + t_p \phi(2B) \right) \\
& + \prod_{k=1}^{p} \|A_k\| t_p \prod_{l=1}^{p} \|B_l\| \left(M_1 + t_p \phi(2B) \right) + \phi(2B) T^2 \Bigg] \\
\le\ & \left\| \left(M - \prod_{k=1}^{p} A_k \right)^{-1} \right\| \Bigg[\max\left\{ \prod_{i=m}^{p} \|B_i\|, 1 \right\} \max\left\{ \prod_{k=1}^{p} \|A_k\|, 1 \right\} T \left(M_1 + t_p \phi(2B) \right) \\
& + \phi(2B) T^2 \Bigg]
\end{aligned}
$$

such that $|x_{\lambda f_q}(0)| \le M_0$ and $|\dot{x}_{\lambda f_q}(0)| \le M_1$, for every $\lambda \in [0,1]$, $q \in Q$. Hence, condition (iv) in Proposition 2.2 is satisfied.

ad (v) Let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot, \lambda)$. We will show now that q_* can not lay in ∂Q.

At first, let us investigate the case when $\lambda = 0$. Then problem (24) transforms into the b.v.p. (30) which has only the trivial solution. Therefore, for $\lambda = 0$, it holds that $q_* \equiv 0$ which lays in $Int\ Q$. Hence, if $\lambda = 0$, condition (v) in Proposition 2.2 is satisfied.

Secondly, let us assume that $\lambda \in (0,1)$. If q_* laid in ∂Q, then there would exist $t_0 \in [0,T]$ such that $q_*(t_0) \in \partial K$ or $|\dot{q}_*(t_0)| = 2B$. Since, for a.a. $t \in [0,T]$, we have

$$|\ddot{q}_*(t)| = \lambda |\bar{F}(t, q_*(t), \dot{q}_*(t))| \le \phi(|\dot{q}_*(t)|)$$

and $|q_*(t)| \le R$, for every $t \in [0,T]$, Proposition 2.3 implies that $|\dot{q}_*(t)| \le B < 2B$, for every $t \in [0,T]$. Hence, it would hold that $q_*(t_0) \in \partial K$.

But this is impossible, since $\bar{F}(t,c,d) = F(t,c,d)$ when $c \in \overline{K}$, and hypotheses (iii), (iv), (v) and (vi) guarantee that K is a bound set for (24) (according to Proposition 3.1 and

Remark 3.2). Therefore, it holds that $q_*(t) \in K$, for all $t \in [0,T]$, and subsequently $q_* \in Int\ Q$.

Thus, condition (v) from Proposition 2.2 is satisfied, for all $\lambda \in [0,1)$, which guarantees that (24) has a solution x with $x(t) \in \overline{K}$ for every $t \in [0,T]$. Recalling that $\bar{F}(t,c,d) = F(t,c,d)$ when $c \in \overline{K}$, it implies that x is a solution of (1)-(4).

□

Remark 5.1 When V is of class C^2, then, according to Corollary 3.1, condition (iv) in Theorem 5.1 is equivalent to requiring that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$,

$$\langle HV(x)v, v\rangle + \lambda \langle \nabla V(x), w\rangle > 0, \text{ for every } \lambda \in (0,1) \text{ and } w \in F(t,x,v). \tag{31}$$

Since the function $g(\lambda) = \lambda \langle \nabla V(x), w\rangle$ is monotone in $[0,1]$, (31) is then equivalent to require $g(0) \ge 0$ and $g(1) \ge 0$ with at least one of the inequalities to be strict, i.e. requiring one of the following two conditions

$$\langle HV(x)v, v\rangle \ge 0 \text{ and } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{32}$$

or

$$\langle HV(x)v, v\rangle > 0 \text{ and } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle \ge 0$$

that do not depend on λ.

Remark 5.2 If the multivalued mapping $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ in (1) is globally u.s.c., then it is possible to replace conditions $(iv)-(vi)$ in Theorem 5.1 by the following assumptions directly imposed on the boundary of the bound set K:

(iv_{usc}) for all $\lambda \in (0,1), x \in \partial K,\ t \in (0,T) \setminus \{t_1, \dots t_p\}$ and $v \in \mathbb{R}^n,\ |v| \le \phi^{-1}(\phi(2R) + 2R)$ with

$$\langle \nabla V(x), v\rangle = 0,$$

the following condition holds

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} > 0,$$

for all $w \in \lambda F(t,x,v)$.

(v_{usc}) for all $\lambda \in (0,1), x \in \partial K$ and $v \in \mathbb{R}^n$ with $|v| \le \phi^{-1}(\phi(2R) + 2R)$ satisfying

$$\langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle, \text{ for some } i = 1, \dots, p,$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} > 0$$

holds, for all $w \in \lambda F(t_i, x, v)$;

(vi_{usc}) for all $\lambda \in (0,1), x \in \partial K$ and $v \in \mathbb{R}^n$ with $|v| \le \phi^{-1}(\phi(2R) + 2R)$ satisfying

$$\langle \nabla V(x), v\rangle \le 0 \le \langle \nabla V(Mx), Nv\rangle,$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(Mx + hNv), Nv + hw\rangle}{h} > 0$$

holds, for all $w \in \lambda F(T, Mx, Nv)$.

Remark 5.3 If the bounding function V is of class C^2, then conditions (iv_{usc}), (v_{usc}) and (vi_{usc}) can be (according to Remark 4.2) rewritten in terms of gradients and Hessian matrices as follows:

(iv'_{usc}) for all $\lambda \in (0,1), x \in \partial K,\ t \in (0,T) \setminus \{t_1, \dots t_p\}$ and $v \in \mathbb{R}^n,\ |v| \leq \phi^{-1}(\phi(2R) + 2R)$ with

$$\langle \nabla V(x), v \rangle = 0,$$

the following condition holds

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0,$$

for all $w \in \lambda F(t, x, v)$.

(v'_{usc}) for all $\lambda \in (0,1), x \in \partial K$ and $v \in \mathbb{R}^n$ with $|v| \leq \phi^{-1}(\phi(2R) + 2R)$ satisfying

$$\langle \nabla V(A_i x), B_i v \rangle \leq 0 \leq \langle \nabla V(x), v \rangle, \text{ for some } i = 1, \dots, p,$$

the following condition

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0$$

holds, for all $w \in \lambda F(t_i, x, v)$;

(vi'_{usc}) for all $\lambda \in (0,1), x \in \partial K$ and $v \in \mathbb{R}^n$ with $|v| \leq \phi^{-1}(\phi(2R) + 2R)$ satisfying

$$\langle \nabla V(x), v \rangle \leq 0 \leq \langle \nabla V(Mx), Nv \rangle,$$

the following condition

$$\langle HV(Mx)Nv, Nv \rangle + \langle \nabla V(Mx), w \rangle > 0$$

holds, for all $w \in \lambda F(T, Mx, Nv)$.

Remark 5.4 If we compare the results from Theorem 5.1 and Remark 5.2 for the impulsive Floquet problem with the previous ones for the Dirichlet boundary conditions, we will find out that in case of Dirichlet conditions, it is possible to omit condition (vi) in Theorem 5.1 and (vi_{usc}) in Remark 5.2 (see Theorem 4.1 in [25] and Theorem 4.3 in [26])

6 An application to a Liénard type equation

As an application of the previous existence and localization results, let us study a generalization of the Liénard equation which is widely studied in literature (see, e.g., [10, 11, 23]) and, in turn, is a generalization of the Duffing equation, the Josephson equation, the Van der Pol equation, the pendulum equation.

For this purpose, let us consider the second-order equation

$$\ddot{x}(t) = g(x(t))f(t, \dot{x}(t)) + h(t, x(t)), \text{ for a.a. } t \in [0,T], \tag{33}$$

together with antiperiodic impulses

$$x(t_i^+) = -x(t_i),\ i = 1, \dots, p, \tag{34}$$

$$\dot{x}(t_i^+) = -\dot{x}(t_i),\ i = 1, \dots, p, \tag{35}$$

where $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T,\ p \in \mathbb{N}$. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and that $f, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Carathédory functions. Suppose moreover that there exist $k \in C(\mathbb{R}, [0, +\infty))$, $q \in C([0, +\infty), [0, +\infty))$, with q increasing, such that

$$\lim_{s\to+\infty} \frac{s^2}{q(s)} = +\infty, \tag{36}$$

$$|f(t, d)| \le q(|d|) \tag{37}$$

and

$$|h(t, c)| \le k(c) \tag{38}$$

for every $t \in [0, T], c, d \in \mathbb{R}$. Associated to (33)-(35) we consider the following boundary conditions

$$x(T) = ax(0), \quad \dot{x}(T) = b\dot{x}(0) \tag{39}$$

with $a, b \in \mathbb{R}, a \ne (-1)^p, b \ne (-1)^p$ and $ab \ge 0$, which includes both periodic and antiperiodic conditions, respectively when p is odd and p is even.

We will show now that, under very general conditions, the problem (33), (39) together with impulse conditions (34), (35) satisfies all the assumptions of Theorem 5.1. On this purpose, let us consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-R, R)$, with R to be specified later, and the C^2-function $V(x) = \frac{1}{2}(x^2 - R^2)$ that trivially satisfies conditions $(H1)$ and $(H2)$.

In order to verify condition (i), let us define the continuous and increasing function

$$\phi(d) = g_R\, q(d) + k_R,$$

where $g_R = \max_{|x|\le R} |g(x)|, k_R = \max_{|x|\le R} |k(x)|$. Since q is monotone, there exists

$$\lim_{s\to+\infty} q(s) = \overline{q} \in (0, +\infty].$$

Then the limit

$$\lim_{s\to+\infty} \frac{q(s)}{g_R\, q(s) + k_R}$$

is finite in both cases when $\overline{q}$ is finite or non finite. Hence, according to (36), the function ϕ satisfies (21). Moreover $F(t, c, d) := g(c)f(t, d) + h(t, c)$ satisfies (22), for all $t \in [0, T]$ and all $c,\ d \in \mathbb{R}$, with $|c| \le R$.

Assumption (ii) follows from the conditions put on a and b.

Condition (iii) follows from the fact that $\dot{V}(x) = x$ and $\ddot{V}(x) = 1$, for every $x \in \mathbb{R}$.

Notice moreover that $(-x)(-v)xv = x^2v^2 > 0$ whenever $xv \ne 0$ and that $(ax)(bv)xv = abx^2v^2 \ge 0$ for every $x, v \in \mathbb{R}$, according to properties of a and b. Hence, also conditions $(v) - (vi)$ hold.

It remains to prove condition (iv), which, according to Remark 5.1, is equivalent to (32). Since

$$\phi^{-1}(d) = q^{-1}\left(\frac{d - k_R}{g_R}\right),$$

we easily get that

$$\phi^{-1}(\phi(2R) + 2R) = q^{-1}\left(q(2R) + \frac{2R}{g_R}\right).$$

Thus condition (iv) reads as

$$v^2 + xg(x)f(t, v) + xh(t, x) > 0 \tag{40}$$

for every $t \in (0, T), x = \pm R, v$ with $|v| \le q^{-1}(q(2R) + \frac{2R}{g_R})$.

The previous result can be stated in the form of the following theorem.

Theorem 6.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, $f, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ be Carathédory functions, $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$ and $a, b \in \mathbb{R}, a \neq (-1)^p, b \neq (-1)^p$ and $ab \geq 0$. Suppose that there exist $k \in C(\mathbb{R}, [0, +\infty))$, $q \in C([0, +\infty), [0, +\infty))$, with q increasing, satisfying (36)-(37) and (38) for every $t \in [0, T], c, d \in \mathbb{R}$. If there exists $R > 0$ such that condition (40) holds for every $t \in (0, T), x = \pm R$ and $|v| \leq q^{-1}(q(2R) + \frac{2R}{g_R})$, with $g_R = \max_{|x| \leq R} |g(x)|$, then problem (33), (39) together with impulse conditions (34), (35) has a solution x such that $|x(t)| \leq R$, for every $t \in [0, T]$.*

Remark 6.1 We stress that condition (40) is satisfied in many situations. For example, assume that there exists $r > 0$ such that $c \cdot h(t, c) \geq 0$ for every $t \in (0, T)$ and c with $|c| < r$, an usual sign condition in applications. Then, it is sufficient to assume a sign condition on g and f, i.e. that, for some $R < r$, $c \cdot g(c) > 0$ when $|c| = R$ and $f(t, v) > 0$ when $t \in (0, T), v \in \mathbb{R}$ with $|v| \leq q^{-1}(q(2R) + \frac{2R}{g_R})$. Alternatively we can suppose a growth condition on f and g i.e. that

$$\lim_{c \to 0} \frac{c}{g_c} = 0 \tag{41}$$

and that

$$\lim_{d \to 0} \frac{q(d)}{d^2} = l < \frac{1}{r g_r}.$$

Fixed $\epsilon \in (0, \frac{1}{r g_r} - l)$, there exists $\delta > 0$ such that, for every $d \in (0, \delta)$ it holds $q(d) < (\epsilon + l) d^2$. Since q is continuous and increasing, also q^{-1} is continuous and increasing. Therefore, according to (41),

$$\lim_{c \to 0} q^{-1}\left(q(2c) + \frac{2c}{g_c}\right) = q^{-1}(q(0)) = 0.$$

Now choose $R \leq r$ such that $q^{-1}(q(2R) + \frac{2R}{g_R}) \leq \delta$. Then, according to (37), for every $t \in (0, T), x = \pm R, v \in \mathbb{R}$ with $|v| \leq q^{-1}(q(2R) + \frac{2R}{g_R})$,

$$\begin{aligned} v^2 + x g(x) f(t, v) + x h(t, x) &\geq v^2 - |x g(x) f(t, v)| + x h(t, x) \geq v^2 - R g_R q(|v|) + x h(t, x) \\ &> v^2 [1 - R g_R (\epsilon + l)] + x h(t, x) > 0. \end{aligned}$$

Remark 6.2 For the sake of simplicity, we considered the single-valued case, but we stress that our results can be extended to the multivalued case, for which the literature is rare and which can be e.g. used for modelling optimal control problems. Indeed, if we suppose, for example, that f and h are upper-Carathéodory multimaps, Theorem 6.1 holds under the same conditions as well, just replacing (40) by

$$v^2 + x g(x) w + x z > 0$$

for every $t \in (0, T), x = \pm R, v$ with $|v| \leq q^{-1}(q(2R) + \frac{2R}{g_R})$, $w \in f(t, v)$, $z \in h(t, x)$.

Remark 6.3 In literature, only few results devoted to impulsive periodic or anti-periodic solutions of the Liénard equation can be found. The majority of papers dealing with this topic are related to non-impulsive solutions, sometimes also with delay. In both cases, however, the authors usually consider g constant and/or $f(t, x') = x'$ with additional strong regularity conditions on the other terms of the equation, i.e. the continuity of g, f and h.
For example, in [23], the existence of an impulsive periodic solution is investigated under the presence of a L^1 forcing term, in the case when $g(x) = 1$ and f is a continuous and sublinear function. Moreover, it is assumed there that $x f(t, x') \geq \beta |x|^2$ for some positive constant β, while $h = h(x)$ is supposed to be continuous and superlinear. We point out that these conditions directly guarantee (37)-(40).

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