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Cap partitions of the Segre variety $\mathscr{S}_{1,3}$

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Abstract

We prove that the Segre variety $\mathscr{S}_{1,3}$ of $PG(7,q)$ can be partitioned into caps of size $(q^4-1)/(q-1)$. It can also be partitioned into three-dimensional elliptic quadrics or into twisted cubics.

Keywords: Partition; Cap in a projective space; Singer cycle; Segre variety

1. Preliminaries

We begin with a property of linear collinations of an odd-dimensional projective space over $GF(q)$. For $(k,q)\neq(3,2)$ it is a special case of Lemma 2.3 in [6].

Lemma 1. *Let T be a transformation in $GL(2k,q)$, $k\geqslant 2$, inducing a collineation of order q^k+1 which fixes no r-dimensional subspace of $PG(2k-1,q)$ for $r=0,1,\ldots,k-1$. Then T is a power of a Singer cycle of $GL(2k,q)$.*

Proof. Let $m(x)$ be the minimal polynomial of T over $GF(q)$. We want to show that $m(x)$ is irreducible of degree $2k$. We have that T^{q^k+1} is a scalar transformation and so $m(x)$ divides the polynomial $x^{(q^k+1)(q-1)}-1$, which in turn divides $x^{q^{2k}-1}-1$. In particular $m(x)$ splits into linear factors in $GF(q^{2k})[x]$.

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Let $f(x)$ be an irreducible divisor of $m(x)$ and assume the splitting field of $f(x)$ over $GF(q)$ is a proper subfield of $GF(q^{2k})$, hence one of $GF(q)$, $GF(q^2)$, $GF(q^k)$, $GF(q^h)$ or $GF(q^{2h})$ for some proper divisor h of k. In each one of these cases the rational canonical form for T over $GF(q)$ would have an $r\times r$ block (the companion matrix of $f(x)$) for some $r\leqslant k$, hence the collineation induced by T on $PG(2k-1,q)$ would have an invariant $(r-1)$-dimensional projective subspace, contradicting our assumption. Hence $f(x)$ has degree $2k$. Since $f(x)$ is a divisor of $m(x)$, which in turn divides the characteristic polynomial of T which has degree $2k$, we conclude that $m(x)$ is irreducible of degree $2k$ and coincides with the characteristic polynomial of T. The rational canonical form of T over $GF(q)$ is thus simply the companion matrix of $m(x)$.

Represent the underlying $2k$-dimensional vector space V as $GF(q^{2k})$. Let β be an element of $GF(q^{2k})$ having $m(x)$ as minimal polynomial over $GF(q)$. Consider the $GF(q)$-linear transformation M given by $V\to V$, $v\mapsto \beta v$. The minimal polynomial of M over $GF(q)$ is precisely $m(x)$, which is thus also the characteristic polynomial of M, hence the rational canonical form of M over $GF(q)$ is again the companion matrix of $m(x)$.

We conclude that M and T are conjugate in $GL(2k,q)$. The transformation M is obviously a power of the Singer cycle given by $V\to V$, $v\mapsto\omega v$ where ω is a primitive element of $GF(q^{2k})$. Hence T is also a power of a Singer cycle. □

Let $PG(m,q)$ and $PG(k,q)$ be projective spaces over $GF(q)$ with $m\geqslant 1$, $k\geqslant 1$. Set $n=(m+1)(k+1)-1$. For each $\mathbf{u}=(u_0,u_1,\dots,u_m)\in GF(q)^{m+1}$ and $\mathbf{w}=(w_0,w_1,\dots,w_k)\in GF(q)^{k+1}$ define

$$(\mathbf{u}\otimes\mathbf{w})=(u_0w_0,u_0w_1,\dots,u_0w_k,u_1w_0,u_1w_1,\dots,u_1w_k,\dots,u_mw_0,u_mw_1,\dots,u_mw_k).$$

The *Segre variety* of the two projective spaces is the variety $\mathscr{S}=\mathscr{S}_{m,k}$ of $PG(n,q)$ consisting of all points represented by the vectors $(\mathbf{u}\otimes\mathbf{w})$ as $\mathbf{u}$ and $\mathbf{w}$ vary over all non-zero vectors of $GF(q)^{m+1}$ and $GF(q)^{k+1}$, respectively. For more details see [4, Section 25].

The Segre variety $\mathscr{S}$ has two families of maximal subspaces with dimensions m and k respectively, say $\mathscr{M}$ and $\mathscr{K}$, each of which forms a cover of $\mathscr{S}$. Two maximal subspaces from one and the same family are skew; two maximal subspaces from distinct families meet in exactly one point [4, Theorems 25.5.2 and 25.5.3]. We have

$$\mathscr{M}=\{PG(m,q)\otimes\mathbf{w}|\mathbf{w}\in PG(k,q)\},$$

$$\mathscr{K}=\{\mathbf{u}\otimes PG(k,q)|\mathbf{u}\in PG(m,q)\}.$$

Let S and T be Singer cycles in $GL(m+1,q)$ and $GL(k+1,q)$, respectively. Then the Kronecker product $S\otimes T$ yields a linear collineation of $PG(n,q)$ fixing $\mathscr{S}$ setwise [4, Theorem 25.5.9]. We will need the following result.

Lemma 2. *Each point orbit of $\langle S\otimes T\rangle$ contained in $\mathscr{S}$ meets each member of $\mathscr{M}\cup\mathscr{K}$ in at least one point.*

Proof. Take a point in $PG(m,q)$ represented by the vector $\mathbf{u}=(u_0,u_1,\ldots,u_m)$, and take a point in $PG(k,q)$ represented by the vector $\mathbf{w}=(w_0,w_1,\ldots,w_k)$. By [4] we have $(\mathbf{u}\otimes\mathbf{w})((S\otimes T)^j)=(\mathbf{u}S^j)\otimes(\mathbf{w}T^j)$ for each $j\geqslant 1$.

Let $\mathbf{x}\otimes PG(k,q)$ be a member of $\mathscr{K}$. Since S induces a transitive collineation group on $PG(m,q)$, there exists an index j such that the point represented by the vector $\mathbf{x}$ is equal to the point represented by the vector $\mathbf{u}S^j$, and so the point of $\mathscr{S}$ represented by the vector $(\mathbf{u}\otimes\mathbf{w})((S\otimes T)^j)$ lies in $\mathbf{x}\otimes PG(k,q)$. We have proved that the $\langle S\otimes T\rangle$-orbit of the point of $\mathscr{S}$ represented by the vector $\mathbf{u}\otimes\mathbf{w}$ meets $\mathbf{x}\otimes PG(k,q)$.

A similar argument holds for the members of $\mathscr{M}$. $\square$

2. The construction

Let T be a Singer cycle in $GL(4,q)$. The matrix T is conjugate in $GL(4,q^4)$ to the diagonal matrix $D_2=\mathrm{diag}(\omega,\omega^q,\omega^{q^2},\omega^{q^3})$ by the matrix

$$E=\begin{pmatrix} 1 & 1 & 1 & 1\\ \omega & \omega^q & \omega^{q^2} & \omega^{q^3}\\ \omega^2 & \omega^{2q} & \omega^{2q^2} & \omega^{2q^3}\\ \omega^3 & \omega^{3q} & \omega^{3q^2} & \omega^{3q^3}\end{pmatrix}$$

for some primitive element ω of $GF(q^4)$ [5]. Since ω^{q^2+1} is a primitive element of $GF(q^2)$, we have that there exists a Singer cycle S in $GL(2,q)$ which is conjugate to $D_1=\mathrm{diag}(\omega^{q^2+1},\omega^{q^3+q})$ in $GL(2,q^2)$.

The Kronecker product $S\otimes T$ is conjugate in $GL(8,q^4)$ to the Kronecker product

$$\begin{aligned} &D_1\otimes D_2\\ &\quad=\mathrm{diag}(\omega^{q^2+2},\omega^{q^2+q+1},\omega^{2q^2+1},\omega^{q^3+q^2+1},\omega^{q^3+q+1},\omega^{q^3+2q},\omega^{q^3+q^2+q},\omega^{2q^3+q}).\end{aligned}$$

The elements

$$\omega^{q^2+2},\omega^{q^3+2q},\omega^{2q^2+1},\omega^{2q^3+q}\quad\text{and}\quad\omega^{q^2+q+1},\omega^{q^3+q+1},\omega^{q^3+q^2+1},\omega^{q^3+q^2+q}$$

form two full sets of elements of $GF(q^4)$ which are conjugate over $GF(q)$. That means the rational canonical form of $S\otimes T$ over $GF(q)$ is a block diagonal matrix

$$R=\begin{pmatrix} C_1 & 0\\ 0 & C_2\end{pmatrix}$$

consisting of two 4×4 blocks C_1 and C_2, each of which is the companion matrix of an irreducible quartic polynomial over $GF(q)$. It follows that the linear collineation g induced by R on $PG(7,q)$ fixes (setwise) two projective 3-dimensional subspaces, say Σ_1 and Σ_2.

Lemma 3. *The order of the collineation g induced by R is $(q^4-1)/(q-1)$.*

Proof. The eigenvalues of R are $\omega^{q^2+2}, \omega^{q^2+q+1}$ and their conjugates over $GF(q)$. It is easily seen that the equality $\omega^{(q^2+2)(q^4-1)/(q-1)} = \omega^{(q^2+q+1)(q^4-1)/(q-1)}$ holds and that this is an element in $GF(q)\backslash\{0\}$. Hence the order b of the collineation g is at most $(q^4-1)/(q-1)$. Assume $b<(q^4-1)/(q-1)$. It follows that $\omega^{(q^2+2)b} = \omega^{(q^2+q+1)b} \in GF(q)\backslash\{0\}$. Hence $\omega^{(q-1)b}=1$ and q^4-1 divides $b(q-1)$, implying $(q^4-1)/(q-1)$ divides b, a contradiction. □

Lemma 4. *The collineation group G generated by g acts semiregularly on $PG(7,q)\backslash(\Sigma_1 \cup \Sigma_2)$.*

Proof. Let P be a point neither on Σ_1 nor on Σ_2, represented by the vector $\mathbf{x}=(u_1,u_2,u_3,u_4,w_1,w_2,w_3,w_4)$. In particular, we have $\mathbf{u}=(u_1,u_2,u_3,u_4)\neq(0,0,0,0)$ and $\mathbf{w}=(w_1,w_2,w_3,w_4)\neq(0,0,0,0)$. Assume that $\mathbf{x}$ is proportional to $\mathbf{x}\cdot R^i$ for some index i with $0\leqslant i<(q^4-1)/(q-1)$. Then there exists a non-zero element $\lambda\in GF(q)$ such that $\lambda\mathbf{u}=\mathbf{u}\cdot C_1^i$, $\lambda\mathbf{w}=\mathbf{w}\cdot C_2^i$, which means that C_1^i and C_2^i have a common eigenvalue in $GF(q)\backslash\{0\}$. Hence we have $\lambda=\omega^{(q^2+2)i}=\omega^{(q^2+q+1)iq^j}$ for some $j\in\{0,1,2,3\}$. Since λ is in $GF(q)$, we have $\lambda=\lambda^{q^{4-j}}=\omega^{(q^2+q+1)i}$ and so $\omega^{(q^2+2)i}=\omega^{(q^2+q+1)i}$. This implies that q^4-1 divides $(q^2+q+1)i-(q^2+2)i=(q-1)i$, whence $(q^4-1)/(q-1)$ divides i, a contradiction. □

As described in Section 1, G leaves invariant a Segre variety $\mathscr{S}=\mathscr{S}_{1,3}$, disjoint from $\Sigma_1\cup\Sigma_2$.

Theorem 5. *Each point orbit of G on $\mathscr{S}$ is a cap of size $(q^4-1)/(q-1)$.*

Proof. Let $\mathscr{O}$ be one such orbit. Denote by $\mathscr{M}$ the family of maximal subspaces of dimension 1 on $\mathscr{S}$, and denote by $\mathscr{K}$ the family of maximal subspaces of dimension 3 on $\mathscr{S}$. The collineation g leaves each family $\mathscr{M}$ and $\mathscr{K}$ invariant.

We have seen that each G-orbit on $\mathscr{S}$ meets each line in $\mathscr{M}$ in at least one point. Moreover, it follows from the above lemmas that $|\mathscr{O}|=q^3+q^2+q+1=|\mathscr{M}|$. Hence, since any two lines in $\mathscr{M}$ are disjoint, each line in $\mathscr{M}$ meets $\mathscr{O}$ in exactly one point. Furthermore, since each solid in $\mathscr{K}$ meets $\mathscr{O}$ in at least one point, we see that the group G is transitive on $\mathscr{K}$. As the family $\mathscr{K}$ consists of $q+1$ solids, the stabilizer of a solid in $\mathscr{K}$ under G is the subgroup $H=\langle g^{q+1}\rangle$ of order q^2+1 of G, and so H fixes the family $\mathscr{K}$ elementwise.

Since the group G is semiregular on $PG(7,q)\backslash(\Sigma_1\cup\Sigma_2)$, so is the subgroup H and each point orbit of H inside a solid in $\mathscr{K}$ has length q^2+1. We conclude that H induces a cyclic linear collineation group of order q^2+1 on each solid Π of $\mathscr{K}$ with point orbits of equal size q^2+1. In particular, H fixes no point or line of Π and so the action of H on Π is induced by a power of a Singer cycle, see Lemma 1 or [6, Lemma 2.3].

By [2] we have that each H-orbit on Π is an elliptic quadric, hence a cap of Π and thus of $PG(7,q)$ as well. We also see that the orbit $\mathscr{O}$ meets each solid of $\mathscr{K}$ in an elliptic quadric.

Suppose now that a line ℓ of $PG(7,q)$ meets $\mathcal{O}$ in three distinct points. As $\mathcal{S}$ is the intersection of quadrics, we have that ℓ is entirely contained in $\mathcal{S}$. Hence ℓ is either a line in $\mathcal{M}$ or lies in some solid Π of $\mathcal{K}$. The former case cannot occur, as each line in $\mathcal{M}$ meets $\mathcal{O}$ in precisely one point. In the latter case the line ℓ lies entirely in Π, and hence ℓ meets the elliptic quadric $\mathcal{O} \cap \Pi$ in three distinct points, a contradiction. □

The proof of the above theorem immediately implies the following result.

Theorem 6. *The Segre variety $\mathcal{S}_{1,3}$ in $PG(7,q)$ can be partitioned into caps of size $(q^4-1)/(q-1)$. Moreover, it can also be partitioned into $(q+1)^2$ elliptic quadrics.* □

We now show that the Segre variety $\mathcal{S} = \mathcal{S}_{1,3}$ can be partitioned in yet another way.

Theorem 7. *The Segre variety $\mathcal{S}_{1,3}$ in $PG(7,q)$ can be partitioned into $(q^4-1)/(q-1)$ twisted cubics.*

Proof. Let F be the subgroup of G generated by g^{q^2+1}. If we let ω^{q^2+2} and ω^{q^2+q+1} be eigenvalues of C_1 and C_2, respectively, then by looking at the eigenvalues of R^{q^2+1} we see that the linear collineation g^{q^2+1} induces a collineation of order $(q+1)/\gcd(q+1,3)$ on Σ_1 and a collineation of order $(q+1)/\gcd(q+1,q+2) = q+1$ on Σ_2, the induced collineation being a power of a Singer cycle in either case. The theorem in the appendix of [3] yields for $i = 1,2$ the existence of a regular spread $\mathcal{R}_i$ in Σ_i which is linewise fixed by F: each line of $\mathcal{R}_1$ is a full point orbit under F if $\gcd(q+1,3) = 1$ or is the union of three point orbits under F if $\gcd(q+1,3) = 3$; each line of $\mathcal{R}_2$ is always a full point orbit under F. Let P be a point on $\mathcal{S}$ represented by the vector $\mathbf{x} = (u_1,u_2,u_3,u_4,w_1,w_2,w_3,w_4)$. Since $\mathbf{u} = (u_1,u_2,u_3,u_4) \neq (0,0,0,0)$ and $\mathbf{w} = (w_1,w_2,w_3,w_4) \neq (0,0,0,0)$, there exist uniquely determined lines $\ell_1 \in \mathcal{R}_1$ and $\ell_2 \in \mathcal{R}_2$ containing the points of Σ_1 and Σ_2 represented by $\mathbf{u}$ and $\mathbf{w}$, respectively. Then the 3-subspace Σ spanned by ℓ_1 and ℓ_2 is fixed by F since so are both lines ℓ_1 and ℓ_2. Again, we consider the eigenvalues of R^{q^2+1} and see by [1] that the action of F on Σ has ℓ_1 and ℓ_2 as fixed lines and is semiregular on the remaining points, yielding orbits of equal length $q+1$ which are twisted cubics. Since F also fixes the Segre variety $\mathcal{S}$, we see that $\mathcal{S} \cap \Sigma$ is partitioned into F-orbits that are necessarily twisted cubics, and so in particular the F-orbit of P is a twisted cubic on $\mathcal{S}$. □

References

[1] A. Cossidente, Mixed partitions of $PG(3,q)$, J. Geom. 68 (2002) 48–57.

[2] G.L. Ebert, Partitioning projective geometries into caps, Canad. J. Math. 37 (1985) 1163–1175.

[3] J.W.P. Hirschfeld, Projective Spaces of Square Size, Simon Stevin 65 (1991) 319–329.
[4] J.W.P. Hirschfeld, J.A. Thas, General Galois Geometries, Oxford University Press, Oxford, 1991.
[5] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.
[6] C. Suetake, Flag transitive planes of order q^n with a long cycle l_∞ as a Collineation, Graphs Combin. 7 (1991) 183–195.