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Wonderful compactifications and Kontsevich moduli spaces of conics

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A mio papà Daniele,
sei con me in questo traguardo, come in ogni cosa,
con tutto l'amore e i tanti insegnamenti che mi hai dato
nel poco tempo che abbiamo avuto a nostra disposizione.

ABSTRACT

In this thesis we study certain algebraic varieties from the point of view of birational geometry. Given a variety we want to describe all its birational models. In general, this is a very difficult problem, but for a special class of varieties, called Mori dream spaces, the birational geometry is encoded in a decomposition into convex sets of their effective cone. Mori dream spaces have been introduced by Y. Hu and S. Keel, and are named so since they behave in the best possible way from the point of view of the minimal model program.

The first part of the thesis is dedicated to the construction of wonderful compactifications of spaces of linear maps. We recall the construction, due to I. Vainsencher, of the spaces of complete collineations and quadrics of maximal rank and then we generalize it to spaces of linear maps of any rank, and to the wonderful compactification of the space of symmetric and symplectic matrices. By a result of D. Luna, wonderful varieties are spherical and hence Mori dream spaces. So, we take advantage of the spherical structure of these spaces to study their birational geometry from the point of view of Mori theory and in the cases of small Picard rank we give a complete description of the decomposition of the effective cone.

In the second part, we relate our wonderful compactification to other moduli spaces such as Hilbert schemes and Kontsevich spaces of stable maps. In fact, we get several results on the birational geometry of Kontsevich moduli spaces of conics in Grassmannians, Lagrangian Grassmannians and of stable maps of bi-degree $(1,1)$ in a product of two projective spaces.

Keywords: Compactifications; Mori dream spaces; Cox rings; Spherical varieties; Stable maps.

RIASSUNTO

In questa tesi studiamo alcune varietà algebriche dal punto di vista della geometria birazionale. Data una varietà vogliamo descrivere tutti i suoi modelli birazionali. In generale, questo é un problema molto difficile, ma per una classe speciale di varietà, chiamate Mori dream spaces, la geometria birazionale é codificata in una decomposizione in insiemi convessi del loro cono effettivo. I Mori dream spaces sono stati introdotti da Y. Hu e S. Keel, e sono chiamati cosí poiché si comportano nel miglior modo possibile dal punto di vista del programma dei modelli minimali.

La prima parte della tesi é dedicata alla costruzione di compattificazioni wonderful di spazi di mappe lineari. Riprendiamo la costruzione, dovuta a I. Vainsencher, degli spazi delle collineazioni e delle quadriche complete di rango massimo e poi la generalizziamo a spazi di mappe lineari di qualsiasi rango. Costruiamo poi la campattificazione wonderful dello spazio delle matrici simmetriche e simplettiche. Grazie ad un risultato di D. Luna, le varietà wonderful sono varietà sferiche e quindi Mori dream spaces. Approfittiamo quindi della struttura sferica di questi spazi per studiarne la loro geometria birazionale dal punto di vista della teoria di Mori e nei casi di rango di Picard basso diamo una descrizione completa della decomposizione del cono effettivo.

Nella seconda parte, mettiamo in relazione le nostre nuove compattificazioni wonderful con altri spazi di moduli come gli schemi di Hilbert e gli spazi di Kontsevich di mappe stabili. Infatti, otteniamo in questo modo molti risultati sulla geometria birazionale degli spazi di moduli di Kontsevich di coniche in Grassmanniane, Grassmanniane Lagrangiane e di mappe stabili di bi-grado (1,1) in un prodotto di due spazi proiettivi.

Parole chiave: Compattificazioni; Mori dream spaces; Anelli di Cox; Varietà sferiche; Mappe stabili.

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Finally, many thanks to my family for always being there for me. Everything would have been more complicated without the certainty that I can always count on their love and encouragement.

CONTENTS

1

INTRODUCTION

The *wonderful compactification* of a symmetric space was introduced by C. De Concini and C. Procesi in [CP83].

Let $\mathscr{G}$ be a reductive group, and $\mathscr{B} \subset \mathscr{G}$ a Borel subgroup. A *spherical variety* is a variety admitting an action of $\mathscr{G}$ with an open dense $\mathscr{B}$-orbit. For *wonderful varieties* we require in addition the existence of an open orbit whose complementary set is a simple normal crossing divisor $D_1 \cup \cdots \cup D_r$, where the D_i are the $\mathscr{G}$-invariant prime divisors in X. The number r is called the rank of X. Note that $\mathscr{G}$ has 2^r orbits in X given by all the possible intersections among the D_i. The unique closed orbit is $\bigcap_{i=1}^{r} D_i$.

D. Luna proved that all wonderful varieties are spherical in [Lun96]. Apart from their role in group theory, wonderful varieties proved themselves important in enumerative geometry and recently also in birational geometry. We refer to [BL11], [Per14], [Pez18] for comprehensive treatments of these topics.

Classical examples of wonderful varieties are the spaces of complete quadrics and of complete collineations. These spaces have been studied both from the geometrical and enumerative point of view [Sem48], [Sem51], [Sem52], [Tyr56], [Vai82], [Vai84], [KT88], [LLT89], [Tha99]. An aspect that will be fundamental in this thesis is that spaces of complete quadrics and collineations play a role in the study of other moduli spaces such as Hilbert schemes and Kontsevich spaces of stable maps [Alg56], [Pie82], [Cav16]. The birational geometry of the spaces of complete quadrics and collineations, mostly from the point of view of Mori theory, has recently been studied in [Hue15], [Mas20a], [Mas20b].

In Chapter 3 we investigate spaces of complete forms. The spaces of complete collineations and quadrics have been constructed, as a sequence of blow-ups, by I. Vainsencher in [Vai84], [Vai82], and a similar construction for complete skew-forms has been carried out by M. Thaddeus in [Tha99].

Let $\mathcal{S}^{n,m}$ be the image of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, and $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ the h-secant variety of $\mathcal{S}^{n,m}$, that is the subvariety of $\mathbb{P}^N$ obtained as the closure of the union of all $(h-1)$-planes spanned by h general points of $\mathcal{S}^{n,m}$.

We summarize here the main results in Theorem 3.1.18 and Propositions 3.1.22, 3.1.28.

Theorem 1.0.1. *Consider the following sequence of blow-ups*

$$\mathcal{C}(n,m,h) := \mathbb{S}ec_h^{(h-1)}(\mathcal{S}^{n,m}) \to \mathbb{S}ec_h^{(h-2)}(\mathcal{S}^{n,m}) \to \cdots \to \mathbb{S}ec_h^{(0)}(\mathcal{S}^{n,m}) := \mathbb{S}ec_h(\mathcal{S}^{n,m})$$

where $\mathbb{S}ec_h^{(k)}(\mathcal{S}^{n,m}) \to \mathbb{S}ec_h^{(k-1)}(\mathcal{S}^{n,m})$ is the blow-up of $\mathbb{S}ec_h^{(k-1)}(\mathcal{S}^{n,m})$ along the strict transform of $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ for $k = 1,\dots,h-1$. Denote by $E_k^{\mathcal{C}} \subset \mathcal{C}(n,m,h)$ the exceptional divisor over $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ for $k = 1,\dots,h-1$.

The $(SL(n+1)\times SL(m+1))$-action

$$\begin{array}{ccc} (SL(n+1)\times SL(m+1))\times \mathbb{P}^N & \longrightarrow & \mathbb{P}^N \\ ((A,B),Z) & \longmapsto & AZB^t \end{array}$$

induces an $(SL(n+1)\times SL(m+1))$-action on $\mathcal{C}(n,m,h)$, and $\mathcal{C}(n,m,h)$ is wonderful.

Assume that $h < n+1$ and fix homogeneous coordinates $[z_{0,0}:\dots:z_{n,m}]$ on $\mathbb{P}^N$. For $i = 1,\dots,h$ we define the divisors $D_i^{\mathcal{C}}$ as the strict transforms in $\mathcal{C}(n,m,h)$ of the divisor given by the intersection of

$$\det\begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{i-1,0} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0$$

with $\mathcal{C}(n,m,h)$.

The divisor $D_h^{\mathcal{C}}$ in $\mathcal{C}(n,m,h)$ has two irreducible components $H_1^{\mathcal{C}}, H_2^{\mathcal{C}}$, and the Picard rank of $\mathcal{C}(n,m,h)$ is $\rho(\mathcal{C}(n,m,h)) = h+1$. Moreover, the effective cone $\mathrm{Eff}(\mathcal{C}(n,m,h))$ is generated by $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}}, H_1^{\mathcal{C}}, H_2^{\mathcal{C}}$ and the nef cone $\mathrm{Nef}(\mathcal{C}(n,m,h))$ is generated by $D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}}, H_1^{\mathcal{C}}, H_2^{\mathcal{C}}$.

In the case $h = n+1$ we present similar results. Furthermore, we extend the construction in Theorem 1.0.1, by replacing $\mathcal{S}^{n,m}$ with the Veronese variety $\mathcal{V}^n$, to the space $\mathcal{Q}(n,h)$ of rank h symmetric complete collineations.

Note that both $\mathrm{Sec}_h(\mathcal{S}^{n,m})$ and $\mathrm{Sec}_h(\mathcal{V}^n)$ are singular, the wonderful varieties $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ are examples of the process producing a wonderful compactification from a conical one in [MP98].

Furthermore, we construct the wonderful compactification of the space of symmetric and symplectic matrices. More precisely, we summarize our main results in Propositions 3.2.6, 3.2.13, and Theorem 3.2.19 as follows:

Theorem 1.0.2. *Let $\mathbb{P}^{N_+}$ be the projective space parametrizing $2r\times 2r$ symmetric matrices modulo scalar, consider the following $Sp(2r)$-action:*

$$\begin{array}{ccc} Sp(2r)\times \mathbb{P}^{N_+} & \longrightarrow & \mathbb{P}^{N_+} \\ (M,Z) & \longmapsto & MZM^t \end{array}$$

and denote by $X_{2r}\subset \mathbb{P}^{N_+}$ the closure of the $Sp(2r)$-orbit of the identity. Then X_{2r} admits a stratification

$$Y_1\subset Y_2\subset \dots Y_r\subset X_{2r}$$

where the variety Y_k parametrizes matrices in X_{2r} of rank at most k, $\dim(Y_k) = 2rk+k-k^2-1$ for $k = 1,\dots,r$, and $\dim(X_{2r}) = r(r+1)$.

Furthermore, consider the following sequence of blow-ups:

$$\mathcal{S}_{2r} := X_{2r}^{(r-1)} \to X_{2r}^{(r-2)} \to X_{2r}^{(r-3)} \to \dots \to X_{2r}^{(1)} \to X_{2r}^{(0)} := X_{2r}$$

where $X_{2r}^{(k)} \to X_{2r}^{(k-1)}$ is the blow-up of the strict transform of Y_k in $X_{2r}^{(k-1)}$ for $k = 1,\dots,r-1$. Denote by $E_k\subset \mathcal{S}_{2r}$ the exceptional divisor over Y_k for $k = 1,\dots,r-1$, and by $S_r^{(r-1)}(\mathcal{V}^{2r-1})$ the strict transform of the divisor $Y_r\subset X_{2r}$.

Then $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ are smooth and intersect transversally. Furthermore, the closures of the orbits of the $Sp(2r)$-action on $\mathcal{S}_{2r}$ induced by the $Sp(2r)$-action above are given by all the possible intersections among $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ and $\mathcal{S}_{2r}$ itself. Therefore $\mathcal{S}_{2r}$ is wonderful.

We will call $\mathcal{S}_{2r}$ the space of *complete symplectic quadrics* of dimension $2r-2$. By Proposition 3.2.13 Y_k is the intersection of X_{2r} with the secant variety $\mathbb{S}ec_k(\mathcal{V}^{2r-1})$, that is the closure of the union of the $(k-1)$-planes generated by k general points on the Veronese variety $\mathcal{V}^{2r-1}$ of degree two and dimension $2r-1$.

Note that the formula for the dimension of Y_k in Theorem 1.0.2 yields that $\mathcal{V}^{2r-1}$ is entirely contained in X_{2r}, while for $r \geqslant 2$ the orbit closure X_{2r} intersects $\mathbb{S}ec_k(\mathcal{V}^{2r-1})$ in a proper subvariety. Furthermore, by Proposition 3.2.15 we have that set-theoretically $\mathbb{S}ec_k(\mathcal{V}^{2r-1}) \cap X_{2r} = \mathbb{S}ec_r(\mathcal{V}^{2r-1}) \cap X_{2r}$ for $k \geqslant r$. Interestingly, this means that if M is a symmetric $2r \times 2r$ matrix that is a limit of a family of symplectic matrices then either $1 \leqslant \operatorname{rank}(M) \leqslant r$ or $\operatorname{rank}(M) = 2r$.

For instance, by Proposition 3.2.16 X_4 is the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$. In this case by Theorem 1.0.2 we have that $\mathcal{S}_4$ is the blow-up of $\mathbb{G}(1,4)$ along the Veronese 3-fold $\mathcal{V}^3 \subset \mathbb{G}(1,4)$. This is a wonderful variety of rank two. As remarked in [Was96] wonderful varieties of rank two are a building block in the theory of spherical varieties. The wonderful compactification $\mathcal{S}_4$ is the sixth variety in [Was96, Table C], and will be a central character in this study.

Remark 1.0.3. The use of wonderful compactifications in enumerative geometry dates back to the solution of M. Chasles to a problem posed by J. Steiner asking how many conics in the plane are tangent to five given general conics [Kle80]. Steiner's answer, which then turned out to be wrong, was $6^5 = 7776$. Later on Chasles computed the right number which is 3264.

Although enumerative problems are not within the scope of this thesis, we give a simple application of our construction in enumerative geometry. It is well known that there are 92 quadric surfaces in $\mathbb{P}^3$ that are tangent to nine general lines [BFS20, Remark 4.3]. The points of $\mathcal{S}_4$ in a divisor of class $2H - E_1$, where H is the pull-back of the hyperplane class of X_4, correspond to the symplectic quadrics in $\mathbb{P}^3$ that are tangent to a general line. We have that $(2H - E_1)^6 = 40$. From the enumerative point of view this means that there are exactly 40 symplectic quadrics in $\mathbb{P}^3$ that are tangent to six general lines.

The variety X_{2r} is singular for $r \geqslant 3$. Then, also the wonderful variety $\mathcal{S}_{2r}$ may be seen as an incarnation, in the singular setting, of the process producing a wonderful compactification from a conical one in [MP98]. Furthermore, by Proposition 3.2.18 $\mathcal{S}_{2r}$ provides a resolution of a variety with conical singularities as remarked in [MP98, Section 3.3].

In Section 3.2.1 and 3.2.2 we take advantage of the spherical structure of $\mathcal{S}_{2r}$ to study its birational geometry from the point of view of Mori theory. Roughly speaking, a *Mori dream space* is a projective variety X whose cone of effective divisors $\operatorname{Eff}(X)$ admits a well-behaved decomposition into convex sets, called Mori chamber decomposition, and these chambers are the nef cones of birational models of X. These varieties, introduced by Y. Hu and S. Keel in [HK00], are named so because they behave in the best possible way from the point of view of the minimal model program. In general, to determine whether or not a variety is a Mori dream space, and in case to study in detail its Mori chamber decomposition is a hard problem. This has been done for instance when X is obtained by blowing-up points in a projective space [Muk01], [CT06], [AM16], [AC17], [BM21], [LP17].

Spherical varieties are Mori dream spaces, we refer to [Per14] for a comprehensive treatment of these topics. Cox rings were first introduced by D. A. Cox for toric varieties [Cox95], and then his construction was generalized to projective varieties in [HK00]. These algebraic objects are basically universal homogeneous coordinate rings of projective varieties, defined as the direct sum of the spaces of sections of all isomorphism classes of line bundles on them. We have that a normal $\mathbb{Q}$-factorial projective variety X, over an algebraically closed field, with finitely generated Picard group is a Mori dream space if and only if its Cox ring is finitely generated [HK00, Proposition 2.9].

In Propositions 3.1.32 and Propositions 3.1.33 we give a detailed description of the Mori chamber decompositions of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ when their Picard rank is at most three. Furthemore, summing-up the results in Propositions 3.2.26, 3.2.28, 3.2.31 and Theorem 3.2.33 we have the following:

Theorem 1.0.4. *Fix homogeneous coordinates $[z_{0,0} : \cdots : z_{n,n}]$ on $\mathbb{P}^{N_+}$, and consider the blow-up $f\colon \mathcal{S}_{2r} \to X_{2r} \subset \mathbb{P}^{N_+}$ with exceptional divisors $E_1,\dots,E_{r-1}$ in Theorem 1.0.2. For $i = 1,\dots,r$ we define the divisors D_i as the strict transforms in $\mathcal{S}_{2r}$ of the divisor given by the intersection of*

$$\det\begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{0,i-1} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0$$

with X_{2r}, and let H be the pull-back of the hyperplane section of $X_{2r} \subset \mathbb{P}^{N_+}$ to $\mathcal{S}_{2r}$.

The Picard rank of $\mathcal{S}_{2r}$ is $\rho(\mathcal{S}_{2r}) = r$ and $\mathrm{Pic}(\mathcal{S}_{2r})$ is generated by $H, E_1,\dots,E_{r-1}$. Furthermore, the effective cone $\mathrm{Eff}(\mathcal{S}_{2r})$ is generated by $E_1,\dots,E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$, the nef cone $\mathrm{Nef}(\mathcal{S}_{2r})$ is generated by $D_1,\dots,D_r$, and the Cox ring of $\mathcal{S}_{2r}$ is generated by the sections of $E_1,\dots,E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1}), D_1,\dots,D_r$.

Finally, the Mori chamber decomposition of the $\mathrm{Eff}(\mathcal{S}_4)$ has three chambers, and the Mori chamber decomposition of the $\mathrm{Eff}(\mathcal{S}_6)$ has nine chambers.

We refer to Proposition 3.2.31 and Theorem 3.2.33 for a detailed description of the Mori chamber decompositions.

In Chapter 4 we investigate the relation among the spaces of complete forms we introduced and moduli spaces of stable maps. These spaces are denoted by $\overline{M}_{g,n}(X,\beta)$ where X is a projective scheme and $\beta \in H_2(X,\mathbb{Z})$ is the homology class of a curve in X. A point in $\overline{M}_{g,n}(X,\beta)$ corresponds to a holomorphic map α from an n-pointed genus g curve C to X such that $\alpha_*([C]) = \beta$. If X is a homogeneous variety then there exists a smooth, irreducible Deligne-Mumford stack $\overline{\mathcal{M}}_{0,n}(X,\beta)$ whose coarse moduli space is $\overline{M}_{0,n}(X,\beta)$, as stated in [FP97]. The Mori theory of the spaces $\overline{M}_{0,n}(X,\beta)$, especially when the target variety is a projective space or a Grassmannian, has been widely investigated in a series of papers [CS06], [Che08], [CHS08], [CHS09], [CC10], [CC11], [CM17].

When X is a projective space or a Grassmannians the class β is completely determined by its degree and we will write $\beta = d[L]$, where $[L]$ is the class of a line or the class of a line in the Plücker embedding. Similarly, when X is the product of two projective spaces we identify the class β with its bidegree (a, b). Summing-up Propositions 4.1.1, 4.2.2, 4.3.1, 4.3.5, and Corollary 4.3.4 we have the following:

Theorem 1.0.5. *There are isomorphisms*

$$\mathcal{C}(n, m, 2) \xrightarrow{\sim} \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1, 1))$$

and

$$\mathcal{S}ec_3^{(1)}(\mathcal{V}^n) \xrightarrow{\sim} \overline{M}_{0,0}(\mathbb{P}^n, 2).$$

Furthermore, there is a 2*-to-*1 *morphism*

$$\overline{M}_{0,0}(\mathbb{G}(1, n), 2) \to \mathcal{S}ec_4^{(2)}(\mathcal{V}^n).$$

For the automorphism groups we have that

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1, 1))) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m \geqslant 2; \end{cases}$$

and $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))) \cong PGL(4)$.

Furthermore, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n, 2)) \cong PGL(n+1)$ *for* $n \geqslant 3$, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^2, 2)) \cong PGL(3) \rtimes S_2$*, and* $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1, 2)) \cong PGL(3)$*.*

Finally,

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1, n), 2)) \cong \begin{cases} S_2 \ltimes PGL(n+1) & \text{if } n > 3; \\ S_2 \ltimes (S_2 \ltimes PGL(n+1)) & \text{if } n = 3. \end{cases}$$

As an application of Theorem 1.0.5 we recover some of the results in Proposition 4.1.2, and Remarks 4.1.4, 4.2.3. In particular, Theorem 1.0.5 gives an explicit description of the birational contraction of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ in [CHS09, Theorem 1.2] as the blow-down $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n) \to \mathcal{S}ec_3(\mathcal{V}^n)$.

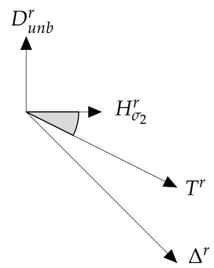
When X is a Lagrangian Grassmannian the class β is also completely determined by its degree and we will write $\beta = d[L]$, where $[L]$ is the class of a line in the Plücker embedding. On the Kontsevich space $\overline{M}_{0,0}(LG(r, 2r), 2)$ of conics in the Lagrangian Grassmannian $LG(r, 2r)$, parametrizing Lagrangian subspaces of a $2r$-dimensional symplectic vector space, we consider the divisor classes: Δ^r of maps with reducible domain, T^r of conics tangent to a fixed hyperplane section of $LG(r, 2r)$, $H^r_{\sigma_2}$ of conics intersecting a fixed codimension two Schubert variety $\Sigma_2^r \subset LG(r, 2r)$, and D^r_{unb} which we now define. A stable map $\alpha \colon \mathbb{P}^1 \to LG(r, 2r)$ induces a rank r subbundle $\mathcal{E}_\alpha \subset \mathcal{O}_{\mathbb{P}^1} \otimes K^{2r}$. If $r = 2$ we define D_{unb} as the closure of the locus of maps $[\mathbb{P}^1, \alpha] \in \overline{M}_{0,0}(LG(2, 4), 2)$ such that $\mathcal{E}_\alpha \neq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. If $r \geqslant 3$ there is a trivial subbundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus r-2} \subset \mathcal{E}_\alpha$ which induces a $(r-3)$-dimensional subspace $H_\alpha \subset \mathbb{P}^{2r-1}$. We define D^r_{unb} as the closure of the locus of maps $[\mathbb{P}^1, \alpha] \in \overline{M}_{0,0}(LG(r, 2r), 2)$ such that H_α intersects a fixed $(r+1)$-dimensional subspace of $\mathbb{P}^{2r-1}$.

The main results in Proposition 4.4.11, Theorem 4.4.14, Remark 4.4.13 and Corollary 4.4.18 can be summarized in the following statement:

Theorem 1.0.6. *Let $\overline{M}_{0,0}(LG(r,2r),2)$ be the Kontsevich space of conics in the Lagrangian Grassmannian $LG(r,2r)$, parametrizing Lagrangian subspaces of a $2r$-dimensional symplectic vector space, with $r \geqslant 2$.*

The effective cone $\mathrm{Eff}(\overline{M}_{0,0}(LG(r,2r),2))$ *is generated by Δ^r and D^r_{unb}, and the nef cone* $\mathrm{Nef}(\overline{M}_{0,0}(LG(r,2r),2))$ *is generated by $H^r_{\sigma_2}$ and T^r.*

The Mori chamber decomposition of $\mathrm{Eff}(\overline{M}_{0,0}(LG(r,2r),2))$ *has three chambers as displayed in the following picture:*



where $H^r_{\sigma_2} \sim \frac{1}{2}(\Delta^r + 2D^r_{unb})$ and $T^r \sim \Delta^r + D^r_{unb}$.

Furthermore, if $r > 2$ then $\mathrm{Mov}(\overline{M}_{0,0}(LG(r,2r),2))$ *is generated by T^r and D^r_{unb} while* $\mathrm{Mov}(\overline{M}_{0,0}(LG(2,4),2))$ *is generated by T^r and $H^r_{\sigma_2}$.*

The divisor $H^r_{\sigma_2}$ induces a birational morphism

$$f_{H^r_{\sigma_2}} : \overline{M}_{0,0}(LG(r,2r),2) \to \widetilde{Chow}(LG(r,2r),2)$$

which is an isomorphism away form the locus $Q^r(1)$ of double covers of a line in $LG(r,2r)$, and contracts $Q^r(1)$ so that the locus of double covers with the same image maps to a point, where $\widetilde{Chow}(LG(r,2r),2)$ is the normalization of the Chow variety of conics in $LG(r,2r)$.

The divisor T^r induces a morphism

$$f_{T^r} : \overline{M}_{0,0}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2,1)$$

which is an isomorphism away from Δ^r and contracts the locus of maps with reducible domain $[C_1 \cup C_2, \alpha]$ to $\alpha(C_1 \cap C_2)$, where $\overline{M}_{0,0}(LG(r,2r),2,1)$ is the moduli space of weighted stable maps to $LG(r,2r)$.

The birational model X_r corresponding to the chamber delimited by $H^r_{\sigma_2}$ and D^r_{unb} is a fibration $X_r \to SG(r-2,2r)$ with fibers isomorphic to the Grassmannian $\mathbb{G}(2,4)$ parametrizing plane in $\mathbb{P}^4$, where $SG(r-2,2r)$ is the symplectic Grassmannian parametrizing isotropic subspaces of dimension $r-2$. Moreover, D^r_{unb} contracts $\overline{M}_{0,0}(LG(r,2r),2)$ onto $SG(r-2,2r)$.

Finally, $\overline{M}_{0,0}(LG(r,2r),2)$ is Fano for $2 \leqslant r \leqslant 6$, weak Fano, that is $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ is nef and big, for $r = 7$, and $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ is not ample for $r \geqslant 8$.

Moreover, Lemma 4.4.6, Proposition 4.4.8, Remark 4.4.13 and Corollary 4.4.19 provide additional information for the case $r = 2$.

Theorem 1.0.7. *The following $Sp(4)$-action*

$$\begin{array}{ccc} Sp(4) \times \overline{M}_{0,0}(LG(2,4),2) & \longrightarrow & \overline{M}_{0,0}(LG(2,4),2) \\ (M,[C,\alpha]) & \longmapsto & [C, \wedge^2 M \circ \alpha] \end{array}$$

induces on $\overline{M}_{0,0}(LG(2,4),2)$ *a structure of spherical variety. Furthermore, there exists an isomorphism*

$$\varphi\colon \overline{M}_{0,0}(LG(2,4),2) \to \mathcal{S}_4$$

where $\mathcal{S}_4$ *is the wonderful compactification of the space of symplectic quadrics of* $\mathbb{P}^3$*, mapping a smooth conic* $C \subset LG(2,4)$ *to the quadric* $\bigcup_{[L]\in C} L \subset \mathbb{P}^3$*. The Cox ring* $\mathrm{Cox}(\overline{M}_{0,0}(LG(2,4),2))$ *is generated by the sections of* $\Delta^2, D^2_{unb}, H^2_{\sigma_2}, T^2$*.*

The moduli space $\overline{M}_{0,0}(LG(2,4),2)$ *identifies with the blow-up of* $\mathbb{G}(1,4)$ *along the Veronese* $\mathcal{V}^3$*. With this identification the morphism associated to* $H^2_{\sigma_2}$ *is the blow-down and* $\widetilde{Chow}(LG(2,4),2) \cong \mathbb{G}(1,4)$*, while the morphism associated to* T^2 *is induced by the strict transform on* $\mathcal{S}_4$ *of the linear system of quadrics containing* $\mathcal{V}^3$*, and its image is a* 6*-fold of degree* 40 *in* $\mathbb{P}^{14}$ *isomorphic to* $\overline{M}_{0,0}(LG(2,4),2,1)$*.*

Finally, it holds $\mathrm{PsAut}(\overline{M}_{0,0}(LG(2,4),2)) \cong \mathrm{Aut}(\overline{M}_{0,0}(LG(2,4),2)) \cong PSp(4)$ *where* $PSp(4)$ *is the projective symplectic group, and* $\mathrm{PsAut}(\overline{M}_{0,0}(LG(2,4),2))$ *is the group of birational self-maps of* $\overline{M}_{0,0}(LG(2,4),2)$ *inducing automorphisms in codimension one.*

2 MORI DREAM SPACES AND SPHERICAL VARIETIES

Throughout the thesis X will be a normal projective variety over an algebraically closed field of characteristic zero. We denote by $N^1(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\mathrm{Nef}(X) \subset N^1(X)$ generated by classes of nef divisors. The *effective cone* of X is the convex cone $\mathrm{Eff}(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions $\mathrm{Nef}(X) \subset \overline{\mathrm{Eff}(X)}$. We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

2.1 SPHERICAL VARIETIES

Recall that an algebraic group G is solvable when it is solvable as an abstract group. A Borel subgroup B of an algebraic group G is a subgroup which is maximal among the connected solvable algebraic subgroups of G. The radical $R(G)$ of an algebraic group is the identity component of the intersection of all Borel subgroups of G. We say that G is semi-simple if $R(G)$ is trivial. We say that G is reductive if the unipotent part of $R(G)$, i.e. the subgroup of unipotent elements of $R(G)$, is trivial.

Given an algebraic group G there is a single conjugacy class of Borel subgroups. For instance, in the group GL_n of $n \times n$ invertible matrices, the subgroup of invertible upper triangular matrices is a Borel subgroup. The radical of GL_n is the subgroup of scalar matrices, therefore GL_n is reductive but not semi-simple. On the other hand, SL_n is semi-simple.

Definition 2.1.1. A *spherical variety* is a normal variety X together with an action of a connected reductive affine algebraic group G, a Borel subgroup $B \subset G$, and a base point $x_0 \in X$ such that the B-orbit of x_0 in X is a dense open subset of X.

The *complexity* $c(X)$ of a normal variety X with an action of a connected reductive affine algebraic group G is the minimal codimension in X of an orbit of a Borel subgroup $B \subset G$. Therefore, a spherical variety is a normal G-variety of complexity zero.

Next, we recall that the effective cone of a spherical variety can be described in terms of divisors which are invariant under the action of the Borel subgroup.

Definition 2.1.2. Let (X, G, B, x_0) be a spherical variety. We distinguish two types of B-invariant prime divisors:

- A *boundary divisor* of X is a G-invariant prime divisor on X.
- A *color* of X is a B-invariant prime divisor that is not G-invariant.

For instance, any toric variety is a spherical variety with $B = G$ equal to the torus. For a toric variety there are no colors, and the boundary divisors are the usual toric invariant divisors. For a spherical variety we have to take into account the colors as well.

Proposition 2.1.3 ([ADHL15, Proposition 4.5.4.4]). *Let (X, G, B, x_0) be a spherical variety.*

- *There are finitely many boundary divisors $E_1, \dots, E_r$ and finitely many colors $D_1, \dots, D_s$ on X. Furthermore, $X \setminus B \cdot x_0 = E_1 \cup \cdots \cup E_r \cup D_1 \cup \cdots \cup D_s$.*
- *The classes of the E_k's and of the D_i's generate $\mathrm{Eff}(X) \subset N^1(X)$ as a cone.*

Definition 2.1.4. A *wonderful variety* is a smooth projective variety X with the action of a semi-simple simply connected group G such that:

- there is a point $x_0 \in X$ with open G orbit and such that the complement $X \setminus G \cdot x_0$ is a union of prime divisors $E_1, \cdots, E_r$ having simple normal crossing;
- the closures of the G-orbits in X are the intersections $\bigcap_{i \in I} E_i$ where I is a subset of $\{1, \dots, r\}$.

As proven by D. Luna in [Lun96] wonderful varieties are in particular spherical.

2.2 MORI DREAM SPACES

The *stable base locus* $\mathbf{B}(D)$ of a $\mathbb{Q}$-divisor D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$\mathbf{B}(D) = \bigcap_{s>0} B(sD). \tag{2.1}$$

The *movable cone* of X is the convex cone $\mathrm{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X. We have inclusions

$$\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}.$$

Since stable base loci do not behave well with respect to numerical equivalence [Laz04, Example 10.3.3], we will assume that $h^1(X, \mathcal{O}_X) = 0$ so that linear and numerical equivalence of $\mathbb{Q}$-divisors coincide.

Then, numerically equivalent $\mathbb{Q}$-divisors on X have the same stable base locus, and the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of X can be decomposed into chambers depending on the stable base locus of the corresponding linear series. The resulting decomposition is called *stable base locus decomposition*.

Remark 2.2.2. Recall that two divisors D_1, D_2 are said to be *Mori equivalent* if $\mathbf{B}(D_1) = \mathbf{B}(D_2)$ and the following diagram of rational maps is commutative

$$\begin{tikzcd} & X \arrow[dl, dashed, "\phi_{D_1}"'] \arrow[dr, dashed, "\phi_{D_2}"] & \\ X(D_1) \arrow[rr, "\sim"'] & & X(D_2) \end{tikzcd}$$

where the horizontal arrow is an isomorphism. Therefore, the Mori chamber decomposition is a, possibly trivial, refinement of the stable base locus decomposition.

Let X be a normal $\mathbb{Q}$-factorial variety with free and finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup G of the group of Weil divisors on X such that the canonical map $G \to \mathrm{Cl}(X)$, mapping a divisor $D \in G$ to its class $[D]$, is an isomorphism. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where $D \in G$ represents $[D] \in \mathrm{Cl}(X)$, and the multiplication in $\mathrm{Cox}(X)$ is defined by the standard multiplication of homogeneous sections in the field of rational functions on X. If $\mathrm{Cox}(X)$ is finitely generated as an algebra over the base field, then X is said to be a *Mori dream space*. A perhaps more enlightening definition, especially for the relation with the minimal model program, is the following.

Definition 2.2.3. A normal projective $\mathbb{Q}$-factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\mathrm{Pic}\,(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$,
- $\mathrm{Nef}\,(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i\colon X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and $\mathrm{Mov}\,(X) = \bigcup_i f_i^*(\mathrm{Nef}\,(X_i))$.

The collection of all faces of all cones $f_i^*(\mathrm{Nef}\,(X_i))$ above forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}\,(X_i))$ and $f_j^*(\mathrm{Nef}\,(X_j))$, meet along a facet, then there exist a normal projective variety Y, a small modification $\varphi\colon X_i \dashrightarrow X_j$, and $h_i\colon X_i \to Y$, $h_j\colon X_j \to Y$ small birational morphisms of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 2.2.4. Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i\colon X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form: $\mathcal{C}_i = \langle g_i^*(\mathrm{Nef}(Y_i)), \mathrm{Exc}(g_i)\rangle$. We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^\circ$ a *maximal chamber* of $\mathrm{Eff}(X)$.

If X is a Mori dream space, satisfying then the condition $h^1(X, \mathcal{O}_X) = 0$, determining the stable base locus decomposition of $\mathrm{Eff}(X)$ is a first step in order to compute its Mori chamber decomposition.

Remark 2.2.5. By the work of M. Brion [Bri93] we have that $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [Per14, Section 4].

Remark 2.2.6. Recall that by [HK00, Proposition 2.11] given a Mori Dream Space X there is an embedding $i\colon X \to \mathcal{T}_X$ into a simplicial projective toric variety $\mathcal{T}_X$ such that $i^*\colon \operatorname{Pic}(\mathcal{T}_X) \to \operatorname{Pic}(X)$ is an isomorphism inducing an isomorphism $\operatorname{Eff}(\mathcal{T}_X) \to \operatorname{Eff}(X)$. Furthermore, the Mori chamber decomposition of $\operatorname{Eff}(\mathcal{T}_X)$ is a refinement of the Mori chamber decomposition of $\operatorname{Eff}(X)$. Indeed, if $\operatorname{Cox}(X) \cong \frac{K[T_1,\dots,T_s]}{I}$, where the T_i are homogeneous generators with non-trivial effective $\operatorname{Pic}(X)$-degrees, then $\operatorname{Cox}(\mathcal{T}_X) \cong K[T_1,\dots,T_s]$.

Since the variety $\mathcal{T}_X$ is toric, the Mori chamber decomposition of $\operatorname{Eff}(\mathcal{T}_X)$ can be computed by means of the Gelfand–Kapranov–Zelevinsky, GKZ for short, decomposition [ADHL15, Section 2.2.2]. Let us consider the family $\mathcal{W}$ of vectors in $\operatorname{Pic}(\mathcal{T}_X)$ given by the generators of $\operatorname{Cox}(\mathcal{T}_X)$, and let $\Omega(\mathcal{W})$ be the set of all convex polyhedral cones generated by some of the vectors in $\mathcal{W}$. By [ADHL15, Construction 2.2.2.1] the GKZ chambers of $\operatorname{Eff}(\mathcal{T}_X)$ are given by the intersections of all the cones in $\Omega(\mathcal{W})$ containing a fixed divisor in $\operatorname{Eff}(\mathcal{T}_X)$.

Remark 2.2.7. Let (X, G, B, x_0) be a projective spherical variety. Consider a divisor D on X, and let f_D be the, unique up to constants, section of $\mathcal{O}_X(D)$ associated to D. We will denote by $\operatorname{lin}_K(G \cdot D) \subseteq \operatorname{Cox}(X)$ the finite-dimensional vector subspace of $\operatorname{Cox}(X)$ spanned by the orbit of f_D under the action of G that is the smallest linear subspace of $\operatorname{Cox}(X)$ containing the G-orbit of f_D.

By [ADHL15, Theorem 4.5.4.6] if G is a semi-simple and simply connected algebraic group and (X, G, B, x_0) is a spherical variety with boundary divisors $E_1, \dots, E_r$ and colors $D_1, \dots, D_s$ then $\operatorname{Cox}(X)$ is generated as a K-algebra by the canonical sections of the E_i's and the finite dimensional vector subspaces $\operatorname{lin}_K(G \cdot D_i) \subseteq \operatorname{Cox}(X)$ for $1 \leqslant i \leqslant s$.

Definition 2.2.8. Let X be a normal projective $\mathbb{Q}$-factorial variety. We say that X is *weak Fano* if $-K_X$ is nef and big. We say X is *log Fano* if there exists an effective $\mathbb{Q}$-divisor Δ such that $-(K_X + \Delta)$ is ample and the pair (X, Δ) is klt.

By [BCHM10, Corollary 1.3.2] log Fano and weak Fano varieties are Mori dream spaces.

Example 2.2.9. Let write X^n_k for the blow up of $\mathbb{P}^n$ at k points in general position. Results of [Muk01] and [CT06] show that:

- For $n = 4$, X^n_k is a Mori dream space if and only if $k \leqslant 8$.
- For $n > 4$, X^n_k is a Mori dream space if and only if $k \leqslant n + 3$.

Moreover, for $n = 2$ and $k \leqslant 8$, $-K_{X^n_k}$ is ample and and for $n = 3$ and $k \leqslant 7$, X^n_k is log Fano, then X^n_k is a Mori dream space also in these cases.

Example 2.2.10. Let us work out explicitly the cone of effective divisors and the Mori cone of curves of $X := X^n_2$, the blow-up of $\mathbb{P}^n$ at two points $p, q \in \mathbb{P}^n$, with $n > 1$.

Let $H, H_p, H_q, H_{p,q}$ be the strict transforms respectively of a hyperplane, a hyperplane passing through p, through q, and through both p and q. Moreover, let E_p, E_q be the exceptional divisors over p and q respectively. Note that $H_p = H - E_p$, $H_q = H - E_q$ and $H_{p,q} = H - E_p - E_q$. Then $N^1(X) \cong \mathbb{Z}[H, E_p, E_q]$.

We will denote by h the strict transform of a general line in $\mathbb{P}^n$, and by e_p, e_q classes of lines in E_p and E_q respectively. The intersection pairing is given by

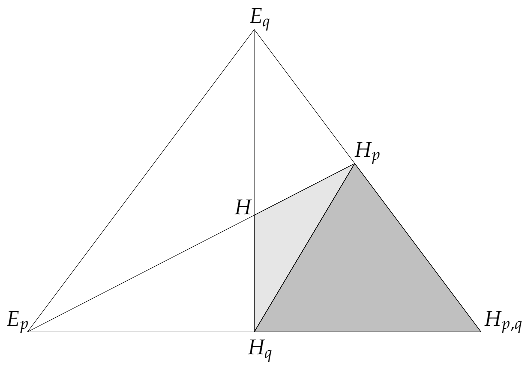
$H \cdot h = 1, H \cdot e_p = H \cdot e_q = 0, E_p \cdot e_q = E_q \cdot e_p = 0, E_p \cdot e_p = E_q \cdot e_q = -1$. The last two intersections numbers might be not obvious from a geometrical point of view. To compute them one may reason as follows: the divisor $H - E_p$ represents the strict transform of a general hyperplane through p, and $h - e_p$ represents the strict transform of a general line through p. In the blow-up X these strict transforms do not intersect anymore, so $0 = (H - E_p) \cdot (h - e_p) = H \cdot h - H \cdot e_p - E_p \cdot h + E_p \cdot e_p$ and hence $E_p \cdot e_p = -H \cdot h = -1$.

Now, let $C \subset X$ be an irreducible curve. Then either C is contained in an exceptional divisor and then it is numerically equivalent to a positive multiple of e_p or e_q, or it is mapped by the blow-down map to an irreducible curve $\Gamma \subset \mathbb{P}^n$. Let d, m_p, m_q be respectively the degree of Γ and the multiplicities of Γ at p and q. Then $C \equiv dh - m_p e_p - m_q e_q$. We may write $C \equiv d(h - e_p - e_q) + (d - m_p)e_p + (d - m_q)e_q$. Furthermore, $d - m_p > 0$ otherwise by Bézout's theorem Γ would contain a line through p as a component, and similarly $d - m_q > 0$. Hence $\mathrm{NE}(X)$ is closed and generated by the classes e_p, e_q and $h - e_p - e_q$. Note that the latter is the strict transform of the line in $\mathbb{P}^n$ through p, q. Similarly, it can be shown that $\mathrm{Eff}(X)$ is closed and generated by the classes of E_p, E_q and $H_{p,q}$.

Let us work out the nef cone of X when $n > 2$. This is the cone of divisors intersecting non negatively all the irreducible curves in X. Since any curve in X can be written as a linear combination with non-negative coefficients of the generators of $\mathrm{NE}(X)$, it is enough to check when a divisor intersects non-negatively these generators. Let us write $D \equiv aH + bE_p + cE_q$. Then $D \cdot (h - e_p - e_q) = a + b + c$, $D \cdot e_p = -b$ and $D \cdot e_p = -c$, and $\mathrm{Nef}(X)$ is defined in $N^1(X)_{\mathbb{R}} \cong \mathbb{R}^3$ by the inequalities $a + b + c \geqslant 0, b \leqslant 0, c \leqslant 0$. Hence $\mathrm{Nef}(X)$ is generated by $\langle H, H_p, H_q \rangle$.

Finally, we determine the movable cone of X. The divisor $H_{p,q}$ represents the hyperplanes of $\mathbb{P}^n$ passing through p, q. Hence the stable base locus of $H_{p,q}$ consists of the strict transform of the line through p, q. The stable base locus of all divisors in the cone generated by $\langle H_p, H_q, H_{p,q} \rangle$ is contained in such a strict transform as H_p, H_q have no base loci. Hence all the divisors in this cone are movable when $n > 2$. On the other hand, all divisors in the interior of the cone $\langle H, H_p, E_q \rangle$ contain E_q, all divisors in the interior of the cone $\langle H, H_q, E_p \rangle$ contain E_p, and all divisors in the interior of the cone $\langle H, E_p, E_q \rangle$ contain $E_p \cup E_q$. Therefore, $\mathrm{Mov}(X)$ is the cone generated by $\langle H, H_p, H_q, H_{p,q} \rangle$.

The following picture is a two dimensional cross-section of $\mathrm{Eff}(X)$ displaying its Mori chamber decomposition:



The divisors $H, H_p, H_q, H_{p,q}$ generate $\mathrm{Mov}(X)$, and H, H_p, H_q generate $\mathrm{Nef}(X)$. The chamber delimited by H, H_q, E_p corresponds to the contraction of E_p, similarly the chamber delimited by H, H_p, E_q corresponds to the contraction of E_q, and chamber delimited by H, E_p, E_q corresponds to the contraction of both E_p and E_q.

In the case $n \geqslant 3$, X admits only one small $\mathbb{Q}$-factorial modification X' corresponding to the chamber delimited by $H_p, H_q, H_{p,q}$. In what follows in this example, we will investigate the geometry of X'.

Consider a divisor lying on the wall delimited by H_p and H_q, for instance $D = H_p + H_q = 2H - E_p - E_q$, and let L be the strict transform of the line through p and q. Then $D \cdot L = 0$ and the linear system of quadrics in $\mathbb{P}^n$ through p and q induces a morphism $h_D \colon X \to Y$ contracting L to a point.

On the other hand, a divisor in the maximal chamber delimited by $H_p, H_q, H_{p,q}$ must be ample on X'. We can write such a divisor as $aH_p + bH_q + cH_{p,q}$ with $a, b, c > 0$ and observe that $(aH_p + bH_q + cH_{p,q}) \cdot L = -c < 0$.

Note that the curve L prevents divisors in the chamber $\langle H_p, H_q, H_{p,q} \rangle$ from being ample.

Let $g \colon W \to X$ be the blow-up of X along L with exceptional divisor $E_L \subset W$. Observe that E_L is a $\mathbb{P}^{n-2}$-bundle over L.

There is a morphism $g' \colon W \to X'$ contracting E_L, in the direction of L, onto a subvariety $Z \subset X'$ such that $Z \cong \mathbb{P}^{n-2}$. Consider the divisor $D' \equiv H_p + H_q + H_{p,q} \equiv 3H - 2E_p - 2E_q$. The linear system of D' induces a rational map $\phi_{D'} \colon X \dashrightarrow X'$, and we have the following commutative diagram

$$\begin{tikzcd}
& W \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
X \arrow[rr, dashed, "\phi_{D'}"] \arrow[dr, "h_D"'] & & X' \arrow[dl, "h"] \\
& Y &
\end{tikzcd}$$

where $h \colon X' \to Y$ is a small modification contracting $Z \subset X'$ to $h_D(L)$. The rational map $\phi_{D'} \colon X \dashrightarrow X'$ is an isomorphism between $X \backslash L$ and $X' \backslash Z$ and replaces L with the variety Z which is covered by curves having non-negative intersection with all divisors in the chamber $\langle H_p, H_q, H_{p,q} \rangle$. Concretely, in the case $n = 3$ for instance, we can fix homogeneous coordinates $[x : y : z : w]$ on $\mathbb{P}^3$, assume that $p = [1 : 0 : 0 : 0]$, $q = [0 : 0 : 0 : 1]$, and consider the rational maps

$$\alpha \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^7$$

defined by $\alpha([x : y : z : w]) = [xy : xz : xw : y^2 : yz : yw : z^2 : zw]$, that is induced by the quadrics of $\mathbb{P}^3$ passing through p and q, and

$$\beta \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^{11}$$

defined by $\beta([x : y : z : w]) = [xy^2 : xz^2 : xyz : xyw : xzw : y^3 : y^2z : y^2w : yz^2 : yzw : z^3 : z^2w]$, that is induced by the cubics of $\mathbb{P}^3$ having at least double points at p and q. Then Y is the closure of the image of α and X' is the closure of the image of β.

Let us give a geometric description of X'. Let $\Pi \subset X$ be the strict transform of a 2-plane through the line $\overline{pq}$. The plane Π is contracted to a point by the map $\pi_{H_{p,q}}\colon X \dashrightarrow \mathbb{P}^{n-2}$ induced by $H_{p,q}$. Indeed, $\pi_{H_{p,q}}$ is induced by the linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$ with center $\overline{pq}$. Observe that a divisor in the linear system of D' has a base component when restricted to Π, namely the curve L.

Therefore, $\phi_{D'|\Pi}$ is the rational map induced by the linear system of conics through p and q, hence its image is a smooth quadric surface $Q_\Pi \cong \mathbb{P}^1 \times \mathbb{P}^1$. The quadric Q_Π intersects Z at a point. The morphism $\tilde{\pi}_{H_{p,q}}\colon X' \dashrightarrow \mathbb{P}^{n-2}$, induced by the strict transform of $H_{p,q}$ on X', contracts Q_Π to the point $\pi_{H_{p,q}}(\Pi)$ and maps Z isomorphically onto $\mathbb{P}^{n-2}$. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi_{D'}"] \arrow[dr, dashed, "\pi_{H_{p,q}}"'] & & X' \arrow[dl, "\tilde{\pi}_{H_{p,q}}"] \\ & \mathbb{P}^{n-2} & \end{tikzcd}$$

and X' has a structure of $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle over $\mathbb{P}^{n-2}$. Summing up, the birational model of X corresponding to the chamber $\langle H_p, H_q, H_{p,q}\rangle$ is a quadric bundle over $\mathbb{P}^{n-2}$ and, as we already noticed, the other chambers $\langle H, H_p, E_q\rangle$, $\langle H, H_q, E_p\rangle$ and $\langle H, E_p, E_q\rangle$ corresponds respectively to $\mathbb{P}^n$ blown-up at q, $\mathbb{P}^n$ blown-up at p and $\mathbb{P}^n$. The chamber $\langle H, H_p, H_q\rangle$ corresponds to X itself.

2.3 MODULI SPACES OF STABLE MAPS

Let X be a projective variety, $\beta \in H_2(X,\mathbb{Z})$ be a homology class, and $Z_1, \dots, Z_n \subset X$ cycles in general position. We want to study the following set of curves

$$\{C \subset X \textit{ of genus } g,\ \textit{homology } \beta, \textit{ and } C \cap Z_i \neq \emptyset \textit{ for any } i\}. \tag{2.2}$$

In [Kon95], M. Kontsevich observed that the curve $C \subset X$ should be replaced by a pointed curve $(C, \{x_1, \dots, x_n\})$ and a holomorphic map $f\colon C \to X$ such that $f(x_i) \in Z_i$ for any $i = 1, \dots, n$. The key idea, in order to give an algebraic definition of Gromov-Witten classes and invariant, is to introduce a suitable compactification done by stable maps of the space of curves (2.2).

Definition 2.3.1. An n-pointed, genus g, *quasi-stable* curve $[C, \{x_1, \dots, x_n\}]$ is a projective, connected, reduced, at most nodal curve of arithmetic genus g, with n distinct, and smooth marked points.

A family of n-pointed genus g quasi-stable curve parametrized by a scheme S over $\mathbb{C}$ is a flat, projective morphism $\pi\colon \mathcal{C} \to S$, with n-sections $x_1, \dots, x_n : S \to \mathcal{C}$, such that the fiber $[C_s, \{x_1(s), \dots, x_n(s)\}]$ is a n-pointed, genus g, quasi-stable curve, for any geometric point $s \in S$.

Definition 2.3.2. Let X be a scheme over $\mathbb{C}$. A family of maps over S to X is a collection

$$(\pi\colon \mathcal{C} \to S, \{x_1, \dots, x_n\}, \alpha\colon \mathcal{C} \to X)$$

such that:

- $(\pi\colon \mathcal{C} \to S, \{x_1, ..., x_n\})$, is a family of n-pointed genus g quasi-stable curve parametrized by S.
- $\alpha\colon \mathcal{C} \to X$ is a morphism.

The families $(\pi\colon \mathcal{C} \to S, \{x_1, ..., x_n\}, \alpha)$ and $(\pi'\colon \mathcal{C}' \to S, \{x_1', ..., x_n'\}, \alpha')$ are isomorphic if there is a isomorphism of schemes $\phi\colon \mathcal{C} \to \mathcal{C}'$ such that $\pi = \pi' \circ \phi$, $x_i' = \phi \circ x_i$ for any $i = 1, ..., n$, and $\alpha = \alpha' \circ \phi$.
Let $(C, \{x_1, ..., x_n\}, \alpha)$ be a map from an n-pointed genus g curve to X, the *special points* of an irreducible component $E \subseteq C$ are the marked points of C on E and the points in $E \cap \overline{C \backslash E}$.

Definition 2.3.3. A map $(C, \{x_1, ..., x_n\}, \alpha)$ from an n-pointed genus g quasi-stable curve to X is stable if:

- any component $E \cong \mathbb{P}^1$ of C contracted by α contains at least three special points,
- any component $E \subseteq C$ of arithmetic genus 1 contracted by α contains at least one special point.

A family $(\pi\colon \mathcal{C} \to S, \{x_1, ..., x_n\}, \alpha)$ is stable if each geometric fiber is stable.

Remark 2.3.4. In the case $X = \mathbb{P}^N$ the map $(\pi\colon \mathcal{C} \to S, \{x_1, ..., x_n\}, \alpha)$ is stable if and only if $\omega_{\mathcal{C}/S}(x_1 + ... + x_n) \otimes \alpha^*(\mathcal{O}_{\mathbb{P}^N}(3))$ is π-ample.

Let X be a scheme over $\mathbb{C}$, and let $\beta \in A_1X$. To any scheme S over $\mathbb{C}$ we associate the set of isomorphism classes of stable families $(\pi\colon \mathcal{C} \to S, \{x_1, ..., x_n\}, \alpha\colon \mathcal{C} \to X)$ parametrized by S of n-pointed genus g curves to X such that $\alpha_*(C_s) = [\beta]$, where $[\beta]$ denotes the fundamental class of β. In this way we get a controvariant functor

$$\overline{\mathcal{M}}_{g,n}(X, \beta)\colon \mathfrak{Schemes} \to \mathfrak{Sets}.$$

By [FP97, Theorem 1], if X is a projective scheme over $\mathbb{C}$ then there exists a projective scheme $\overline{M}_{g,n}(X, \beta)$ coarsely representing the functor $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The spaces $\overline{M}_{g,n}(X, \beta)$ are called *moduli spaces of stable maps*, or *Kontsevich moduli spaces*.
Recall that a smooth variety X is said to be convex if $H^1(\mathbb{P}^1, \alpha^*T_X) = 0$ for any morphism $\alpha\colon \mathbb{P}^1 \to X$.

Remark 2.3.5. The tangent bundle of an homogeneous variety is generated by global section, so it is convex. On the other hand to be convex for an uniruled variety is a strong condition, as instance the blow-up of a convex variety is not convex.

Theorem 2.3.6 ([FP97, Theorem 2]). *Let X be a projective, nonsingular, convex variety, then $\overline{M}_{0,n}(X, \beta)$ is a normal, projective variety of pure dimension*

$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Furthermore $\overline{M}_{0,n}(X, \beta)$ is locally a quotient of a nonsingular variety by a finite group, that is $\overline{M}_{0,n}(X, \beta)$ has at most finite quotient singularities.

In the special case $X = \mathbb{P}^N$ we have $\beta \sim d[\text{line}]$ for some integer d and the scheme $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is irreducible.

Examples

In the following we give a list of examples in which moduli of stable maps have a clear geometric description.

- The moduli space of stable maps to a point is isomorphic to the moduli space of curves
$$\overline{M}_{g,n}(\mathbb{P}^0, 0) \cong \overline{M}_{g,n}.$$
For the space of degree zero stable maps we have:
$$\overline{M}_{g,n}(X, 0) \cong \overline{M}_{g,n} \times X.$$

- The moduli space of degree one maps to $\mathbb{P}^N$ is the Grassmannian
$$\overline{M}_{0,0}(\mathbb{P}^N, 1) \cong \mathbb{G}(1, N),$$
and similarly the moduli space of degree one maps to a smooth quadric hypersurface $Q \subset \mathbb{P}^N$, with $N \geqslant 3$, is the orthogonal Grassmannian
$$\overline{M}_{0,0}(Q, 1) \cong \mathbb{OG}(1, N).$$

- The Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics that is to the blow up of the $\mathbb{P}^5$ parametrizing conics in $\mathbb{P}^2$ along the Veronese surface $\mathcal{V}^2$ of double lines:
$$\overline{M}_{0,0}(\mathbb{P}^2, 2) \cong Bl_{\mathcal{V}^2}\mathbb{P}^5.$$

- Consider now $\overline{M}_{1,0}(\mathbb{P}^2, 3)$. Smooth plane cubic are parametrized by an open subset of $\mathbb{P}^9 = \mathbb{P}(k[x_0, x_1, x_2]_3)$. On the other hand we have maps from a reducible curve with a component of genus zero and a component of genus one, contracting the genus one component and of degree three on the genus zero component.
For any curve of genus one we have a 1-dimensional choice for the genus zero component, namely the connecting node. So we get a component of dimension 10 of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$. Finally we have curve with three component: an elliptic curve and two rational tails. The map contracts the elliptic curve and maps the rational tails to a line and a conic.
Here we have a 2-dimensional choice for the two nodes on the elliptic curve, a 2-dimensional choice for the line, and a 5-dimensional choice for the conic. We conclude that $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ has three irreducible component: two of dimension 9 and one of dimension 10.

- Let $X \subset \mathbb{P}^7$ be a smooth degree seven hypersurface containing a $\mathbb{P}^3$. Writing down an explicit equation for X one can see that $\overline{M}_{0,0}(X, 2)$ has two irreducible component: one component is 5-dimensional and cover X, the second component parametrizes conics in the $\mathbb{P}^3$ and so has dimension $5 + 3 = 8$.
Generalizing this construction one can show that $\overline{M}_{0,0}(X, 2)$ can have a component of dimension arbitrary larger that the dimension of the main component even if X is a Fano hypersurface in $\mathbb{P}^N$.

Natural maps

Kontsevich moduli spaces, as moduli spaces of curves, admits natural morphisms.

- Forgetful morphisms
$$\pi_I \colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n-|I|}(X,\beta),$$
forgetting the the points marked by $I = \{i_1, ..., i_j\} \subseteq \{1, \dots, n\}$.
- Evaluation morphisms
$$ev_i \colon \overline{M}_{g,n}(X,\beta) \to X,$$
mapping $(C, \{x_1, ..., x_n, \alpha\})$ to $\alpha(x_i)$.
- If $2g + n - 3 \geqslant 0$ we have morphisms forgetting the map α:
$$\rho \colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}.$$

2.3.1 *The stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$*

In this section we follow the clear and detailed discussion worked out by F. Poma in [Pom11]. The construction of the moduli of stable maps can be transposed into the realm of algebraic stacks. Let k be a field and consider the functor:

$$\mathcal{F} \colon \mathfrak{Schemes}_{/k} \to \mathfrak{Groupoids},$$

associating to a scheme S the groupoids $\mathcal{F}(S)$ of flat projective families $\pi \colon C \to S$ of nodal curves of genus g,

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\pi"] & X \\ S \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

where s_i are disjoint smooth sections of π, $\alpha_*[C_s] = \beta$ for any fiber $C_s = \pi^{-1}(s)$, and $\mathrm{Aut}(C, \alpha, \pi, s_i)$ is finite over S.

Theorem 2.3.7 ([AO01, Section 2.5])**.** *There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of finite type over k which represents $\mathcal{F}$.*

Theorem 2.3.8 ([Kon95, Section 1.3.1])**.** *If* $\mathrm{char}\, k = 0$*, then $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is of Deligne-Mumford type.*

Recall that a Dedekind domain D is an integral domain which is not a field, satisfying one of the following equivalent conditions:

- D is noetherian, and the localization at each maximal ideal is a Discrete Valuation Ring.
- D is an integrally closed, noetherian domain with Krull dimension one.
- Every nonzero proper ideal of D factors into primes ideals.

- Every fractional ideal of D is invertible.

Example 2.3.9. Let C be an affine smooth curve over a field k. The coordinate ring $A(C)$ of C is a finitely generated k-algebra, and so noetherian, and it has dimension one since C is a curve. Moreover, C is smooth and then normal, so $A(C)$ is integrally closed. In particular, $A(C)$ is a Dedekind domain.

Consider now the functor

$$\mathcal{F}_D \colon \mathfrak{Schemes}_{/D} \to \mathfrak{Groupoids},$$

exactly defined as $\mathcal{F}$ but from the category of schemes over a Dedekind domain D.

Theorem 2.3.10 ([AO01, Section 2.5]). *There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of finite type over D which represents $\mathcal{F}_D$.*

Remark 2.3.11. In the case $\operatorname{char} k = p$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not in general of Deligne-Mumford type, but it is a proper Artin stack.

For example, consider the element $(\mathbb{P}^1, \alpha) \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, p)$ given by

$$\begin{array}{rccc} \alpha\colon & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & [x_0, x_1] & \longmapsto & [x_0^p, x_1^p]. \end{array}$$

Then $\operatorname{Aut}(\mathbb{P}^1, \alpha) = \mu_p = \operatorname{Spec} k[\xi]/(\xi^p - 1) = \operatorname{Spec} k[\xi]/(\xi - 1)^p$, which is not reduced over $\operatorname{Spec} k$.

However, even in the characteristic p case the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a global quotient stack and the functor

$$\theta \colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$$

is representable.

This led A. Kresch to define an intersection theory for Artin stacks over a field in [Kre99].

2.3.2 *Virtual dimension of $\overline{M}_{g,n}(X,\beta)$*

If X is a homogeneous variety then it is smooth and its tangent bundle is generated by global sections, in particular X is convex. In this case, as seen in Theorem 2.3.6, $\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension. Furthermore if $X = \mathbb{P}^N$ then $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is irreducible. On the other hand, when $g \geqslant 1$, and even when $g = 0$ for most schemes $X \neq \mathbb{P}^N$, the space $\overline{M}_{g,n}(X,\beta)$ may have many components of dimension greater that the expected dimension. To overcome this gap and to give a rigorous definition of Gromov-Witten invariants we have to introduce the notions of virtual fundamental class and virtual dimension as in [BF97]. Recently, F. Poma in [Pom11] extended this construction of the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ also to schemes in positive and mixed characteristic and lead to a rigorous definition of Gromov-Witten invariants for these classes of schemes.

2.3.2.1 *The normal cone*

In this section we follow [BF97]. Let E be a rank r vector bundle on a smooth variety Y, $s \in H^0(E)$ a section, and $Z = Z(s) \subset Y$ the zero scheme of s. As s varies, Z can become reducible or even of non pure dimension.

Definition 2.3.12. Let $\mathcal{I}$ be the ideal sheaf of Z in Y, the *normal cone* of Z in Y is the affine cone over Z defined by:

$$C_Z Y = \mathrm{Spec}(\bigoplus_{k=0}^{\infty} \mathcal{I}^k/\mathcal{I}^{k+1}).$$

Note that the normal cone $C_Z Y$ has pure dimension $n = \dim Y$. Multiplication by s induces a surjective map:

$$\bigoplus_k Sym^k(\mathcal{O}(E^*/\mathcal{I}\mathcal{O}(E^*))) \to \bigoplus_k \mathcal{I}^k/\mathcal{I}^{k+1},$$

and applying Spec we get an embedding

$$C_Z Y \to E_{|Z}.$$

The normal cone gives a class $[C_Z Y] \in A_n(E_{|Z})$, so we have $s^*[C_Z Y] \in A_{n-r}(Z)$. Let $\mathcal{M}$ be a Deligne-Mumford stack. Since $\mathcal{M}$ admits an étale open cover by schemes we can consider a scheme U and take an embedding $U \hookrightarrow W$, where W is a smooth scheme. Now, consider the ideal sheaf $\mathcal{I}$ of U in W, and form the normal cone $C_U W$. The differentiation map:

$$\begin{array}{ccc} \bigoplus_k \mathcal{I}^k & \longrightarrow & \Omega^1_W \\ f & \longmapsto & df \end{array}$$

induces a map

$$\bigoplus_k \mathcal{I}^k/\mathcal{I}^{k+1} \to \bigoplus_k Sym^k(\Omega^1_W/\mathcal{I}\Omega^1_W),$$

and finally applying Spec we get a map

$$T_{W|U} = \mathrm{Spec}(\bigoplus_k Sym^k(\Omega^1_W/\mathcal{I}\Omega^1_W)) \to C_U W.$$

Definition 2.3.13. The *intrinsic normal cone* $\mathcal{C}_U$ is defined as the stack quotient $[C_U W/T_{W|U}]$.

Now, given an étale open cover $\{U_i\}$ of $\mathcal{M}$ the intrinsic normal cones C_{U_i} glue to give the intrinsic normal cone $\mathcal{C}_{\mathcal{M}}$ of $\mathcal{M}$.

Definition 2.3.14. If $L^\bullet_{\mathcal{M}}$ is the cotangent complex of $\mathcal{M}$, an *obstruction theory* for $\mathcal{M}$ is a complex of sheaves $\mathcal{E}^\bullet$ on $\mathcal{M}$ with a morphism $\mathcal{E}^\bullet \to L^\bullet_{\mathcal{M}}$, which is an isomorphism on h^0 and a surjection on h^{-1}.

Given an arbitrary complex $\mathcal{E}^\bullet$ we define $h^1/h^0(\mathcal{E}^\bullet)$ to be the quotient stack of the kernel of $\mathcal{E}^1 \to \mathcal{E}^2$ by the cokernel of $\mathcal{E}^{-1} \to \mathcal{E}^0$.

By the definition of perfect obstruction theory the intrinsic normal cone $\mathcal{C}_{\mathcal{M}}$ embeds in $h^1/h^0((\mathcal{E}^\bullet)^*)$.

Let C be the fiber product of $(E^{-1})^*$ with $\mathcal{C}_{\mathcal{M}}$ over $h^1/h^0((\mathcal{E}^\bullet)^*)$, where $\mathcal{O}(E^{-1}) = \mathcal{E}^{-1}$. Then C is a cone contained in the vector bundle $(E^{-1})^*$.

Definition 2.3.15. The *virtual fundamental class* is defined to be the intersection of C with the zero section of $(E^{-1})^*$.

In this part we mainly follow [Deb01] and [Pom11].

Let X be a smooth connected projective scheme, $\mathfrak{M}_{g,n}$ the Artin stack parametrizing pre-stable n-pointed genus g connected nodal curves, and C its universal curve.

Definition 2.3.16. We define an algebraic stack $Mor(C, X)$ as follows:

- for any scheme S objects in $Mor(C, X)(S)$ are pre-stable curves $(C_S \to S, s_i)$ over S with a morphism $f_S \colon C_S \to X$,
- for any scheme S a morphism from $(C_S \to S, s_i)$ to $(C'_S \to S, s'_i)$ is an isomorphism α of pre-stable curves such that $f'_S \circ \alpha = f_S$.

There is a natural functor $\theta \colon Mor(C, X) \to \mathfrak{M}_{g,n}$ forgetting the map to X and $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is an open substack of $Mor(C, X)$.

The fiber product $\overline{\mathcal{C}} \times_{\mathfrak{M}_{g,n}} Mor(C, X)$ is an universal family for $Mor(C, X)$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[rr, bend left, "\psi"] \arrow[d, "\pi"] & \overline{\mathcal{C}} \arrow[r, "\overline{\psi}"'] \arrow[d, "\overline{\pi}"] & X \\
\overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[r] \arrow[u, bend left, "s_i"] & Mor(C, X) \arrow[u, bend left, "\overline{s}_i"] &
\end{tikzcd}$$

where $\mathcal{C} = \overline{\mathcal{C}} \times_{Mor(C,X)} \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the universal stable map.

It turns out that considering the complex $F^\bullet = (R\overline{\pi}_* \overline{\psi}^* T_X)^*$ we get a vector bundle stack $h^1/h^0(F^\bullet)$. Similarly $E^\bullet = (R\pi_* \psi^* T_X)^*$ gives a perfect obstruction theory for θ, and so a virtual fundamental class for $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

In what follows we try to understand more concretely the tangent and the obstruction spaces to $Mor(Y, X)$, where X, Y are projective varieties over a field. The scheme $Mor(Y, X)$ parametrizing morphisms $Y \to X$ is a locally noetherian scheme having countably many components. However fixing an ample divisor H on X we can consider the scheme $Mor(P)(Y, X)$ parametrizing morphism $Y \to X$ with fixed Hilbert polynomial $P(m) = \chi(Y, mf^*H)$. This is a quasi-projective scheme.

The tangent space $T_{[f]}Mor(Y, X)$ in a point $[f] \in Mor(Y, X)$ parametrizes morphisms $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to Mor(Y, X)$, and hence $k[\epsilon]/(\epsilon^2)$-morphisms

$$f_\epsilon \colon Y \times \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X \times \operatorname{Spec} k[\epsilon]/(\epsilon^2),$$

which should be interpreted as first order deformations of f.

Proposition 2.3.17. *Let X, Y be projective varieties. The tangent space to $Mor(Y, X)$ in a point $[f]$ is given by*

$$T_{[f]}Mor(Y, X) = H^0(Y, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y)).$$

Proof. Assume $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ to be affine, where A, B are finitely generated k-algebras. Let $f^\sharp \colon A \to B$ be the morphism induced by f. We are looking for $k[\epsilon]/(\epsilon^2)$-algebras homomorphisms $f^\sharp_\epsilon \colon A[\epsilon] \to B[\epsilon]$ of the type $f^\sharp_\epsilon(a) = f^\sharp(a) + \epsilon g(a)$. Notice that since $f^\sharp_\epsilon(aa') = f^\sharp_\epsilon(a)f^\sharp_\epsilon(a')$ we get $\epsilon g(aa') = (f^\sharp(a) + \epsilon g(a))(f^\sharp(a') + \epsilon g(a')) - f^\sharp(a)f^\sharp(a') = \epsilon(f^\sharp(a)g(a') + f^\sharp(a')g(a))$. Then $f^\sharp_\epsilon(aa') = f^\sharp_\epsilon(a)f^\sharp_\epsilon(a')$ is equivalent to

$$g(aa') = f^\sharp(a)g(a') + f^\sharp(a')g(a),$$

that is $g \colon A \to B$ is a k-derivation of the A-module B and then it has to factorize as $g \colon A \to \Omega_A \to B$. Such extensions are therefore parametrized by $Hom_A(\Omega_A, B) = Hom_B(\Omega_A \otimes_A B, B)$.
In general cover X by open affine $U_i = \mathrm{Spec}(A_i)$ and Y by open affine $V_i = \mathrm{Spec}(B_i)$ such that $f(V_i) \subseteq U_i$. By the previous part of the proof first order deformations of $f_{|V_i}$ are parametrized by $h_i \in Hom_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y))$. To glue these together we need the compatibility condition $h_{i|V_{ij}} = h_{j|V_{ij}}$ which means that the collection $\{h_i\}$ defines a global section on Y. □

Notice that when X is smooth along the image of f we have

$$T_{[f]}Mor(Y, X) = H^0(Y, f^*T_X).$$

Furthermore, when Y is smooth, $H^0(Y, T_Y)$ is the tangent space to the automorphisms group of Y at the identity. Its element are called infinitesimal automorphisms. The images of the morphism $H^0(Y, T_Y) \to H^0(Y, f^*T_X)$ parametrizes deformation of f by reparametrizations.

Let $0 \mapsto I \to R \to R/I \mapsto 0$ be a semi-small extension in the category of local artinian k-algebras, that is $I \subseteq \mathfrak{M}$ and $I\mathfrak{M} = 0$, where $\mathfrak{M}$ is the maximal ideal of R. Let $f \colon Y \to X$ be a morphism. Assume as before X, Y affine. Since X is smooth along the image of f and $I^2 = 0$ by the infinitesimal lifting property [Har77, Chapter 2, Exercise 8.6], there exists a lifting of $f^\sharp_{R/I} \colon A \otimes_k R/I \to B \otimes_k R/I$ to a morphism $f^\sharp_R \colon A \otimes_k R \to B \otimes_k R$, and two different liftings differ by an R-derivation $A \otimes_k R \to B \otimes_k I$, that is by an element of $H^0(Y, f^*T_X) \otimes_k I$.

In the general case we need to glue two extensions h_i, h_j on each $V_i \cap V_j$. These two extension differs by an element $\nu_{ij} \in H^0(V_i \cap V_j, f^*T_X) \otimes_k I$. We have $\nu_{ij}h_{i|V_{ij}} = h_{j|V_{ij}}$. On the triple intersection $V_i \cap V_j \cap V_k$ we have $\nu_{jk}\nu_{ij}h_{i|V_{ijk}} = \nu_{jk}h_{j|V_{ijk}} = h_{k|V_{ijk}} = \nu_{ik}h_{i|V_{ijk}}$. So $\nu_{ik} = \nu_{jk}\nu_{ij}$ and the collection $\{\nu_{ij}\} \in C^1(\{V_i\}, f^*T_X \otimes_k I)$ is a cocycle. We have a global lifting if and only if $\nu_{ij} = 0$, and the obstruction space is $H^1(Y, f^*T_X) \otimes I$.

Locally around a point $[f] \in Mor(Y, X)$ the space $Mor(Y, X)$ can be defined by a set of polynomial $\{P_i\}$ is some affine space $\mathbb{A}^N$. The rank r of the Jacobian $J(P_i)$ is the codimension of the Zariski tangent space $T_{[f]}Mor(Y, X) \subseteq k^N$. Let V be a variety defined by r equations among the P_i for which the corresponding rows in the Jacobian have rank r, then V is smooth at $[f]$ and has the same Zariski tangent space of $Mor(Y, X)$. By Proposition 2.3.17 the variety V has dimension $h^0(Y, f^*T_X)$ in $[f]$. We want to show that in the regular local ring $R = \mathcal{O}_{V,[f]}$ the ideal I of regular functions vanishing on $Mor(Y, X)$ can be generated by $h^1(Y, f^*T_X)$ elements.

Since the Zariski tangent spaces are the same, the ideal I is contained in the square of the maximal ideal $\mathfrak{M}$ of R. Furthermore by Nakayama's lemma it is enough to show that the k-vector space $I/\mathfrak{M}I$ has dimension at most h^1.

The morphism $\mathrm{Spec}(R/I) \to Mor(Y,X)$ corresponds to an extension $f_{R/I}\colon Y \times \mathrm{Spec}(R/I) \to X \times \mathrm{Spec}(R/I)$ of f. We know that the obstruction to lift this extension to an extension $f_{R/\mathfrak{M}I}\colon Y \times \mathrm{Spec}(R/\mathfrak{M}I) \to X \times \mathrm{Spec}(R/\mathfrak{M}I)$ lies in

$$H^1(Y, f^*T_X) \otimes_k I/\mathfrak{M}I.$$

Let $\sum_{i=1}^{h_1} a_i \otimes \overline{b}_i$ be the obstruction, where $b_i \in I$. Since the obstruction vanishes modulo the ideal $(b_1, \dots, b_{h^1})$, the morphism $\mathrm{Spec}(R/I) \to Mor(Y,X)$ lifts to a morphism $\mathrm{Spec}(R/\mathfrak{M}I + (b_1, \dots, b_{h^1})) \to Mor(Y,X)$. In other words the identity $R/I \to R/I$ factors through the projection as $R/I \to R/\mathfrak{M}I + (b_1, \dots, b_{h^1}) \to R/I$. Then $I = \mathfrak{M}I + (b_1, \dots, b_{h^1})$, which means that $I/\mathfrak{M}I$ is generated by the classes of $b_1, \dots, b_{h^1}$.

Remark 2.3.18. Locally around $[f]$ the space $Mor(Y,X)$ can be defined by at most $h^1(Y, f^*T_X)$ equations in a smooth variety of dimension $h^0(Y, f^*T_X)$. In particular any irreducible component of $Mor(Y,X)$ through $[f]$ has dimension at least

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

The equations defining $Mor(Y,X)$ locally around $[f]$ can intersect badly so that the actual dimension is not the expected one. A naive way of understanding the deformation to the normal cone and the virtual fundamental class is to imagine a deformation of these equations that make the intersection transverse. If there is such a deformation, which formally means that exists a perfect obstruction theory, then the object we obtain would be a virtual fundamental class.

Spectral sequence of Ext functors

Let $\mathcal{E} \in \mathfrak{Coh}(X)$ be a coherent sheaf on a scheme X. Consider the functor:

$$\begin{aligned} \mathcal{H}om(\mathcal{E}, -)\colon \mathfrak{Coh}(X) &\longrightarrow \mathfrak{Coh}(X) \\ \mathcal{Q} &\longmapsto \mathcal{H}om(\mathcal{E}, \mathcal{Q}) \end{aligned}$$

and the global section functor:

$$\begin{aligned} \Gamma_X\colon \mathfrak{Coh}(X) &\longrightarrow \mathfrak{Ab} \\ \mathcal{Q} &\longmapsto \Gamma_X(\mathcal{Q}) \end{aligned}$$

Note that $\Gamma_X \circ \mathcal{H}om(\mathcal{E}, -) = Hom(\mathcal{E}, -)$. By Grothendieck spectral sequence we have $(R^h\Gamma_X \circ R^k\mathcal{H}om(\mathcal{E}, -))(\mathcal{Q}) \Longrightarrow R^{h+k}(Hom(\mathcal{E}, -)(\mathcal{Q})$ for any $\mathcal{Q} \in \mathfrak{Coh}(X)$, that is

$$H^h(X, \mathcal{E}xt^k(\mathcal{E}, \mathcal{Q})) \Longrightarrow Ext^{h+k}(\mathcal{E}, \mathcal{Q}).$$

The corresponding sequence of low degrees is:

$$\begin{aligned} 0 \to H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^1(\mathcal{E}, \mathcal{Q}) \to \\ \to H^0(X, \mathcal{E}xt^1(\mathcal{E}, \mathcal{Q})) \to H^2(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^2(\mathcal{E}, \mathcal{Q}) \end{aligned} \tag{2.3}$$

Theorem 2.3.20. *Let X be a smooth projective variety. The virtual dimension of the moduli space $\overline{M}_{g,n}(X,\beta)$ is given by*

$$\mathrm{virdim}(\overline{M}_{g,n}(X,\beta)) = (1-g)(\dim(X)-3) - \int_{\beta} \omega_X + n.$$

Proof. Take the stable map $(C,\{x_1,...,x_n\},\alpha\}) \in \overline{M}_{g,n}(X,\beta)$. Let consider the space of first order deformations of $(C,\{x_1,...,x_n\},\alpha\})$: $Def(C,\{x_1,...,x_n\},\alpha\})$, and the space of first order deformations with C held rigid: $Def_\alpha(C,\{x_1,...,x_n\},\alpha\})$. There is an exact sequence

$$0 \mapsto Def(C,\{x_1,...,x_n\}) \to Def(C,\{x_1,...,x_n\},\alpha\}) \to Def_\alpha(C,\{x_1,...,x_n\},\alpha\}) \mapsto 0.$$

Note that since $(C,\{x_1,...,x_n\},\alpha\})$ is stable it does not have infinitesimal automorphisms, and this gives the injectivity of the map on the left.

- First we compute the dimension of $Def(C,\{x_1,...,x_n\})$. The curve C is a stable nodal curve. Since there is no H^2 on a curve, the sequence (2.3) in this case gives:

  $$0 \mapsto H^1(C,\mathcal{H}om(\Omega_C,\mathcal{O}_C)) \to Ext^1(\Omega_C,\mathcal{O}_C) \to H^0(C,\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C)) \mapsto 0.$$

  We denote by δ the number of nodes in C. Since the sheaf Ω_C is locally free on the smooth locus of C, the sheaf $\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C))$ is just k at each node, then $\dim(H^0(C,\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C))) = \delta$. The curve C is l.c.i, then the dualizing sheaf ω_C is an invertible sheaf, and since $\omega_C \cong \Omega_C$ on the open set of regular points, we have an injective morphism $\check{\omega_C} \to \mathcal{H}om(\Omega_C,\mathcal{O}_C)$, and an exact sequence

  $$0 \mapsto \check{\omega_C} \to \mathcal{H}om(\Omega_C,\mathcal{O}_C) \to \mathcal{O}_Z \mapsto 0,$$

  where $Z = Sing(C)$. Since C is stable $h^0(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) = 0$, by the cohomology exact sequence we get $h^0(\check{\omega_C}) = 0$, and

  $$0 \mapsto H^0(C,\mathcal{O}_Z) \to H^1(C,\check{\omega_C}) \to H^1(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) \mapsto 0.$$

  By Riemann-Roch for singular curves we get $h^1(\check{\omega_C}) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) = 3g-3-\delta$. Finally

  $$\dim(Ext^1(\Omega_C,\mathcal{O}_C)) = h^1(T_C) + h^0(\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C)) = 3g-3-\delta+\delta = 3g-3.$$

  and

  $$\dim Def(C,\{x_1,...,x_n\}) = 3g-3+n.$$

- By Remark 2.3.18, the expected dimension of $Def_\alpha(C,\{x_1,...,x_n\},\alpha\})$ is given by $h^0(\alpha^*T_X) - h^1(\alpha^*T_C)$. By Riemann-Roch theorem we get:

  $$\begin{aligned} \mathrm{expdim}\, Def_\alpha(C,\{x_1,...,x_n\},\alpha\}) &= h^0(\alpha^*T_X) - h^1(\alpha^*T_C) \\ &= \chi(\alpha^*T_C) \\ &= -K_X \cdot \alpha_* C + (1-g)\dim(X). \end{aligned}$$

We conclude that

$$\operatorname{expdim} Def(C, \{x_1, ..., x_n\}, \alpha\}) \geqslant -K_X \cdot \alpha_* C + (1-g)\dim(X) + 3g - 3 + n,$$

and the virtual dimension of $\overline{M}_{g,n}(X, \beta)$ is given by

$$\begin{aligned} \operatorname{virdim}(\overline{M}_{g,n}(X,\beta)) &= -K_X \cdot \alpha_* C + (1-g)\dim(X) + 3g - 3 + n \\ &= (1-g)(\dim(X) - 3) - \int_\beta \omega_X + n. \end{aligned}$$

$\square$

3 SPACES OF COMPLETE FORMS

3.1 complete rank h collineations

Let V, W be K-vector spaces of dimension respectively $n+1$ and $m+1$ with $n \leqslant m$, and let $\mathbb{P}^N$ with $N = nm + n + m$ be the projective space parametrizing collineations from V to W that is non-zero linear maps $V \to W$ up to a scalar multiple.

The line bundle $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(1,1) = \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(W)}(1)$ induces an embedding

$$\begin{array}{rccc} \sigma\colon & \mathbb{P}(V) \times \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V \otimes W) = \mathbb{P}^N \\ & ([u],[v]) & \longmapsto & [u \otimes v] \end{array}$$

The image $\mathcal{S}^{n,m} = \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$ is the *Segre variety*. Let $[x_0, \dots, x_n], [y_0, \dots, y_m]$ be homogeneous coordinates respectively on $\mathbb{P}^n$ and $\mathbb{P}^m$. Then the morphism σ can be written as

$$\sigma([x_0, \dots, x_n], [y_0, \dots, y_m]) = [x_0 y_0 : \dots : x_0 y_m : x_1 y_0 : \dots : x_n y_m].$$

We will denote by $[z_{0,0} : \dots : z_{n,m}]$ the homogeneous coordinates on $\mathbb{P}^N$, where $z_{i,j}$ corresponds to the product $x_i y_j$.

A point $p \in \mathbb{P}^N = \mathbb{P}(\mathrm{Hom}(W,V))$ can be represented by an $(n+1) \times (m+1)$ matrix Z. The Segre variety $\mathcal{S}^{n,m}$ is the locus of rank one matrices. More generally, $p \in \mathbb{S}ec_h(\mathcal{S}^{n,m})$ if and only if Z can be written as a linear combination of h rank one matrices that is if and only if $\mathrm{rank}(Z) \leqslant h$. If $p = [z_{0,0} : \dots : z_{n,m}]$ then we may write

$$Z = \begin{pmatrix} z_{0,0} & \dots & z_{0,m} \\ \vdots & \ddots & \vdots \\ z_{n,0} & \dots & z_{n,m} \end{pmatrix}. \tag{3.1}$$

Therefore, the ideal of $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ is generated by the $(h+1) \times (h+1)$ minors of Z.

By [Mas20a, Lemma 3.3] the $(SL(n+1) \times SL(m+1))$-action

$$\begin{array}{ccc} (SL(n+1) \times SL(m+1)) \times (\mathbb{P}^n \times \mathbb{P}^m) & \longrightarrow & \mathbb{P}^n \times \mathbb{P}^m \\ ((A,B),([v],[w])) & \longmapsto & ([Av],[Bw]) \end{array}$$

induces the $(SL(n+1) \times SL(m+1))$-action on $\mathbb{P}^N$ given by

$$\begin{array}{ccc} (SL(n+1) \times SL(m+1)) \times \mathbb{P}^N & \longrightarrow & \mathbb{P}^N \\ ((A,B),Z) & \longmapsto & AZB^t \end{array} \tag{3.2}$$

In order to get a wonderful compactification we must consider the space of complete collineations.

Definition 3.1.2. The space of *complete collineations* from V to W is the closure of the graph of the rational map

$$\begin{array}{ccc} \mathbb{P}(\mathrm{Hom}(V,W)) & \dashrightarrow & \mathbb{P}(\bigwedge^2 W, \bigwedge^2 V) \times \cdots \times \mathbb{P}(\bigwedge^{n+1} W, \bigwedge^{n+1} V) \\ Z & \longmapsto & (\wedge^2 Z, \dots, \wedge^{n+1} Z) \end{array}$$

By [Vai84, Theorem 1] the space of complete collineations can be constructed as a sequence of blow-ups as follows:

Construction 3.1.3. Let us consider the following sequence of blow-ups:

- $\mathcal{X}(n,m)_1$ is the blow-up of $\mathcal{X}(n,m)_0 := \mathbb{P}^N$ along the Segre variety $\mathcal{S}^{n,m}$
- $\mathcal{X}(n,m)_2$ is the blow-up of $\mathcal{X}(n,m)_1$ along the strict transform of $\mathcal{S}ec_2(\mathcal{S}^{n,m})$

  $\vdots$
- $\mathcal{X}(n,m)_i$ is the blow-up of $\mathcal{X}(n,m)_{i-1}$ along the strict transform of $\mathcal{S}ec_i(\mathcal{S}^{n,m})$

  $\vdots$
- $\mathcal{X}(n,m)_n$ is the blow-up of $\mathcal{X}(n,m)_{n-1}$ along the strict transform of $\mathcal{S}ec_n(\mathcal{S}^{n,m})$

Let $g_i\colon \mathcal{X}(n,m)_i \to \mathcal{X}(n,m)_{i-1}$ be the blow-up morphism. We will denote by E_i^c both the exceptional divisor of g_i and its strict transforms in the subsequent blow-ups. We will denote by $\mathcal{X}(n,m)$ the last blow-up $\mathcal{X}(n,m)_n$ and by $g\colon \mathcal{X}(n,m) \to \mathbb{P}^N$ the composition of the g_i.

Then for any $i = 1, \dots, n$ the variety $\mathcal{X}(n,m)_i$ is smooth, the strict transform of $\mathcal{S}ec_{i+1}(\mathcal{S}^{n,m})$ in $\mathcal{X}(n,m)_i$ is smooth, and the divisor $E_1^c \cup E_2^c \cup \cdots \cup E_i^c$ in $\mathcal{X}(n,m)_i$ is simple normal crossing. Furthermore, the variety $\mathcal{X}(n,m)$ is isomorphic to the space of complete collineations from V to W.

For $n = m$, we denote by $\mathbb{P}^{N_+} \subset \mathbb{P}^N$ the subspace of symmetric matrices. Then $\mathcal{S}ec_h(\mathcal{S}^{n,m}) \cap \mathbb{P}^{N_+} = \mathcal{S}ec_h(\mathcal{V}^n)$ for any $h \geqslant 1$, where $\mathcal{V}^n \subset \mathbb{P}^{N_+}$ is the degree two Veronese embedding of $\mathbb{P}^n$.

By [Mas20a, Lemma 3.3] the $SL(n+1)$-action

$$\begin{array}{ccc} SL(n+1) \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ (M,[v]) & \longmapsto & [Mv] \end{array}$$

induces the $SL(n+1)$-action on $\mathbb{P}^{N_+}$ given by

$$\begin{array}{ccc} SL(n+1) \times \mathbb{P}^{N_+} & \longrightarrow & \mathbb{P}^{N_+} \\ (M,Z) & \longmapsto & MZM^t \end{array} \tag{3.3}$$

The orbit closures of the action (3.3) are precisely the secant varieties $\mathcal{S}ec_h(\mathcal{V}^n)$. Note that $\mathbb{P}^{N_+}$ is not a wonderful compactification of $SL(n+1)/H$, where H is the stabilizer of the identity matrix with respect to the $SL(n+1)$-action in (3.3), since for instance the orbit closure $\mathcal{S}ec_n(\mathcal{V}^n)$ is a non smooth divisor. In order to get a wonderful compactification we must consider the space of complete quadrics.

Definition 3.1.5. The space of *complete quadrics* is the closure of the graph of the rational map

$$\begin{array}{ccc} \mathbb{P}(\mathrm{Sym}^2 V) & \dashrightarrow & \mathbb{P}(\mathrm{Sym}^2 \bigwedge^2 V) \times \cdots \times \mathbb{P}(\mathrm{Sym}^2 \bigwedge^n V) \\ Z & \longmapsto & (\wedge^2 Z, \dots, \wedge^n Z) \end{array}$$

By restricting Construction 3.1.3 to $\mathbb{P}^{N_+}$, we get the construction of the space of complete quadrics as a sequence of blow-ups as in [Vai82, Theorem 6.3]:

Construction 3.1.6. Let us consider the following sequence of blow-ups:

- $\mathcal{Q}(n)_1$ is the blow-up of $\mathcal{Q}(n)_0 := \mathbb{P}^{N_+}$ along the Veronese variety $\mathcal{V}^n$;
- $\mathcal{Q}(n)_2$ is the blow-up of $\mathcal{Q}(n)_1$ along the strict transform of $\mathcal{S}ec_2(\mathcal{V}^n)$;

  $\vdots$
- $\mathcal{Q}(n)_i$ is the blow-up of $\mathcal{Q}(n)_{i-1}$ along the strict transform of $\mathcal{S}ec_i(\mathcal{V}^n)$;

  $\vdots$
- $\mathcal{Q}(n)_{n-1}$ is the blow-up of $\mathcal{Q}(n)_{n-2}$ along the strict transform of $\mathcal{S}ec_{n-1}(\mathcal{V}^n)$.

Let $f_i \colon \mathcal{Q}(n)_i \to \mathcal{Q}(n)_{i-1}$ be the blow-up morphism. We will denote by E_i^q both the exceptional divisor of f_i and its strict transforms in the subsequent blow-ups. We will denote by $\mathcal{Q}(n)$ the last blow-up $\mathcal{Q}(n)_{n-1}$ and by $f \colon \mathcal{Q}(n) \to \mathbb{P}^{N_+}$ the composition of the f_i.

Then for any $i = 1, \dots, n-1$ the variety $\mathcal{Q}(n)_i$ is smooth, the strict transform of $\mathcal{S}ec_{i+1}(\mathcal{V}^n)$ in $\mathcal{Q}(n)_i$ is smooth, and the divisor $E_1^q \cup E_2^q \cup \cdots \cup E_i^q$ in $\mathcal{Q}(n)_i$ is simple normal crossing. Furthermore, the variety $\mathcal{Q}(n)$ is isomorphic to the space of complete $(n-1)$-dimensional quadrics.

In the following we will analyse the geometry of the $SL(n+1)$-orbits in the blow-ups $\mathcal{Q}(n)_i$ in Construction 3.1.6. This gives a different way of proving that the space $\mathcal{Q}(n)$ is wonderful and introduces the techniques we will use in all the chapter to analyse the orbits of an action in a blow up variety.

For our propose, we need some facts about the varieties $\mathcal{S}ec_k(\mathcal{V}^n)$.

Notation 3.1.7. We will denote by $\mathcal{S}ec_h(\mathcal{V}^n)^i$ the strict transform of $\mathcal{S}ec_h(\mathcal{V}^n)$ in $\mathcal{Q}(n)_i$ for $h > i$. Furthermore, as already said in Construction 3.1.6, for simplicity of notation we will denote by E_i^q both the exceptional divisor of f_i and its strict transforms in the subsequent blow-ups.

Remark 3.1.8. Recall that $\mathcal{S}ec_h(\mathcal{V}^n)$ identifies with the variety parametrizing $(n+1) \times (n+1)$ symmetric matrices modulo scalar of rank at most h. An argument similar to the one used to estimate the dimension of the spaces of matrices, not necessarily symmetric, of rank at most h in [Har95, Example 12.1] shows that

$$\dim(\mathcal{S}ec_h(\mathcal{V}^n)) = \frac{2nh - h^2 + 3h - 2}{2}$$

for $h \leqslant n$. Furthermore, identifying $\mathcal{S}ec_h(\mathcal{V}^n)$ with the variety parametrizing $(n+1) \times (n+1)$ symmetric matrices modulo scalar of corank at least $n+1-h$, by [HT84, Proposition 12(b)] we get that the degree of $\mathcal{S}ec_h(\mathcal{V}^n)$ is given by

$$\deg(\mathcal{S}ec_h(\mathcal{V}^n)) = \prod_{i=0}^{n-h} \frac{\binom{n+1+i}{n+1-h-i}}{\binom{2i+1}{i}}.$$

In particular, for $h = n$ we get $n+1$, and for $h = 1$ we get 2^n.

Proposition 3.1.9. *The tangent cone of $\mathcal{S}ec_h(\mathcal{V}^n)$ at a point $p \in \mathcal{S}ec_k(\mathcal{V}^n) \backslash \mathcal{S}ec_{k-1}(\mathcal{V}^n)$ for $k \leqslant h$ is a cone with vertex of dimension $\binom{n+2}{2} - 1 - \frac{(n-k+1)(n-k+2)}{2}$ over $\mathcal{S}ec_{h-k}(\mathcal{V}_2^{n-k})$. In particular, for $k < h$ we have*

$$\operatorname{mult}_{\mathcal{S}ec_k(\mathcal{V}^n)\backslash \mathcal{S}ec_{k-1}(\mathcal{V}^n)} \mathcal{S}ec_h(\mathcal{V}^n) = \prod_{i=0}^{n-h} \frac{\binom{n-k+1+i}{n+1-h-i}}{\binom{2i+1}{i}}$$

and $\operatorname{Sing}(\mathcal{S}ec_h(\mathcal{V}^n)) = \mathcal{S}ec_{h-1}(\mathcal{V}^n)$.

Proof. We compute the tangent cone of $\mathcal{S}ec_h(\mathcal{V}^n)$ at

$$p_k = \begin{pmatrix} I_{k,k} & 0_{k,n+1-k} \\ 0_{n+1-k,k} & 0_{n+1-k,n+1-k} \end{pmatrix}$$

where $I_{k,k}$ is the $k \times k$ identity matrix. Consider the affine chart $z_{0,0} \neq 0$ and the change of coordinates $z_{i,i} \mapsto z_{i,i} - 1$ for $i = 1, \dots, k-1$, $z_{i,j} \mapsto z_{i,j}$ if $i \neq j$. Then the matrix Z in (3.1) takes the following form

$$\begin{pmatrix} 1 & z_{0,1} & \dots & z_{0,k-1} & z_{0,k} & \dots & z_{0,n} \\ z_{0,1} & z_{1,1}-1 & \dots & z_{1,k-1} & z_{1,k} & \dots & z_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{0,k-1} & z_{1,k-1} & \dots & z_{k-1,k-1}-1 & z_{k-1,k} & \dots & z_{k-1,n} \\ z_{0,k} & z_{1,k} & \dots & z_{k-1,k} & z_{k,k} & \dots & z_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{0,n} & z_{1,n} & \dots & z_{k-1,n} & z_{k,n} & \dots & z_{n,n} \end{pmatrix}$$

Recall that $\mathcal{S}ec_h(\mathcal{V}^n) \subseteq \mathbb{P}^{N_+}$ is cut out by the $(h+1) \times (h+1)$ minors of Z. Now, the lowest degree terms of these minors are given by the $(h+1-k) \times (h+1-k)$ minors of the following matrix

$$\begin{pmatrix} z_{k,k} & \dots & z_{k,n} \\ \vdots & \ddots & \vdots \\ z_{k,n} & \dots & z_{n,n} \end{pmatrix}$$

So, the tangent cone $TC_{p_k}\mathcal{S}ec_h(\mathcal{V}^n)$ is contained in the cone C over $\mathcal{S}ec_{h-k}(\mathcal{V}^{n-k})$ with vertex the linear subspace of $\mathbb{P}^{N_+}$ given by $\{z_{k,k} = \dots = z_{k,n} = z_{k+1,k+1} = \dots = z_{k+1,n} = \dots = z_{n,n} = 0\}$. Now, Remark 3.1.8 yields

$$\begin{aligned} \dim(C) &= \binom{n+2}{2} - 1 - \frac{(n-k+1)(n-k+2)}{2} + \dim(\mathcal{S}ec_{h-k}(\mathcal{V}_2^{n-k})) + 1 \\ &= \dim(\mathcal{S}ec_h(\mathcal{V}^n)) \end{aligned}$$

and hence $TC_{p_k}\mathcal{S}ec_h(\mathcal{V}^n) = C$. Finally, to get the formula for the multiplicity it is enough to observe that

$$\mathrm{mult}_{p_k}\mathcal{S}ec_h(\mathcal{V}^n) = \mathrm{mult}_{p_k} TC_{p_k}\mathcal{S}ec_h(\mathcal{V}^n) = \deg(\mathcal{S}ec_{h-k}(\mathcal{V}^{n-k}))$$

and to apply the formula for the degree of the secant varieties of $\mathcal{V}^n$ in Remark 3.1.8. □

Moreover, we will need the following result on fibrations with smooth fibers on a smooth base.

Proposition 3.1.10. *Let $f\colon X \to Y$ be a surjective morphism of varieties over an algebraically closed field with equidimensional smooth fibers. If Y is smooth then X is smooth as well.*

Proof. By [Sch10, Theorem 3.3.27] the morphism $f\colon X \to Y$ is flat. Finally, since all the fibers of $f\colon X \to Y$ are smooth and of the same dimension [Mum99, Theorem 3′, Chapter III, Section 10] yields that X is smooth.

However, a direct proof is at hand and we present it in what follows. Since the problem is local on both X and Y we may assume that $X \subset K^N$ is an affine variety cut out by polynomials $g_1,\dots,g_a$, $Y = K^m$, and $f\colon X \to Y$ is given by $f(x) = (f_1(x),\dots,f_m(x))$.

Consider a point $p \in X$. Without loss of generality we may assume that $f(p) = 0$. Then the fiber X_0 of f through p is given by

$$f^{-1}(0) = \{x \in K^N \mid g_1(x) = \cdots = g_a(x) = f_1(x) = \cdots = f_m(x) = 0\}.$$

Now, since X_0 is smooth at p there are $b \leqslant a$ polynomials among $g_1,\dots,g_a$ and $l \leqslant m$ polynomials among $f_1,\dots,f_m$ such that $b + l = m + N - \dim(X)$ and the vectors

$$(\nabla g_1)(p),\dots,(\nabla g_b)(p),(\nabla f_1)(p),\dots,(\nabla f_l)(p)$$

are linearly independent. Now, $l \leqslant m$ yields $b \geqslant N - \dim(X)$. On the other hand, X is irreducible of codimension $N - \dim(X)$ and hence $b \leqslant N - \dim(X)$. We conclude that $b = N - \dim(X)$ and the vectors

$$(\nabla g_1)(p),\dots,(\nabla g_{N-\dim(X)})(p)$$

are linearly independent. So X is smooth at p. □

Proposition 3.1.11. *For any $i = 0,\dots,n-1$ the variety $\mathcal{Q}(n)_i$ is smooth and the divisors $E_1^q,\dots,E_i^q$ are smooth and intersect transversally in $\mathcal{Q}(n)_i$. Furthermore, the strict transform $\mathcal{S}ec_{i+1}(\mathcal{V}^n)^i$ of $\mathcal{S}ec_{i+1}(\mathcal{V}^n)$ in $\mathcal{Q}(n)_i$ is smooth and the intersections among $\mathcal{S}ec_{i+1}(\mathcal{V}^n)^i, E_1^q,\dots,E_i^q$ are transversal. The closures of the orbits of the $SL(n+1)$-action on $\mathcal{Q}(n)_i$ induced by the action in (3.3) are given by all the possible intersections of $E_1^q,\dots,E_i^q$,$\mathcal{S}ec_{i+1}(\mathcal{V}^n)^i,\dots,\mathcal{S}ec_n(\mathcal{V}^n)^i$ and $\mathcal{Q}(n)_i$ itself.*

In particular, the variety $\mathcal{Q}(n)$ is smooth, the divisors $E_1^q,\dots,E_{n-1}^q,\mathcal{S}ec_n(\mathcal{V}^n)^{n-1}$ are smooth and the intersections among them are transversal, the closures of the orbits of the $SL(n+1)$-action on $\mathcal{Q}(n)$ induced by (3.3) are given by all the possible intersections of the divisors $E_1^q,\dots,E_{n-1}^q,\mathcal{S}ec_n(\mathcal{V}^n)^{n-1}$ and $\mathcal{Q}(n)$ itself. Hence, $\mathcal{Q}(n)$ is wonderful.

Proof. We will proceed as follows. For $i = 0, 1$ we will prove the statement for any n. Then we will prove that if for $i < j$ the statement holds for any n then it also holds for $i = j$ and any n. This will prove the statement for any $n \geqslant 1$ and $i = 0, \dots, n-1$.

For $i = 0$ we have $\mathcal{Q}(n)_0 \cong \mathbb{P}^{N_+}$, there are no exceptional divisors, and the closures of the orbits of the action (3.3) are the secant varieties of $\mathcal{V}^n$. Therefore, for $i = 0$ the statements holds for any n. Even though we could use the case $i = 0$ as the first step of the proof, to get acquainted with the arguments we will apply, we develop in full detail the case $i = 1$ as well.

The variety $\mathcal{Q}(n)_1$ is the blow-up of $\mathbb{P}^{N_+}$ along the Veronese variety $\mathcal{V}^n$. Hence it is smooth. By Proposition 3.1.9 $\mathcal{S}ec_2(\mathcal{V}^n)$ is smooth away from $\mathcal{V}^n$ and $\mathcal{S}ec_2(\mathcal{V}^n)^1 \cap E_1^q \to \mathcal{V}^n$ is a fibration whose fibers are isomorphic to $\mathcal{V}^{n-1}$. Hence, Proposition 3.1.10 yields that $\mathcal{S}ec_2(\mathcal{V}^n)^1 \cap E_1^q$ is smooth and since $\dim(\mathcal{S}ec_2(\mathcal{V}^n)^1 \cap E_1^q) = n + n - 1 = 2n - 1 = \dim(\mathcal{S}ec_2(\mathcal{V}^n)^1) - 1$ we conclude that $\mathcal{S}ec_2(\mathcal{V}^n)^1$ is smooth and the intersection $\mathcal{S}ec_2(\mathcal{V}^n)^1 \cap E_1^q$ is transversal.

Now, via the action of $SL(n+1)$ in (3.3) we can translate any fiber of E_1^q over $\mathcal{V}^n$ to any other fiber. Fix one such fiber $E_{1,p}^q$. By Proposition 3.1.9 we have that $\mathcal{S}ec_h(\mathcal{V}^n)^1 \cap E_{1,p}^q = \mathcal{S}ec_{h-1}(\mathcal{V}^{n-1})$ and the action of $SL(n+1)$ in (3.3) restricts on $E_{1,p}^q$ to the corresponding action of $SL(n)$. This proves the statement about the orbits for $\mathcal{Q}(n)_1$ for any $n \geqslant 1$.

Assume that for any $i < j$ the statement holds for any n. Since $\mathcal{Q}(n)_{j-1}$ and $\mathcal{S}ec_j(\mathcal{V}^n)^{j-1} \subset \mathcal{Q}(n)_{j-1}$ are smooth the blow-up $\mathcal{Q}(n)_j$ of $\mathcal{Q}(n)_{j-1}$ along $\mathcal{S}ec_j(\mathcal{V}^n)^{j-1}$ is smooth as well. Furthermore, since all the intersections among $\mathcal{S}ec_j(\mathcal{V}^n)^{j-1}, E_1^q, \dots, E_{j-1}^q$ in $\mathcal{Q}(n)_{j-1}$ are transversal we have that all the intersections among $E_1^q, \dots, E_j^q$ in $\mathcal{Q}(n)_j$ are transversal as well.

Now, consider an intersection of the form $\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E_{j_1}^q \cap \dots \cap E_{j_t}^q$. By Proposition 3.1.9 the restriction of the blow-down morphism

$$\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E_{j_1}^q \cap \dots \cap E_{j_t}^q \to E_{j_1}^q \cap \dots \cap E_{j_{t-1}}^q \cap \mathcal{S}ec_{j_t}(\mathcal{V}^n)^{j_t-1}$$

has fibers isomorphic to $\mathcal{S}ec_{j-j_t+1}(\mathcal{V}^{n-j_t})^{j-j_t}$. Since both $\mathcal{S}ec_{j-j_t-1}(\mathcal{V}^{n-j_t})^{j-j_t}$ and $E_{j_1}^q \cap \dots \cap E_{j_{t-1}}^q \cap \mathcal{S}ec_{j_t}(\mathcal{V}^n)^{j_t-1}$ are smooth Proposition 3.1.10 yields that $\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E_{j_1}^q \cap \dots \cap E_{j_t}^q$ is smooth as well. Moreover, note that

$$\begin{aligned}\dim(\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E_{j_1}^q \cap \dots \cap E_{j_t}^q) = {} & \dim(E_{j_1}^q \cap \dots \cap E_{j_{t-1}}^q \cap \mathcal{S}ec_{j_t}(\mathcal{V}^n)^{j_t-1}) \\ & + \dim(\mathcal{S}ec_{j-j_t+1}(\mathcal{V}^{n-j_t})^{j-j_t})\end{aligned}$$

and

$$\dim(E_{j_1}^q \cap \dots \cap E_{j_{t-1}}^q \cap \mathcal{S}ec_{j_t}(\mathcal{V}^n)^{j_t-1}) = \frac{2nj_t - j_t^2 + 3j_t - 2}{2} - (t-1).$$

So, $\dim(\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E_{j_1}^q \cap \dots \cap E_{j_t}^q)$ is given by:

$$\begin{aligned}&\frac{2nj_t - j_t^2 + 3j_t - 2}{2} - (t-1) + \frac{2(n-j_t)(j-j_t+1) - (j-j_t+1)^2 + 3(j-j_t+1) - 2}{2} \\ &= \frac{2n(j+1) - (j+1)^2 + 3(j+1) - 2}{2} - t = \dim(\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j) - t\end{aligned}$$

and hence the intersection $\mathcal{S}ec_{j+1}(\mathcal{V}^n)^j \cap E^q_{j_1} \cap \cdots \cap E^q_{j_t}$ is transversal.

In the following we prove the claim about the orbit closures. If an orbit closure in $\mathcal{Q}(n)_i$ is not contained in the exceptional divisor E^q_i then it is the strict transform of an orbit closure in $\mathcal{Q}(n)_i$, and hence it is given as an intersection among $E^q_1, \dots, E^q_{i-1}, \mathcal{S}ec_{i+1}(\mathcal{V}^n)^i, \dots, \mathcal{S}ec_n(\mathcal{V}^n)^i$.

Now, let us analyze the orbit closures in the exceptional divisor E^q_i. The fibers of E^q_i over $\mathcal{S}ec_i(\mathcal{V}^n)^{i-1}$ are projective spaces of dimension

$$N_{n-i} = \binom{n-i+2}{2} - 1.$$

Moreover, $SL(n+1)$ acts transitively on fibers that lie over the same orbit in $\mathcal{Q}(n)_{i-1}$. Note that by Lemma 3.1.9 $\mathcal{S}ec_h(\mathcal{V}^n)^i$ intersects each of these N_{n-i}-dimensional projective spaces along $\mathcal{S}ec_{h-i}(\mathcal{V}^{n-i})$, and the $SL(n+1)$-action on $\mathcal{Q}(n)_i$ in (3.3) induces the corresponding $SL(n-i+1)$-action on the fibers of E^q_i. Finally, the statement on the orbit closures in $\mathcal{Q}(n)_{i-1}$ follows then from the statement on the orbit closures in $\mathcal{Q}(n-i)_0$. □

3.1.1 *Complete singular forms*

Definition 3.1.12. The space of *complete rank h collineations* is the variety $\mathcal{C}(n,m,h)$ obtained by blowing-up $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ along the strict transforms of the secant varieties $\mathcal{S}ec_k(\mathcal{S}^{n,m})$ for $k < h$ in order of increasing dimension. When $n = m$ we will denote $\mathcal{C}(n,n,h)$ simply by $\mathcal{C}(n,h)$. Furthermore, we will denote by $E^{\mathcal{C}}_1, \dots, E^{\mathcal{C}}_{h-1}$ the exceptional divisors.

Similarly, for $n = m$ the space of *complete rank h quadrics* is the variety $\mathcal{Q}(n,h)$ obtained by blowing-up $\mathcal{S}ec_h(\mathcal{V}^n)$ along the strict transforms of the secant varieties $\mathcal{S}ec_k(\mathcal{V}^n)$ for $k < h$ in order of increasing dimension. We will denote by $E^{\mathcal{Q}}_1, \dots, E^{\mathcal{Q}}_{h-1}$ its exceptional divisors.

Remark 3.1.13. The case $\mathcal{C}(n,m,n+1)$ and $\mathcal{Q}(n,n+1)$ are respectively the space of complete collineations from V to W in Construction 3.1.3 and the space of complete quadrics of V in Construction 3.1.6. By [Vai84, Theorem 1] and [Vai82, Theorem 6.3] they are wonderful varieties and their birational geometry has been studied in [Mas20a].

Notation 3.1.14. For $k \leqslant h$, we will denote by $\mathcal{S}ec^{(k)}_h(\mathcal{S}^{n,m})$ the blow-up of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ along the strict transforms of the secant varieties $\mathcal{S}ec_i(\mathcal{S}^{n,m})$ for $i = 1, \dots, k$, and by $\mathcal{S}ec^{(k)}_h(\mathcal{V}^n)$ the blow-up of $\mathcal{S}ec_h(\mathcal{V}^n)$ along the strict transforms of the secant varieties $\mathcal{S}ec_i(\mathcal{V}^n)$ for $i = 1, \dots, k$.

Note that there is an embedding

$$i\colon \mathcal{Q}(n,h) \hookrightarrow \mathcal{C}(n,h). \tag{3.4}$$

The $(SL(n+1) \times SL(m+1))$-action (3.2) induces an $(SL(n+1) \times SL(m+1))$-action on $\mathcal{C}(n,m,h)$ and similarly, when $n = m$, the $SL(n+1)$-action (3.3) induces an $SL(n+1)$-action on $\mathcal{Q}(n,h)$.

We recall here the analogous to Remark 3.1.8 and Propoposition 3.1.9 for the varieties $\mathcal{S}ec_h(\mathcal{S}^{n,m})$.

Remark 3.1.16. Since $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ can be identified with the variety of $(n+1)\times(m+1)$ matrices modulo scalar of rank at most h, [Har95, Example 12.1], [HT84, Proposition 12(a)] give

$$\dim(\mathcal{S}ec_h(\mathcal{S}^{n,m})) = h(m+n+2-h)-1, \quad \deg(\mathcal{S}ec_h(\mathcal{S}^{n,m})) = \prod_{i=0}^{n-h}\frac{\binom{m+1+i}{n-i}}{\binom{m+1-h+i}{n-h-i}}.$$

Proposition 3.1.17. *The tangent cone $TC_p\mathcal{S}ec_h(\mathcal{S}^{n,m})$ of the secant variety $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ at a point $p \in \mathcal{S}ec_k(\mathcal{S}^{n,m})\backslash\mathcal{S}ec_{k-1}(\mathcal{S}^{n,m})$ for $k \leqslant h$ is a cone with vertex of dimension $nm+n+m-(m+1-k)(n+1-k)$ over $\mathcal{S}ec_{h-k}(\mathcal{S}^{n-k,m-k})$.*

Proof. It is enough to compute the tangent cone of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ at

$$p_k = \begin{pmatrix} I_{k,k} & 0_{k,m+1-k} \\ 0_{n+1-k,k} & 0_{n+1-k,m+1-k} \end{pmatrix}$$

where $I_{k,k}$ is the $k\times k$ identity matrix. Consider the affine chart $z_{0,0}\neq 0$ and the change of coordinates $z_{i,i}\mapsto z_{i,i}-1$ for $i=1,\dots,k-1$, $z_{i,j}\mapsto z_{i,j}$ otherwise. Then the matrix Z in (3.1) takes the following form

$$\begin{pmatrix}
1 & z_{0,1} & \dots & z_{0,k-1} & z_{0,k} & \dots & z_{0,m} \\
z_{1,0} & z_{1,1}-1 & \dots & z_{1,k-1} & z_{1,k} & \dots & z_{1,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{k-1,0} & z_{k-1,1} & \dots & z_{k-1,k-1}-1 & z_{k-1,k} & \dots & z_{k-1,m} \\
z_{k,0} & z_{k,1} & \dots & z_{k,k-1} & z_{k,k} & \dots & z_{k,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{n,0} & z_{n,1} & \dots & z_{n,k-1} & z_{n,k} & \dots & z_{n,m}
\end{pmatrix}.$$

Recall that $\mathcal{S}ec_h(\mathcal{S}^{n,m})\subseteq\mathbb{P}^N$ is cut out by the $(h+1)\times(h+1)$ minors of Z. Now, the lowest degree terms of these minors are given by the $(h+1-k)\times(h+1-k)$ minors of the following matrix

$$\begin{pmatrix}
z_{k,k} & \dots & z_{k,m} \\
\vdots & \ddots & \vdots \\
z_{n,k} & \dots & z_{n,m}
\end{pmatrix}.$$

Therefore, the tangent cone $TC_{p_k}\mathcal{S}ec_h(\mathcal{S}^{n,m})$ is contained in the cone C over the variety $\mathcal{S}ec_{h-k}(\mathcal{S}^{n-k,m-k})$ with vertex the linear subspace of $\mathbb{P}^N$ given by $\{z_{k,k} = \dots = z_{k,m} = z_{k+1,k} = \dots = z_{k+1,m} = \dots = z_{n,k} = \dots = z_{n,m} = 0\}$. Finally, by Remark 3.1.16 we conclude that $TC_{p_k}\mathcal{S}ec_h(\mathcal{S}^{n,m}) = C$. □

Theorem 3.1.18. *The variety $\mathcal{C}(n,m,h)$ is smooth and the divisors $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}}$ are smooth and intersect transversally. The closures of the orbits of the $SL(n+1)\times SL(m+1)$-action on $\mathcal{C}(n,m,h)$ induced by (3.2) are given by all the possible intersections of $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}}$ and $\mathcal{C}(n,m,h)$. Furthermore, the analogous statements hold for $\mathcal{Q}(n,h)$. Hence $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ are wonderful.*

Proof. We will proceed as follows. For $h=1$ we will prove the statement for any n and m. Then we will prove that if for $h<j$ the statement holds for any n and m then it also holds for $h=j$ and any n and m. This will prove the statement for any n,m and $h=0,\dots,n+1$.

For $h=1$ we have $\mathcal{C}(n,m,1)=\mathcal{S}^{n,m}$. Hence, the statements holds for any n and m. Assume that for any $h<j$ the statement holds for any n and m and consider $\mathcal{C}(n,m,j)$. For any $i=1,\dots,j-1$, Proposition 3.1.17 gives a fibration $E_i^{\mathcal{C}}\rightarrow \mathbb{S}ec_i^{(i-1)}(\mathcal{S}^{n,m})=\mathcal{C}(n,m,i)$ whose fibers are isomorphic to $\mathbb{S}ec_{j-i}^{(j-i-1)}(\mathcal{S}^{n-i,m-i})=\mathcal{C}(n-i,m-i,j-i)$. Then, by induction hypothesis and Proposition 3.1.10 the exceptional divisors $E_1^{\mathcal{C}},\dots,E_{j-1}^{\mathcal{C}}$ in $\mathcal{C}(n,m,j)$ are smooth. Moreover, by Proposition 3.1.17, $\mathcal{C}(n,m,j)$ is smooth away from $E_1^{\mathcal{C}},\dots,E_{j-1}^{\mathcal{C}}$ and for $i=1,\dots,j-1$ there is a fibration $\mathcal{C}(n,m,j)\cap E_i^{\mathcal{C}}\rightarrow\mathcal{C}(n,m,i)$ whose fibers are isomorphic to $\mathcal{C}(n-i,m-i,j-i)$. Hence, by induction and Proposition 3.1.10 we get that $\mathcal{C}(n,m,j)\cap E_i^{\mathcal{C}}$ is smooth and

$$\begin{aligned}\dim(\mathcal{C}(n,m,j)\cap E_i^{\mathcal{C}}) &= i(n+m-i)-1+(j-i)(n-i+m-i-j+i)-1\\ &= \dim(\mathcal{C}(n,m,j))-1\,.\end{aligned}$$

So $\mathcal{C}(n,m,j)$ is smooth and the intersection $\mathcal{C}(n,m,j)\cap E_i^{\mathcal{C}}$ is transversal for any $i=1,\dots,j-1$.

Now, consider an intersection of the following form $E_{j_1}^{\mathcal{C}}\cap\dots\cap E_{j_t}^{\mathcal{C}}$. By Proposition 3.1.17 the restriction of the blow-down morphism

$$E_{j_1}^{\mathcal{C}}\cap\dots\cap E_{j_t}^{\mathcal{C}}\rightarrow E_{j_1}^{\mathcal{C}}\cap\dots\cap E_{j_{t-1}}^{\mathcal{C}}\cap\mathcal{C}(n,m,j_t)$$

has fibers isomorphic to $\mathcal{C}(n-j_t,m-j_t,j-j_t)$. Again by induction hypothesis and Proposition 3.1.10 $E_{j_1}^{\mathcal{C}}\cap\dots\cap E_{j_t}^{\mathcal{C}}$ is smooth of dimension

$$(j-j_t)(n-j+m-j-j+j_t)-1+j_t(n+m-j_t)-1-(t-1)=\dim(\mathcal{C}(n,m,j))-t$$

and hence the intersection is transversal.

The claim about the orbit closures follows from [Vai84, Theorem 1] and the fact that the $SL(n+1)\times SL(m+1)$ action on $\mathcal{C}(n,m,h)$ is given by the restriction of the action (3.2) on the space of complete collineations. With an analogous proof we get the result for $\mathcal{Q}(n,h)$. □

3.1.2 *Divisors on $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$*

In this section we study the Picard groups and the cones of effective and nef divisors of the wonderful varieties introduces in Section 3.1. We will denote by $\mathcal{C}(n,m,h)^o$ and $\mathcal{Q}(n,h)^o$ the orbits of the matrix

$$J_h=\begin{pmatrix} I_{h,h} & 0\\ 0 & 0\end{pmatrix} \tag{3.5}$$

where $I_{h,h}$ is the $h\times h$ identity matrix, under the actions (3.2) and (3.3) respectively.

Proposition 3.1.20. *The Picard groups of $\mathcal{C}(n,m,h)^o$ and $\mathcal{Q}(n,h)^o$ are given by*

$$\operatorname{Pic}(\mathcal{C}(n,m,h)^o) \cong \begin{cases} \mathbb{Z} & \text{if } h = n+1 < m+1; \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } h < n+1; \\ \frac{\mathbb{Z}}{(n+1)\mathbb{Z}} & \text{if } h = n+1 = m+1; \end{cases}$$

and

$$\operatorname{Pic}(\mathcal{Q}(n,h)^o) \cong \begin{cases} \mathbb{Z} & \text{if } h < n+1 \text{ is odd;} \\ \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z} & \text{if } h < n+1 \text{ is even;} \\ \frac{\mathbb{Z}}{(n+1)\mathbb{Z}} & \text{if } h = n+1. \end{cases}$$

Proof. Let G_h be the stabilizer of the matrix J_h in (3.5) under the action (3.2). Since the Picard group and the character group of $SL(n+1) \times SL(m+1)$ are trivial [ADHL15, Theorem 4.5.1.2] yields that $\operatorname{Pic}(\mathcal{C}(n,m,h)^o)$ is isomorphic to the character group $\mathbb{X}(G_h)$ of G_h. Write an element $(A,B) \in SL(n+1) \times SL(m+1)$ as

$$A = \begin{pmatrix} A_{h,h} & A_{h,n+1-h} \\ A_{n+1-h,h} & A_{n+1-h,n+1-h} \end{pmatrix}, \quad B = \begin{pmatrix} B_{h,h} & B_{h,m+1-h} \\ B_{m+1-h,h} & B_{m+1-h,m+1-h} \end{pmatrix}. \tag{3.6}$$

Then $(A,B) \in G_h$ if and only if $A_{n+1-h,h} = 0$, $B_{m+1-h,h} = 0$ and $A_{h,h}B^T_{h,h} = \lambda I_{h,h}$. Assume that $h < n+1$ and $h < m+1$. Then $\mathbb{X}(G_h)$ is generated by the characters

$$d_{A_h} := \det(A_{h,h}),\ d_{B_h} := \det(B_{h,h}),\ \lambda,$$

$$d_{A_{n+1-h}} := \det(A_{n+1-h,n+1-h}),\ d_{B_{m+1-h}} := \det(B_{m+1-h,m+1-h})$$

with the following relations

$$d_{A_h} + d_{A_{n+1-h}} = d_{B_h} + d_{B_{m+1-h}} = 0,\ d_{A_h} + d_{B_h} = h\lambda.$$

Hence, $\mathbb{X}(G_h)$ is the free abelian group generated by d_{A_h} and λ.

Now, assume that $h = n+1 < m+1$. Then $d_{A_{n+1-h}} = 0$ and so $d_{A_h} = 0$. Therefore, $\mathbb{X}(G_h)$ is the free abelian group generated by λ.

If $h = n+1 = m+1$ then $d_{A_{n+1-h}} = d_{B_{m+1-h}} = 0$. So $d_{A_h} = d_{B_h} = 0$, and hence $\mathbb{X}(G_h)$ is the abelian group generated by λ with the relation $(n+1)\lambda = 0$.

Now, we consider the symmetric case. We will keep denoting by G_h the stabilizer of the matrix J_h in (3.5) under the action (3.3). Write an element $A \in SL(n+1)$ as in (3.6). Then $A \in G_h$ if and only if $A_{n+1-h,h} = 0$ and $A_{h,h}A^T_{h,h} = \lambda I_{h,h}$. Therefore, $\mathbb{X}(G_h)$ is generated by

$$d_{A_h} := \det(A_{h,h}),\ \lambda$$

with the relation $2d_{A_h} - h\lambda = 0$.

Assume $h < n+1$. If $h = 2k+1$ then $(2,-h) \in \mathbb{Z}^2$ is primitive. Considering the basis $u = 2d_{A_h} - h\lambda, v = d_{A_h} - k\lambda$ of $\mathbb{Z}^2$ we get that $\mathbb{X}(G_h) \cong \mathbb{Z}^2/\langle u \rangle \cong \mathbb{Z}$. If $h = 2k$ then $(2,-h) = 2(1,-k)$, and considering the basis $u = 2d_{A_h} - k\lambda, v = \lambda$ of $\mathbb{Z}^2$ we get that $\mathbb{X}(G_h) \cong \mathbb{Z}^2/\langle 2u \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. Finally, if $h = n+1$ we have $d_{A_h} = 0$, and hence $(n+1)\lambda = 0$. So $\mathbb{X}(G_h) \cong \mathbb{Z}/(n+1)\mathbb{Z}$. $\square$

Proposition 3.1.22. *The Picard rank of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ is given by*

$$\rho(\mathcal{C}(n,m,h)) = \begin{cases} h-1 & \text{if } h = n+1 = m+1; \\ h+1 & \text{if } h < n+1; \\ h & \text{if } h = n+1 < m+1; \end{cases}$$

and

$$\rho(\mathcal{Q}(n,h)) = \begin{cases} h & \text{if } h < n+1; \\ h-1 & \text{if } h = n+1. \end{cases}$$

Proof. Assume that $h < n+1$. Since, by Theorem 3.1.18 the variety $\mathcal{C}(n,m,h)$ is wonderful with boundary divisors $E_1, \dots, E_{h-1}$, [Bri07, Proposition 2.2.1] yields an exact sequence

$$0 \to \mathbb{Z}^{h-1} \to \operatorname{Pic}(\mathcal{C}(n,m,h)) \to \operatorname{Pic}(\mathcal{C}(n,m,h)^o) \to 0$$

where $\mathbb{Z}^{h-1}$ is the free abelian group generated by the boundary divisors. To conclude it is enough to use Proposition 3.1.20. The proof in the symmetric case is similar. $\square$

For $i = 1, \dots, h$, we define the divisors $D_i^{\mathcal{C}}$ in $\mathcal{C}(n,m,h)$ as the strict transform of the divisor given by the intersection of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ with

$$\det \begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{i-1,0} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0.$$

We will keep the same notation for the corresponding divisors in the intermediate blow-ups $\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})$.

Similarly, for $i = 1, \dots, h$ we define the divisors $D_i^{\mathcal{Q}}$ in $\mathcal{Q}(n,h)$ as the strict transform of the divisor given by the intersection of $\mathcal{S}ec_h(\mathcal{V}^n)$ with

$$\det \begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{0,i-1} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0.$$

Again we will keep the same notation for the corresponding divisors in the intermediate blow-ups $\mathcal{S}ec_h^{(k)}(\mathcal{V}^n)$.

Lemma 3.1.23. *Let Z be a $(n+1) \times (m+1)$ matrix of rank $k < \min\{n+1, m+1\}$ such that the determinant of the top left $k \times k$ minor Z_k of Z vanishes. Then, either the first k lines of Z are linearly dependent or the the first k columns of Z are linearly dependent.*

Proof. Assume that both the first k lines and the first k columns of Z are linearly independent. We will then prove that either $\det(Z_k) \neq 0$ or $\operatorname{rank}(Z) > k$. If $\det(Z_k) \neq 0$ the claim follows. So, assume $\det(Z_k) = 0$. We will write $e_1, \dots, e_{m+1}$ for the canonical basis of K^{m+1} and $\bar{e}_1, \dots \bar{e}_{n+1}$ for the canonical basis of K^{n+1}. Since the first k columns of Z are linearly independent, up to a change of coordinates we may assume that these columns are the vectors $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{k-1}, \bar{e}_{k+1}$. The first $k+1$ lines of the matrix Z are in the following form

$$\begin{aligned}
Z_{0,-} &= e_1^t + a_{k+1}^0 e_{k+1}^t + \cdots + a_{m+1}^0 e_{m+1}^t;\\
Z_{1,-} &= e_2^t + a_{k+1}^1 e_{k+1}^t + \cdots + a_{m+1}^1 e_{m+1}^t;\\
&\vdots\\
Z_{k-2,-} &= e_{k-1}^t + a_{k+1}^{k-2} e_{k+1}^t + \cdots + a_{m+1}^{k-2} e_{m+1}^t;\\
Z_{k-1,-} &= a_{k+1}^{k-1} e_{k+1}^t + \cdots + a_{m+1}^{k-1} e_{m+1}^t;\\
Z_{k,-} &= e_k^t + a_{k+1}^k e_{k+1}^t + \cdots + a_{m+1}^k e_{m+1}^t;
\end{aligned}$$

for some $a_i^j \in K$. By assumption, the first k lines are linearly independent and so we must have $a_i^{k-1} \neq 0$ for at least one $i \in \{k+1, \dots m+1\}$. Hence, the $k+1$ lines $Z_{0,-}, \dots, Z_{k,-}$ are linearly independent, and $\operatorname{rank}(Z) \geqslant k+1$. □

Corollary 3.1.24. *For $k < \min\{n+1, m+1\}$, the divisor cut out on $\mathbb{S}ec_k(\mathcal{S}^{n,m})$ by the top left $k \times k$ minor of the matrix in (3.1) has two components H_1 and H_2, where H_1 is cut out by the $k \times k$ minors of the first k lines of Z, and H_2 is cut out by the $k \times k$ minors of the first k columns of Z.*

Proof. The claim follows immediately from Lemma 3.1.23. □

Remark 3.1.25. In $\mathbb{S}ec_k(\mathcal{V}^n)$ the divisor associated to D_k is irreducible. Indeed, in the symmetric case the divisors H_1, H_2 in Corollary 3.1.24 coincide.

In order to further clarify this we explicitly work out the case of 3×3 matrices. The hypersurface $D_2 = \{z_{0,0}z_{1,1} - z_{0,1}z_{1,0} = 0\}$ cuts out on $\mathbb{S}ec_2(\mathcal{S}^{2,2}) \subset \mathbb{P}^8$ a divisor with two irreducible components:

$$H_1 = \{z_{0,1}z_{1,0} - z_{0,0}z_{1,1} = z_{0,2}z_{1,1} - z_{0,1}z_{1,2} = z_{0,2}z_{1,0} - z_{0,0}z_{1,2} = 0\};$$

$$H_2 = \{z_{0,1}z_{1,0} - z_{0,0}z_{1,1} = z_{0,1}z_{2,0} - z_{0,0}z_{2,1} = z_{1,1}z_{2,0} - z_{1,0}z_{2,1} = 0\}.$$

In the symmetric case the divisor $\{z_{0,0}z_{1,1} - z_{0,1}^2 = 0\}$ cuts out on $\mathbb{S}ec_2(\mathcal{V}^2) \subset \mathbb{P}^5$ the irreducible divisor

$$\{z_{0,2}z_{1,1} - z_{0,1}z_{1,2} = z_{0,1}z_{0,2} - z_{0,0}z_{1,2} = z_{0,1}^2 - z_{0,0}z_{1,1} = 0\}$$

with multiplicity two.

Notation 3.1.26. We denote by $H_1^{\mathcal{C}}, H_2^{\mathcal{C}}$ the strict transforms of H_1, H_2 in $\mathcal{C}(n,m,h)$.

Proposition 3.1.27. *The set of colors of $\mathcal{C}(n,m,h)$ is given by*

$$\begin{array}{ll}
\{D_1^{\mathcal{C}}, \dots, D_n^{\mathcal{C}}\} & \textit{if } h = n+1 = m+1;\\
\{H_1^{\mathcal{C}}, H_2^{\mathcal{C}}, D_1^{\mathcal{C}}, \dots, D_{h-1}^{\mathcal{C}}\} & \textit{if } h < n+1;\\
\{D_1^{\mathcal{C}}, \dots, D_{n+1}^{\mathcal{C}}\} & \textit{if } h = n+1 < m+1;
\end{array}$$

while for $\mathcal{Q}(n,h)$ the set of colors is given by

$$\begin{array}{ll}
\{D_1^{\mathcal{Q}}, \dots, D_h^{\mathcal{Q}}\} & \textit{if } h < n+1;\\
\{D_1^{\mathcal{Q}}, \dots, D_n^{\mathcal{Q}}\} & \textit{if } h = n+1.
\end{array}$$

Proof. The claim for $\mathcal{C}(n,m,n+1) = \mathcal{X}(n,m)$ and $\mathcal{Q}(n,n+1) = \mathcal{Q}(n)$ follows from [Mas20a, Proposition 3.6]. In particular, the divisors listed in the statement are stabilized respectively by the action of the Borel subgroups in (3.2) and (3.3). Moreover, $\mathbb{S}ec_h(\mathcal{S}^{n,m})$ and $\mathbb{S}ec_h(\mathcal{V}^n)$ are stabilized respectively by the action (3.2) and (3.3). Then, $D_1^{\mathcal{C}},\dots,D_h^{\mathcal{C}}$ are stabilized by the restriction of the action (3.2), and similarly the strict transform in $\mathcal{Q}(n,h)$ of $D_1^{\mathcal{Q}},\dots,D_h^{\mathcal{Q}}$ are stabilized by the restriction of the action (3.3).

The groups acting are connected, so any reducible divisor which is stabilized must be stabilized component wise. In particular, since by Corollary 3.1.24 in $\mathcal{C}(n,m,h)$ for $h < n+1$ we have $D_h^{\mathcal{C}} = H_1^{\mathcal{C}} \cup H_2^{\mathcal{C}}$ and since $D_h^{\mathcal{C}}$ is stabilized, we have that both $H_1^{\mathcal{C}}$ and $H_2^{\mathcal{C}}$ are stabilized.

As noticed in [ADHL15, Remark 4.5.5.3], if (X, G, B, x_0) is a spherical wonderful variety with colors $D_1,\dots,D_s$ the big cell $X\backslash(D_1 \cup \dots \cup D_s)$ is an affine space. Therefore, it admits only constant invertible global functions and $\mathrm{Pic}(X) = \mathbb{Z}[D_1,\dots,D_s]$.

Now for $h < n+1$ in $\mathcal{C}(n,m,h)$ we have $h+1$ colors and since by Proposition 3.1.22 the Picard rank of $\mathcal{C}(n,m,h)$ is $h+1$, these divisors $D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$ must be all the colors. Similarly, for $h = n+1 < m+1$ we found the divisors $D_1^{\mathcal{C}},\dots,D_{n+1}^{\mathcal{C}}$, and since in this case $\rho(\mathcal{C}(n,m,h)) = h$, they are all the colors. Note that when $h = n+1 = m+1$, the divisor $D_{n+1}^{\mathcal{C}}$ is not a color, since it is stabilized by the whole group. In this case $\rho(\mathcal{C}(n,m,h)) = h-1$ and then $D_1^{\mathcal{C}},\dots,D_n^{\mathcal{C}}$ are the colors. With a similar argument we can compute the colors of $\mathcal{Q}(n,h)$. $\square$

Proposition 3.1.28. *For the effective and the nef cone of $\mathcal{C}(n,m,h)$ we have*

$$\mathrm{Eff}(\mathcal{C}(n,m,h)) = \begin{cases} \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}}\rangle & \text{if } h = n+1 = m+1; \\ \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}\rangle & \text{if } h < n+1; \\ \langle E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},D_{n+1}^{\mathcal{C}}\rangle & \text{if } h = n+1 < m+1; \end{cases}$$

$$\mathrm{Nef}(\mathcal{C}(n,m,h)) = \begin{cases} \langle D_1^{\mathcal{C}},\dots,D_n^{\mathcal{C}}\rangle & \text{if } h = n+1 = m+1; \\ \langle D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}\rangle & \text{if } h < n+1; \\ \langle D_1^{\mathcal{C}},\dots,D_{n+1}^{\mathcal{C}}\rangle & \text{if } h = n+1 < m+1; \end{cases}$$

while for the effective and the nef cone of $\mathcal{Q}(n,h)$ we have

$$\mathrm{Eff}(\mathcal{Q}(n,h)) = \begin{cases} \langle E_1^{\mathcal{Q}},\dots,E_{h-1}^{\mathcal{Q}},D_h^{\mathcal{Q}}\rangle & \text{if } h < n+1; \\ \langle E_1^{\mathcal{Q}},\dots,E_{h-1}^{\mathcal{Q}}\rangle & \text{if } h = n+1; \end{cases}$$

$$\mathrm{Nef}(\mathcal{Q}(n,h)) = \begin{cases} \langle D_1^{\mathcal{Q}},\dots,D_h^{\mathcal{Q}}\rangle & \text{if } h < n+1; \\ \langle D_1^{\mathcal{Q}},\dots,D_n^{\mathcal{Q}}\rangle & \text{if } h = n+1. \end{cases}$$

Proof. The statement for $\mathcal{C}(n,m,n+1) = \mathcal{X}(n,m)$ and $\mathcal{Q}(n,n+1) = \mathcal{Q}(n)$ follows from [Mas20a, Theorem 3.13]. We consider now the case $h < n+1$.

Let start with $\mathcal{C}(n,m,h)$. By [ADHL15, Proposition 4.5.4.4], Theorem 3.1.18 and Proposition 3.1.28 $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$ generate the effective cone of $\mathcal{C}(n,m,h)$. By [Mas20a, Section 5] the divisor $D_i^{\mathcal{C}}$ induces a birational morphism that contracts the exceptional divisor $E_i^{\mathcal{C}}$. Therefore $D_i^{\mathcal{C}}$ lies in the interior of the effective cone for any $i = 1,\dots,h-1$. In particular, since by Proposition 3.1.22

$\rho(\mathcal{C}(n,m,h)) = h+1$, we conclude that the extremal rays of the effective cone are $E_1^{\mathcal{C}},\dots,E_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}},H_2^{\mathcal{C}}$.

Furthermore, by [Bri89, Section 2.6] the nef cone is generated by $D_1^{\mathcal{C}},\dots,D_{h-1}^{\mathcal{C}},H_1^{\mathcal{C}}$, $H_2^{\mathcal{C}}$. A similar argument gives the generators for the effective and nef cones of $\mathcal{Q}(n,h)$. $\square$

3.1.3 *Birational geometry of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$*

Next, we study the birational geometry of $\mathcal{C}(n,m,h)$ and $\mathcal{Q}(n,h)$ when the Picard rank is small.

We begin with $\mathcal{Q}(n,h)$. The varieties $\mathcal{Q}(1,2)$ has Picard rank one and it is isomorphic to $\mathbb{P}^2$, so there is nothing to say on its Mori chamber decomposition. The case of Picard rank two $\mathcal{Q}(2,3)$ of complete quadrics in $\mathbb{P}^2$ is covered in [Mas20a, Section 6], so we mainly focus on the cases $\mathcal{Q}(n,2)$ for $n \geqslant 2$ and $\mathcal{Q}(n,3)$ for $n \geqslant 3$.

Remark 3.1.29. Let Y be a smooth and irreducible subvariety of a smooth variety X, and let $f\colon Bl_YX \to X$ be the blow-up of X along Y with exceptional divisor E. Then for any divisor $D \in \mathrm{Pic}(X)$ in $\mathrm{Pic}(Bl_YX)$ we have

$$\widetilde{D} \sim f^*D - \mathrm{mult}_Y(D)E$$

where $\widetilde{D} \subset Bl_YX$ is the strict transform of D, and $\mathrm{mult}_Y(D)$ is the multiplicity of D at a general point of Y.

Lemma 3.1.30. *For the variety $\mathcal{Q}(n,2)$, for $n \geqslant 2$, we have $D_1^{\mathcal{Q}} \sim H$, $D_2^{\mathcal{Q}} \sim 2H - E_1^{\mathcal{Q}}$, $D_3^{\mathcal{Q}} \sim 3H - 2E_1^{\mathcal{Q}}$; and for $\mathcal{Q}(n,3)$, for $n \geqslant 3$, we have that $D_1^{\mathcal{Q}} \sim H$, $D_2^{\mathcal{Q}} \sim 2H - E_1^{\mathcal{Q}}$, $D_3^{\mathcal{Q}} \sim 3H - 2E_1^{\mathcal{Q}} - E_2^{\mathcal{Q}}$.*

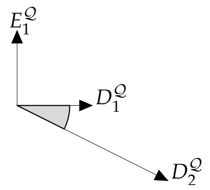
Proof. To compute the divisors classes we can not use directly Remark 3.1.29 since in general we start the blow up construction with a singular variety.

Consider the strict transform $L \subset \mathcal{Q}(n,2)$ of the line parametrizing quadrics of the form $\{\mu x_0^2 + (\lambda(x_1^2+x_2^2) = 0\}$. This line intersects $\mathcal{V}^n$ at a point p, $\mathcal{S}ec_2(\mathcal{V}^n)$ at a point q, and it is not contained neither in the tangent cone of $\mathcal{S}ec_2(\mathcal{V}^n)$ at p nor in the tangent space of $\{z_{0,0}z_{1,1} - z_{0,1}^2 = 0\}$ at p. Write $D_2^{\mathcal{Q}} = 2H - aE_1$.

Then $1 = D_2^{\mathcal{Q}} \cdot L = 2 - a$ yields $a = 1$. Similarly, if $D_3^{\mathcal{Q}} \sim 3H - bE_1$ we get $1 = D_3^{\mathcal{Q}} \cdot L = 3 - b$ and hence $b = 2$.

Since $D_2^{\mathcal{Q}}$ does not contain the strict transform of $\mathcal{S}ec_2(\mathcal{V}^n)$ its expression remains unvaried after the last blow-up. On the other hand, E_2 must appear in the expression of $D_3^{\mathcal{Q}}$. Let us write $D_3^{\mathcal{Q}} \sim 3H - 2E_1 - cE_2$ and keep denoting by L its strict transform in $\mathcal{Q}(n,3)$. Note that the line corresponding to L is not contained in the tangent space of $\{z_{0,2}^2z_{1,1} - 2z_{0,1}z_{0,2}z_{1,2} + z_{0,0}z_{1,2}^2 + z_{0,1}^2z_{2,2} - z_{0,0}z_{1,1}z_{2,2} = 0\}$ at p and q. So $0 = D_3^{\mathcal{Q}} \cdot L = 3 - 2 - c$ and hence $c = 1$. $\square$

Proposition 3.1.31. *When $n \geqslant 2$, the Mori chamber decomposition of* $\mathrm{Eff}(\mathcal{Q}(n,2))$ *has two chambers:*

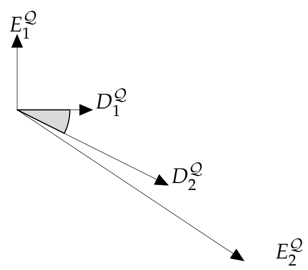


where $\mathrm{Mov}(\mathcal{Q}(n,2))$ *coincides with* $\mathrm{Nef}(\mathcal{Q}(n,2))$ *and is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}$.

Proof. By Theorem 3.1.18, Proposition 3.1.27, Remarks 2.2.6, 2.2.7, and Lemma 3.1.30 the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{Q}(n,2))$ is a possibly trivial coarsening of the decomposition in the statement.

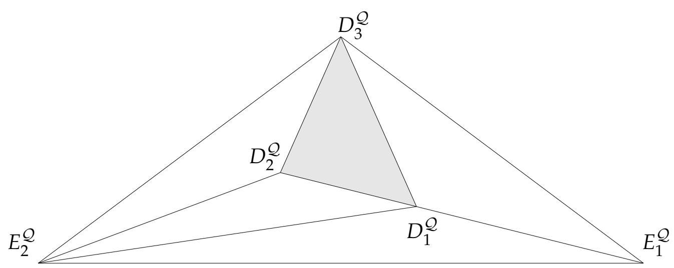
Since by Proposition 3.1.28 $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}$ are the generators of the nef cone of $\mathcal{Q}(n,2)$, the ray $D_1^{\mathcal{Q}}$ can not be removed. □

Proposition 3.1.32. *The Mori chamber decomposition of* $\mathrm{Eff}(\mathcal{Q}(2,3))$ *has three chambers:*



where the divisors classes are $D_1^{\mathcal{Q}} \sim H$, $D_2^{\mathcal{Q}} \sim 2H - E_1^{\mathcal{Q}}$, $E_2^{\mathcal{Q}} \sim 3H - 2E_1^{\mathcal{Q}}$ *and* $\mathrm{Mov}(\mathcal{Q}(2,3))$ *coincides with* $\mathrm{Nef}(\mathcal{Q}(2,3))$ *and is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}$.

For $n \geqslant 3$, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{Q}(n,3))$ *has five chambers as displayed in the following* 2*-dimension section of* $\mathrm{Eff}(\mathcal{Q}(n,3))$



where $\mathrm{Mov}(\mathcal{Q}(n,3))$ *coincides with* $\mathrm{Nef}(\mathcal{Q}(n,3))$ *and is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$.

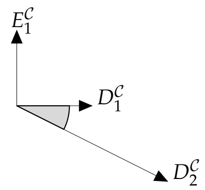
Proof. By Theorem 3.1.18, Proposition 3.1.27, Remarks 2.2.6, 2.2.7, and Lemma 3.1.30 the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{Q}(n,3))$ is a possibly trivial coarsening of the decompositions in the statement.

For the case of $\mathcal{Q}(2,3)$, since $\mathcal{S}ec_2(\mathcal{V}^2)$ is a divisor in $\mathbb{P}^5$, $\mathcal{Q}(2,3)$ is isomorphic to the blow-up of $\mathbb{P}^5$ just along the Veronese variety $\mathcal{V}^2$ and the divisor $E_2^{\mathcal{Q}}$ corresponds to the strict transform of $\mathcal{S}ec_2(\mathcal{V}^2)$ in this blow up. Then, the class of $E_2^{\mathcal{Q}}$ can be computed using Remark 3.1.29, Remark 3.1.8 and Proposition 3.1.9. For the class of $D_2^{\mathcal{Q}}$ we can argue as in Lemma 3.1.30. Moreover, the Mori chamber decomposition is as stated since by Proposition 3.1.28 $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}$ are the generators of the nef cone of $\mathcal{Q}(2,3)$ and so this rays can not be removed. Crossing the ray $D_1^{\mathcal{Q}}$ corresponds to blowing-down $E_1^{\mathcal{Q}}$, while crossing the ray $D_2^{\mathcal{Q}}$ induces the blow-down of $E_2^{\mathcal{Q}}$. In particular, $\mathrm{Mov}(\mathcal{Q}(2,3)) = \mathrm{Nef}(\mathcal{Q}(2,3))$.

When $n \geqslant 3$, since by Proposition 3.1.28 $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$ are the generators of the nef cone of $\mathcal{Q}(n,3)$, this rays can not be removed. Furthermore, since Mori chamber are convex the walls between $E_2^{\mathcal{Q}}, D_2^{\mathcal{Q}}$ and $E_1^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ can not be removed. Finally, to see that the wall between $E_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ can not be removed it is enough to observe that the stable base locus of a divisor in the chamber delimited by $E_2^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}$ is $E_2^{\mathcal{Q}}$, while the stable base locus of a divisor in the chamber delimited by $E_2^{\mathcal{Q}}, D_1^{\mathcal{Q}}, E_1^{\mathcal{Q}}$ is $E_1^{\mathcal{Q}} \cup E_2^{\mathcal{Q}}$. $\square$

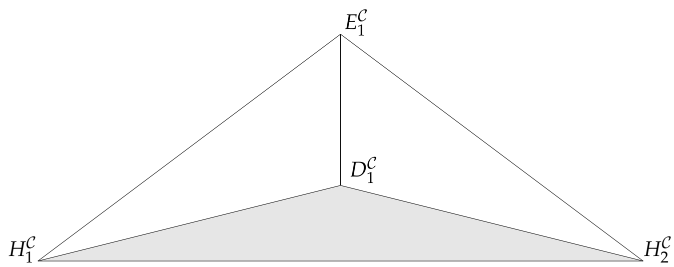
We will study the decomposition of the effective cone of $\mathcal{C}(n,m,2)$. For $n = m = 1$ we have $\mathcal{C}(1,1,2) \cong \mathbb{P}^3$. Hence, the first interesting cases occur for $n = 1$ and $m > 1$.

Proposition 3.1.33. *When $n = 1$ and $m > 1$, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{C}(n,m,2))$ has two chambers:*



where $\mathrm{Mov}(\mathcal{C}(n,m,2))$ coincides with $\mathrm{Nef}(\mathcal{C}(n,m,2))$ and is generated by $D_1^{\mathcal{C}}, D_2^{\mathcal{C}}$.

For $n > 1$ and $m > 1$ the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{C}(n,m,2))$ has three chambers as displayed in the following 2-dimensional section of $\mathrm{Eff}(\mathcal{C}(n,m,2))$

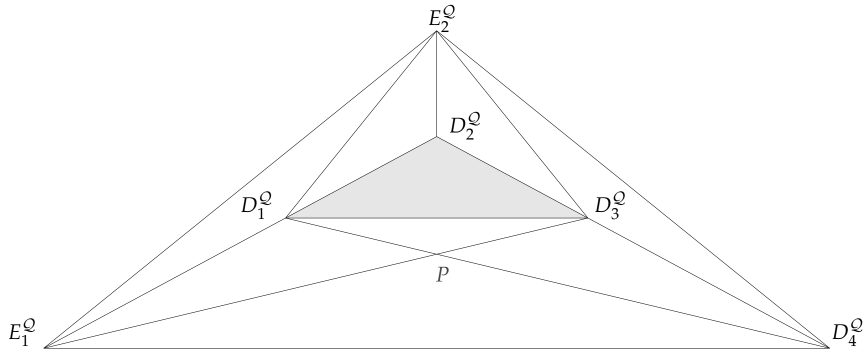


where the movable cone $\mathrm{Mov}(\mathcal{C}(n,m,2))$ coincides with the nef cone $\mathrm{Nef}(\mathcal{C}(n,m,2))$ and is generated by $H_1^{\mathcal{C}}, H_2^{\mathcal{C}}, D_1^{\mathcal{C}}$.

Proof. It is enough to argue as in the proof of Proposition 3.1.32, and to observe that since Mori chamber are convex in the case $n > 1, m > 1$ the wall between $E_1^{\mathcal{C}}, D_1^{\mathcal{C}}$ can not be removed. $\square$

In the following we consider the spherical variety $\mathcal{S}ec_4^{(2)}(\mathcal{V}^n)$ obtained by blowing-up $\mathcal{S}ec_4(\mathcal{V}^n)$ along $\mathcal{V}^n$ and then along the strict transform of $\mathcal{S}ec_2(\mathcal{V}^n)$. We will keep denoting by $D_i^{\mathcal{Q}}, E_j^{\mathcal{Q}}$ the push-forwards of the corresponding divisors via the blow-down $\mathcal{Q}(n,4) \to \mathcal{S}ec_4^{(2)}(\mathcal{V}^n)$.

Proposition 3.1.34. *The Mori chamber decomposition of* $\mathrm{Eff}(\mathcal{S}ec_4^{(2)}(\mathcal{V}^n))$ *has nine chambers as displayed in the following* 2*-dimensional section of* $\mathrm{Eff}(\mathcal{S}ec_4^{(2)}(\mathcal{V}^n))$



where the nef cone $\mathrm{Nef}(\mathcal{S}ec_4^{(2)}(\mathcal{V}^n))$ *is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}$*, and the movable cone* $\mathrm{Mov}(\mathcal{S}ec_4^{(2)}(\mathcal{V}^n))$ *is generated by* $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}, P$ *with* $P \sim 6D_1^{\mathcal{Q}} - 3E_1^{\mathcal{Q}} - 2E_2^{\mathcal{Q}}$.

Proof. Note that the $SL(n+1)$-actions on $\mathcal{S}ec_4^{(2)}(\mathcal{V}^n)$ and $\mathcal{Q}(n,4)$ are equivariant with respect to the blow-down morphism $\mathcal{Q}(n,4) \to \mathcal{S}ec_4^{(2)}(\mathcal{V}^n)$. Hence, by Proposition 3.1.27 the colors of $\mathcal{S}ec_4^{(2)}(\mathcal{V}^n)$ are $D_1^{\mathcal{Q}}, D_2^{\mathcal{Q}}, D_3^{\mathcal{Q}}, D_4^{\mathcal{Q}}$, and its boundary divisors are $E_1^{\mathcal{Q}}, E_2^{\mathcal{Q}}$. Arguing as in the proof of Lemma 3.1.30 we have that $D_4^{\mathcal{Q}} \sim 4H - 3E_1^{\mathcal{Q}} - 2E_2^{\mathcal{Q}}$. Note that $D_4^{\mathcal{Q}}$ is also a boundary divisor when $n = 3$. Now, the claim on the movable cone follows from Remark 2.2.7 and [ADHL15, Proposition 3.3.2.3]. Finally, to conclude it is enough to argue as in the proof of Proposition 3.1.32. □

We conclude this section by computing the automorphism groups of the varieties $\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ and $\mathcal{S}ec_h^{(k)}(\mathcal{V}^n)$.

Proposition 3.1.35. *For all* $h \leqslant n$ *we have*

$$\mathrm{Aut}(\mathcal{S}ec_h(\mathcal{S}^{n,m})) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m; \end{cases}$$

and $\mathrm{Aut}(\mathcal{S}ec_h(\mathcal{V}^n)) \cong PGL(n+1)$.

Proof. Let ϕ be an automorphism of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$. By the stratification of the singular locus of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$ described in Proposition 3.1.17 ϕ must stabilize $\mathcal{S}ec_k(\mathcal{S}^{n,m})$ for all $k \leqslant h$. In particular, ϕ induces an automorphism $\phi_{|\mathcal{S}^{n,m}} \in \mathrm{Aut}(\mathcal{S}^{n,m})$, and by [Mas20a, Lemma 7.4] we have that $\mathrm{Aut}(\mathcal{S}^{n,m}) \cong PGL(n+1) \times PGL(m+1)$ if $n < m$, and $\mathrm{Aut}(\mathcal{S}^{n,n}) \cong S_2 \ltimes (PGL(n+1) \times PGL(n+1))$.

Note that in the case $n = m$ also the involution in S_2 switching the two factors comes from an automorphism of the ambient projective space $\mathbb{P}^N$ and so it induces an automorphism of $\mathcal{S}ec_h(\mathcal{S}^{n,m})$. Let us proceed by induction on h. So $\mathrm{Aut}(\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})) \cong \mathrm{Aut}(\mathcal{S}^{n,m})$, and we have a surjective morphism of groups

$$\begin{array}{cccc} \chi\colon & \mathrm{Aut}(\mathcal{S}ec_h(\mathcal{S}^{n,m})) & \longrightarrow & \mathrm{Aut}(\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})). \\ & \phi & \longmapsto & \phi_{|\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})} \end{array}$$

Recall that $\mathcal{S}ec_h(\mathcal{S}^{n,m}) = \mathrm{Join}(\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m}), \mathcal{S}^{n,m})$. Assume that $\phi_{|\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})} = Id_{\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})}$. Then $\phi_{|\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})}$ fixes $\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})$ and hence $\mathcal{S}^{n,m}$.

Let $p \in \mathcal{S}ec_h(\mathcal{S}^{n,m})$ be a general point. By Remark 3.1.16 the actual dimension of $\mathrm{Join}(\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m}), \mathcal{S}^{n,m})$ is smaller than the expected one. So there are infinitely many lines intersecting $\mathcal{S}^{n,m}$ and $\mathcal{S}ec_{h-1}(\mathcal{S}^{n,m})$ through p. Any two of these lines are stabilized by ϕ and intersect at p, so $\phi(p) = p$. Hence $\phi = Id_{\mathcal{S}ec_h(\mathcal{S}^{n,m})}$ and χ is an isomorphism. The same proof, with the obvious variations, works in the symmetric case as well. □

Theorem 3.1.36. *For all $h \leqslant n$ and $k = 1, \dots, h-1$ we have*

$$\mathrm{Aut}(\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m; \end{cases}$$

$$\mathrm{Aut}(\mathcal{S}ec_h^{(k)}(\mathcal{V}^n)) \cong PGL(n+1);$$

while for $h = n+1$ we have

$$\mathrm{Aut}(\mathcal{C}(n,m,n+1)) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ (S_2 \ltimes (PGL(n+1) \times PGL(n+1))) \rtimes S_2 & \text{if } n = m \geqslant 2; \end{cases}$$

$$\mathrm{Aut}(\mathcal{Q}(n,n+1)) \cong PGL(n+1) \rtimes S_2;$$

$\mathrm{Aut}(\mathcal{C}(1,1,2)) \cong PGL(4)$, *and* $\mathrm{Aut}(\mathcal{Q}(1,2)) \cong PGL(3)$.

Proof. When $h = n+1$ the statement follows from [Mas20a, Theorem 7.5]. Hence we consider the case $h \leqslant n$. We will prove the claim for $\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})$. The argument in the symmetric case is completely analogous.

First, take $k = h-1$. Hence $\mathcal{S}ec_h^{(h-1)}(\mathcal{S}^{n,m}) \cong \mathcal{C}(n,m,h)$. An automorphism $\phi \in \mathrm{Aut}(\mathcal{C}(n,m,h))$ acts on the extremal rays of $\mathrm{Eff}(\mathcal{C}(n,m,h))$ by permutations. If it acts non trivially then it must act non trivially also on the generators of $\mathrm{Nef}(\mathcal{C}(n,m,h))$ in Proposition 3.1.28. However this is not possible since for instance these nef divisors have spaces of global sections of different dimensions. Hence, ϕ stabilizes all the exceptional divisors in Definition 3.1.12, and therefore it induces an automorphism $\tilde{\phi} \in \mathrm{Aut}(\mathcal{S}ec_h(\mathcal{S}^{n,m}))$. The morphism of groups

$$\begin{array}{rccc} \tilde{\chi}\colon & \mathrm{Aut}(\mathcal{C}(n,m,h)) & \longrightarrow & \mathrm{Aut}(\mathcal{S}ec_h(\mathcal{S}^{n,m})) \\ & \phi & \longmapsto & \tilde{\phi} \end{array}$$

is clearly an isomorphism, and we conclude by Proposition 3.1.35.

Now, move to the case $k < h-1$. Recall that the space $\mathcal{C}(n,m,h)$ is obtained from $\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ by blow-ups centered at subvarieties of $\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})$ that are stabilized by all $\phi \in \mathrm{Aut}(\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m}))$. Hence, $\phi \in \mathrm{Aut}(\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m}))$ lifts to an automorphism $\overline{\phi}$ of $\mathcal{C}(n,m,h)$, and we get a morphism of groups

$$\begin{array}{rccc} \overline{\chi}\colon & \mathrm{Aut}(\mathcal{S}ec_h^{(k)}(\mathcal{S}^{n,m})) & \longrightarrow & \mathrm{Aut}(\mathcal{C}(n,m,h)) \\ & \phi & \longmapsto & \overline{\phi} \end{array}$$

which again is an isomorphism. Finally, we conclude by the computation of $\mathrm{Aut}(\mathcal{C}(n,m,h))$ in the first part of the proof. □

3.2 COMPLETE SYMMETRIC SYMPLECTIC FORMS

From now on we will consider the case $n+1 = 2r$ even. Let $Sp(2r)$ be the symplectic group of $2r \times 2r$ symplectic matrices, that is

$$Sp(2r) = \{M \in \mathrm{Hom}(V, V) \mid M^t \Omega M = \Omega\}$$

where

$$\Omega = \begin{pmatrix} 0 & I_{r,r} \\ -I_{r,r} & 0 \end{pmatrix} \tag{3.7}$$

is the standard symplectic form. Over an algebraically closed field of characteristic zero the symplectic group is a non-compact, irreducible, simply connected, simple Lie group.

Remark 3.2.2. Let us write a $2r \times 2r$ matrix M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are four $r \times r$ matrices. The condition of being symplectic translates then into the following system of equations

$$\begin{cases} -C^t A + A^t C & = & 0_{r,r}; \\ -C^t B + A^t D & = & I_{r,r}; \\ -D^t A + B^t C & = & -I_{r,r}; \\ -D^t B + B^t D & = & 0_{r,r}. \end{cases}$$

Considering the transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A - I_{r,r} & B \\ C & D - I_{r,r} \end{pmatrix} \tag{3.8}$$

we get the following relations for the tangent space of $Sp(2r)$ at the identity

$$A = -D^t,\ B = B^t,\ C = C^t.$$

Hence, the tangent space of $Sp(2r)$ at the identity is the Lie algebra $\mathfrak{sp}(2r, K)$ consisting of $2r \times 2r$ matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

with C and B symmetric. In particular, $\dim(Sp(2r)) = r^2 + 2\frac{r(r+1)}{2} = r(2r+1)$.

Remark 3.2.4. By [Ou12, Section 1] the Borel subgroup of the symplectic group can be described as follows:

$$\mathscr{B} = \left\{ \begin{pmatrix} A & 0_{r,r} \\ B & A^{-t} \end{pmatrix} \text{ with } A^t B = B^t A \right\}$$

where $A \in GL(r)$ is lower triangular and B is a general $r \times r$ matrix.

Now, $Sp(2r)$ is a subgroup of $SL(n+1)$ and the $SL(n+1)$-action (3.3) restricts to the following $Sp(2r)$-action:

$$\begin{array}{ccc} Sp(2r) \times \mathbb{P}^{N_+} & \longrightarrow & \mathbb{P}^{N_+} \\ (M, Z) & \longmapsto & MZM^t \end{array} \tag{3.9}$$

We denote by O_{2r} the $Sp(2r)$-orbit of the identity in $\mathbb{P}^{N_+}$ and by $X_{2r} = \overline{O_{2r}} \subseteq \mathbb{P}^{N_+}$ its closure.

Proposition 3.2.6. *Let $Y_k = \overline{O_k} \subset \mathbb{P}^{N_+}$ be the closure of the $Sp(2r)$-orbit of the matrix*

$$I_k = \begin{pmatrix} I_{k,k} & 0_{k,2r-k} \\ 0_{2r-k,k} & 0_{2r-k,2r-k} \end{pmatrix}$$

via the action in (3.9). If $k \leqslant r$ then

$$\begin{aligned} \dim(Y_k) &= r(2r+1) - \frac{k(k-1)}{2} - r(r-k) - \frac{r(r+1)}{2} - \frac{(r-k)(r-k+1)}{2} - 1 \\ &= 2rk + k - k^2 - 1 \end{aligned}$$

Finally, $\dim(Y_{2r}) = r(r+1)$.

Proof. Our aim is to compute the dimension of the stabilizer $H \subset Sp(2r)$ of I_k. Consider the incidence correspondence

$$\begin{tikzcd} & \mathcal{I} = \{(M,\lambda) \mid MI_kM^t = \lambda I_k\} \subseteq Sp(2r) \times K^* \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ Sp(2r) & & K^* \end{tikzcd}$$

Note that the fibers of ψ are isomorphic subgroups of $Sp(2r)$. We will compute the dimension of $H_1 = \psi^{-1}(1)$ and then the dimension of $H = \phi(\mathcal{I})$ will be given by

$$\dim(H) = \dim(\mathcal{I}) = \dim(H_1) + 1. \tag{3.10}$$

Consider first the case $k \leqslant r$. Subdivide as usual the matrices in $Sp(2r)$ in four $r \times r$ blocks and write the matrix whose orbit we want to study as

$$\begin{pmatrix} Z_k & 0_{r,r} \\ 0_{r,r} & 0_{r,r} \end{pmatrix}$$

where Z_k is the following $r \times r$ matrix

$$Z_k = \begin{pmatrix} I_{k,k} & 0_{k,r-k} \\ 0_{r-k,k} & 0_{r-k,r-k} \end{pmatrix}.$$

Now, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Z_k & 0_{r,r} \\ 0_{r,r} & 0_{r,r} \end{pmatrix} \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} = \begin{pmatrix} AZ_kA^t & AZ_kC^t \\ CZ_kA^t & CZ_kC^t \end{pmatrix}.$$

Subdivide the matrix A in blocks as follows

$$A = \begin{pmatrix} A_{k,k} & A_{k,r-k} \\ A_{r-k,k} & A_{r-k,r-k} \end{pmatrix}.$$

Then

$$\begin{pmatrix} A_{k,k} & A_{k,r-k} \\ A_{r-k,k} & A_{r-k,r-k} \end{pmatrix}\begin{pmatrix} I_{k,k} & 0_{k,r-k} \\ 0_{r-k,k} & 0_{r-k,r-k} \end{pmatrix}\begin{pmatrix} A^t_{k,k} & A^t_{r-k,k} \\ A^t_{k,r-k} & A^t_{r-k,r-k} \end{pmatrix} = \begin{pmatrix} A_{k,k}A^t_{k,k} & A_{k,k}A^t_{r-k,k} \\ A_{r-k,k}A^t_{k,k} & A_{r-k,k}A^t_{r-k,k} \end{pmatrix}.$$

Therefore, considering the transformation (3.8) we get the following relations for the tangent space of H_1 at the identity

$$A_{k,k} = -A^t_{k,k},\ A_{r-k,k} = 0_{r-k,k}.$$

Moreover, subdividing C as we did for A, we get that the matrix

$$\begin{pmatrix} A_{k,k} & A_{k,r-k} \\ A_{r-k,k} & A_{r-k,r-k} \end{pmatrix}\begin{pmatrix} I_{k,k} & 0_{k,r-k} \\ 0_{r-k,k} & 0_{r-k,r-k} \end{pmatrix}\begin{pmatrix} C^t_{k,k} & C^t_{r-k,k} \\ C^t_{k,r-k} & C^t_{r-k,r-k} \end{pmatrix} = \begin{pmatrix} A_{k,k}C^t_{k,k} & A_{k,k}C^t_{r-k,k} \\ A_{r-k,k}C^t_{k,k} & A_{r-k,k}C^t_{r-k,k} \end{pmatrix}$$

must be zero. This yields the following further relations for the tangent space of H_1 at the identity

$$C_{k,k} = 0_{k,k},\ C_{r-k,k} = 0_{r-k,k}.$$

Plugging-in the relations for the tangent space at the identity of $Sp(2r)$ in Remark 3.2.2 we get that the tangent space at the identity of H_1 is given by matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where $B = B^t$ and $C = C^t$. Note that $C = C^t, C_{k,k} = 0_{k,k}, C_{r-k,k} = 0_{r-k,k}$ yield $C_{k,r-k} = 0_{k,r-k}$ and $C_{r-k,r-k} = C^t_{r-k,r-k}$. Hence A depends on $\frac{k(k-1)}{2} + k(r-k) + (r-k)^2$ parameters, B depends on $\frac{r(r+1)}{2}$ parameters and C depends on $\frac{(r-k)(r-k+1)}{2}$ parameters. Then by (3.10) we get

$$\dim(H) = \frac{k(k-1)}{2} + k(r-k) + (r-k)^2 + \frac{r(r+1)}{2} + \frac{(r-k)(r-k+1)}{2} + 1$$

and

$$\dim(Y_k) = \dim(Sp(2r)) - \dim(H) = r(2r+1) - \dim(H)$$

yields the formula in the statement.

Finally, consider the case $k = 2r$, and let $H \subset Sp(2r)$ be the stabilizer of the identity matrix. The equality

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{r,r} & 0_{r,r} \\ 0_{r,r} & I_{r,r} \end{pmatrix}\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} &= \begin{pmatrix} AA^t + BB^t & AC^t + BD^t \\ CA^t + DB^t & CC^t + DD^t \end{pmatrix} \\ &= \begin{pmatrix} \lambda I_{r,r} & 0_{r,r} \\ 0_{r,r} & \lambda I_{r,r} \end{pmatrix}\end{aligned}$$

for some $\lambda \in K^*$ yields, applying as usual the transformation (3.8), the following system of equations

$$\begin{cases} -C^t(A-I_{r,r})+(A-I_{r,r})^tC & = & 0_{r,r}; \\ -C^tB^t+(A-I_{r,r})^t(D+I_{r,r}) & = & (D-I_{r,r})^t(A-I_{r,r})-B^tC; \\ -(D-I_{r,r})^tB+B^t(D-I_{r,r}) & = & 0_{r,r}. \end{cases}$$

Note that if $M \in H$ taking the determinants on both sides of $M^T\Omega M = \lambda\Omega$ we see that λ can only take finitely many values. Hence, by Remark 3.2.2 we have the following relations for the tangent space of H at the identity

$$A = -D^t,\ B = B^t,\ C = C^t,\ C = -B^t,\ A = -A^t.$$

Therefore, the tangent space consists of matrices of the following form

$$\begin{pmatrix} A & B \\ -B^t & -A^t \end{pmatrix}$$

with $B = B^t$ and $A = -A^t$. We conclude that

$$\dim(H) = \frac{r(r+1)}{2} + \frac{r(r-1)}{2} = r^2$$

and hence $\dim(Y_{2r}) = r(2r+1) - r^2 = r(r+1)$. $\square$

Corollary 3.2.8. *The projective variety X_{2r} is irreducible and its dimension is given by* $\dim(X_{2r}) = r(r+1)$.

Proof. The variety X_{2r} is the closure of an $Sp(2r)$-orbit, so it is irreducible. Since $X_{2r} = Y_{2r}$ the formula for its dimension follows from Proposition 3.2.6. $\square$

Example 3.2.9. Consider the case $r = 1$. Then Corollary 3.2.8 yields $\dim(X_2) = 2$ and hence $X_2 = \mathbb{P}^2$. Moreover, $O_2 = \mathbb{P}^2 \backslash C$ where $C \subset \mathbb{P}^2$ is the conic parametrizing rank one matrices.

Remark 3.2.10. We work out equations for X_{2r}. The points of the orbit O_{2r} represent symmetric matrices having a scalar multiple that is symplectic, that is $Z^t\Omega Z = \lambda\Omega$ for some $\lambda \in K^*$. The matrix $N = Z^t\Omega Z$ is skew-symmetric and so $N_{i,i} = 0$ for $i = 0, \dots, 2r-1$. Furthermore, for any $i = 0, \dots, 2r-2$ we must have

$$N_{i,i+1} = \cdots = N_{i,r+i-1} = N_{i,r+i+1} = \cdots = N_{i,2r-1} = 0.$$

This gives $2r-i-2$ quadratic equations for any $i = 0, \dots, r-1$, and $2r-i-1$ quadratic equations for any $i = r, \dots, 2r-1$. Moreover, we must have

$$N_{0,r} = N_{1,r+1} = \cdots = N_{r-1,2r-1}$$

and hence we get $r-1$ additional quadratic equations. Summing-up we get

$$\sum_{i=0}^{r-1}(2r-i-2) + \sum_{i=r}^{2r-1}(2r-i-1) + r - 1 = (2r+1)(r-1)$$

quadratic equations for X_{2r} in $\mathbb{P}^{N_+}$.

Now, we explicitly compute these equations. Consider a general symmetric matrix $Z = (z_{i,j})_{i,j=0,\dots,2r-1}$ with $z_{i,j} = z_{j,i}$ and the standard symplectic form Ω. Then

$$c_{i,j} := (Z \cdot \Omega)_{i,j} = \sum_{k=0}^{2r-1} z_{i,k}\Omega_{k,j} = \begin{cases} z_{i,j-r} & \text{for } j \geqslant r; \\ -z_{i,j+r} & \text{for } j < r; \end{cases}$$

and so

$$N_{i,j} := (Z \cdot \Omega \cdot Z)_{i,j} = \sum_{k=0}^{2r-1} c_{i,k}z_{k,j} = \sum_{k=0}^{r-1} c_{i,k}z_{k,j} + \sum_{k=r}^{2r-1} c_{i,k}z_{k,j} = \sum_{k=0}^{r-1} -z_{i,k+r}z_{k,j} + z_{i,k}z_{k+r,j}.$$

Summing-up, the equations

$$\begin{cases} N_{l,r+l} - N_{l+1,r+l+1} = 0 & \text{for } l = 0,\dots,r-2; \\ N_{i,j} = 0 & \text{for } i = 0,\dots,2r-2,\ j > i,\ j \neq r+i; \end{cases}$$

can be explicitly written as follow

$$\begin{cases} \sum_{k=0}^{r-1} -z_{l,k+r}z_{k,r+l} + z_{l,k}z_{k+r,r+l} + z_{l+1,k+r}z_{k,r+l+1} - z_{l+1,k}z_{k+r,r+l+1} = 0 & \text{for } l = 0,\dots,r-2; \\ \sum_{k=0}^{r-1} -z_{i,k+r}z_{k,j} + z_{i,k}z_{k+r,j} = 0 & \text{for } i = 0,\dots,2r-2,\ j > i,\ j \neq r+i \end{cases}$$

Now, our aim is to construct a wonderful compactification of the space of complete symmetric symplectic forms.

Construction 3.2.11. Set $S_h(\mathcal{V}^{2r-1}) := \mathbb{S}ec_h(\mathcal{V}^{2r-1}) \cap X_{2r}$. Let us consider the following sequence of blow-ups:

- $X_{2r}^{(1)}$ is the blow-up of $X_{2r}^{(0)} := X_{2r}$ along the Veronese variety $\mathcal{V}^{2r-1} \subset X_{2r}$;
- $X_{2r}^{(2)}$ is the blow-up of $X_{2r}^{(1)}$ along the strict transform of $S_2(\mathcal{V}^{2r-1})$;

  $\vdots$
- $X_{2r}^{(i)}$ is the blow-up of $X_{2r}^{(i-1)}$ along the strict transform of $S_i(\mathcal{V}^{2r-1})$;

  $\vdots$
- $X_{2r}^{(r-1)}$ is the blow-up of $X_{2r}^{(r-2)}$ along the strict transform of $S_{r-1}(\mathcal{V}^{2r-1})$.

Let $f_i\colon X_{2r}^{(i)} \to X_{2r}^{(i-1)}$ be the blow-up morphism. We will denote by E_i both the exceptional divisor of f_i and its strict transforms in the subsequent blow-ups. We set $\mathcal{S}_{2r} := X_{2r}^{(r-1)}$ and we indicate with $f\colon \mathcal{S}_{2r} \to X_{2r}$ the composition of the f_i.

Let $M_{2r,2r}(K)$ be the space of $2r \times 2r$ matrices. Following [Cb16] we define the operator

$$\begin{array}{rccc} \Phi_\Omega\colon & M_{2r,2r}(K) & \longrightarrow & M_{2r,2r}(K) \\ & A & \mapsto & \Omega^{-1}A^T\Omega \end{array}$$

Definition 3.2.12. A matrix $A \in M_{2r,2r}(K)$ is symplectically congruent to a matrix $B \in M_{2r,2r}(K)$ if there exists a symplectic matrix Q such that $QAQ^T = B$.

By [Cb16, Theorem 21] a matrix $A \in M_{2r,2r}(K)$ is symplectically congruent to a diagonal matrix if and only if A is symmetric and $A\Phi_{\Omega}(A)$ is diagonalizable.

Proposition 3.2.13. *The quadratic equations in Remark 3.2.10 cut out X_{2r} set-theoretically. Furthermore $Y_i = \mathcal{S}ec_i(\mathcal{V}^{2r-1}) \cap X_{2r}$, set-theoretically, and there is a stratification*

$$Y_1 \subset Y_2 \subset \cdots \subset Y_{r-1} \subset Y_r \subset Y_{2r} = X_{2r}.$$

In particular, $\dim(\mathcal{S}ec_i(\mathcal{V}^{2r-1}) \cap X_{2r}) = 2ri + i - i^2 - 1$ *for* $i = 1, \dots, r$ *and* Y_r *is a divisor in* X_{2r}*.*

Proof. Let Z be a symmetric matrix satisfying the equations in Remark 3.2.10. Then we have two cases:

(i) $N_{0,r} = \cdots = N_{r-1,2r-1} = \lambda \in K^*$;

(ii) $N_{0,r} = \cdots = N_{r-1,2r-1} = 0$.

Consider (i). Then $Z^t\Omega Z = \lambda\Omega$ and $\det(Z) \neq 0$. Moreover,

$$Z\Phi_{\Omega}(Z) = Z\Omega^{-1}Z^t\Omega = -Z^t\Omega Z\Omega = -\lambda\Omega^2 = \lambda I_{2r,2r}$$

and by [Cb16, Theorem 21] Z is symplectically congruent to a diagonal matrix.

In case (ii) $Z^t\Omega Z$ is the zero matrix. So $\det(Z) = 0$, and $Z\Phi_{\Omega}(Z)$ is the zero matrix as well. Again, [Cb16, Theorem 21] yields that Z is symplectically congruent to a diagonal matrix.

So if Z is a symmetric matrix satisfying the equations in Remark 3.2.10 there is a symplectic matrix Q such that $QZQ^t = D$ with D diagonal. Our aim is to prove that D can be moved to a matrix of the form I_k, where k is the rank of D, with the action of the symplectic group.

Let $D_\alpha = \mathrm{diag}(\alpha_1, \dots, \alpha_{2r})$ be a diagonal matrix satisfying the equations in Remark 3.2.10. Then either $\alpha_i\alpha_{r+i} = 0$ for all $i = 1, \dots, r$, or $\alpha_i\alpha_{r+i} = \lambda \in K^*$ for all $i = 1, \dots, r$. Write

$$D_\alpha = \begin{pmatrix} D_{\alpha_1,\dots,\alpha_p} & 0_{r,r} \\ 0_{r,r} & D_{\alpha_{p+1},\dots,\alpha_{p+q}} \end{pmatrix}$$

with $p + q \leqslant r$, where $D_{\alpha_1,\dots,\alpha_p}$ is an $r \times r$ diagonal matrix with the α_i appearing on the diagonal, and similarly for $D_{\alpha_{p+1},\dots,\alpha_{p+q}}$. Note that up to permuting the upper and lower diagonal simultaneously we may assume that $\alpha_1, \dots, \alpha_p$ are the first p entries on the diagonal of $D_{\alpha_1,\dots,\alpha_p}$, and $\alpha_{p+1}, \dots, \alpha_{p+q}$ are the last q entries on the diagonal of $D_{\alpha_{p+1},\dots,\alpha_{p+q}}$.

Now, set $p + q = r$. Let A, B, C, D be $r \times r$ matrices defined as follows:

- the first p entries on the diagonal of A are $a_i \in K^*$ for $i = 1, \dots, p$, and the other entries are zero;

- the last q entries on the diagonal of B are $-b_i^{-1} \in K^*$ for $i = p+1, \dots, p+q$, and the other entries are zero;

- the last q entries on the diagonal of C are $b_i \in K^*$ for $i = p+1, \dots, p+q$, and the other entries are zero;

- the first p entries on the diagonal of D are $a_i^{-1} \in K^*$ for $i = 1, \ldots, p$, and the other entries are zero.

Consider the matrix

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and note that P is symplectic. Furthermore, by taking a_i, b_j such that $a_i^2 = \alpha_i$ for $i = 1, \ldots, p$, and $b_j^{-2} = \alpha_j$ for $j = p+1, \ldots, p+q$ we have $P^t I_r P = D_\alpha$ when $p + q = r$.

If $p + q < r$ by permuting the upper diagonal of I_{p+q}, we transform I_{p+q} into the matrix I^*_{p+q} whose entries on the diagonal are $(I^*_{p+q})_{i,i} = 1$ for $i = 1, \ldots, p$, $(I^*_{p+q})_{i,i} = 0$ for $i = p+1, \ldots, p+s$, $(I^*_{p+q})_{i,i} = 1$ for $i = p+s+1, \ldots, p+s+q$, and $(I^*_{p+q})_{i,i} = 0$ for $i = p+s+q+1, \ldots, 2r$, where $p + s + q = r$. In this case consider $r \times r$ diagonal matrices $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ such that

- the first p entries on the diagonal of the matrix $\overline{A}$ are $a_i \in K^*$ for $i = 1, \ldots, p$, $(\overline{A})_{i,i} = 1$ for $i = p+1, \ldots, p+s$, $(\overline{A})_{i,i} = 0$ for $i = p+s+1, \ldots, p+s+q$;
- the first $p+s$ entries on the diagonal of the matrix $\overline{B}$ are zero, followed by $-b_{p+1}^{-1}, \ldots, -b_{p+q}^{-1}$;
- the first $p+s$ entries on the diagonal of the matrix $\overline{C}$ are zero, followed by $b_{p+1}, \ldots, b_{p+q}$;
- the first p entries on the diagonal of the matrix $\overline{D}$ are $a_i^{-1} \in K^*$ for $i = 1, \ldots, p$, $(\overline{D})_{i,i} = 1$ for $i = p+1, \ldots, p+s$, $(\overline{D})_{i,i} = 0$ for $i = p+s+1, \ldots, p+s+q$;

and set

$$\overline{P} = \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}.$$

Then $\overline{P}$ is symplectic and by taking again a_i, b_j such that $a_i^2 = \alpha_i$ for $i = 1, \ldots, p$, and $b_j^{-2} = \alpha_j$ for $j = p+1, \ldots, p+q$ it holds $\overline{P}^t I^*_{p+q} \overline{P} = D_\alpha$.

Furthermore, when D_α is of maximal rank we consider the diagonal symplectic matrix

$$P = \operatorname{diag}(a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}).$$

Note that taking $a_i \in K^*$ such that $a_i^2 = \alpha_i \mu^{-1}$, with $\mu^2 = \lambda$, for $i = 1, \ldots, r$, we get that $P^t I_{2r,2r} P$ is a scalar multiple of D_α. Consider the matrices

$$\Psi_t = \begin{pmatrix} I_{r,r} & 0_{r,r} \\ 0_{r,r} & T_{r,r} \end{pmatrix};$$

$$\Lambda_t = \begin{pmatrix} I_{k,k} & 0 & 0_{k,2r-k-1} \\ 0 & t & 0_{1,2r-k-1} \\ 0_{2r-k-1,k} & 0_{2r-k-1,1} & 0_{2r-k-1,2r-k-1} \end{pmatrix} \text{ for } k = 1, \ldots, r-1$$

where $T_{r,r} = \mathrm{diag}(t, \dots, t)$. By the first part of the proof we have that $\{\Psi_t\}_{t\in K^*}$ is a family of matrices in O_{2r}, and $\lim_{t\mapsto 0}\Psi_t = I_r$. Furthermore, $\{\Lambda_t\}_{t\in K^*}$ is a family of matrices in O_{k+1}, and $\lim_{t\mapsto 0}\Lambda_t = I_k$ for $k = 1, \dots, r-1$.

Summing-up we proved that if Z is a symmetric matrix for rank k with $1 \leqslant k \leqslant r$ or $k = 2r$ satisfying the equations in Remark 3.2.10 then Z can be symplectically transformed into the matrix I_k, and hence it lies in O_k. □

Remark 3.2.14. Proposition 3.2.13 yields that the Veronese variety $\mathcal{V}^{2r-1}$ is contained in X_{2r}. On the other hand, for $h \geqslant 2$ the secant variety $\mathbb{S}ec_h(\mathcal{V}^{2r-1})$ is not contained in X_{2r}.

Proposition 3.2.15. *For any $k \geqslant r$ we have that $X_{2r} \cap \mathbb{S}ec_r(\mathcal{V}^{2r-1}) = X_{2r} \cap \mathbb{S}ec_k(\mathcal{V}^{2r-1})$*

Proof. Assume there is a matrix $M \in X_{2r} \cap \mathbb{S}ec_k(\mathcal{V}^{2r-1})$ of rank $r < k < 2r-1$. Arguing as in the proof of Proposition 3.2.13 we can move M with the action of $Sp(2r)$ in a diagonal matrix D_k of rank k, and $D_k \in X_{2r} \cap \mathbb{S}ec_k(\mathcal{V}^{2r-1})$. However, D_k does not satisfy the equation $N_{j,r+j} - N_{j+1,r+j+1} = 0$ for X_{2r} in Remark 3.2.10. A contradiction. □

We analyze in detail the geometry of the objects we introduced in the first non trivial case, namely $r = 2$.

Proposition 3.2.16. *The variety X_4 is isomorphic to the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ of lines in $\mathbb{P}^4$. Furthermore, $\mathcal{V}^3 \subset X_4$, and $S_2(\mathcal{V}^3) \subset X_4$ is an irreducible and reduced divisor singular along $\mathcal{V}^3$. In particular, the equations in Remark 3.2.10 cut out X_4 scheme-theoretically, and $S_2(\mathcal{V}^3) = Y_2$ scheme-theoretically.*

Proof. Consider homogeneous coordinates $z_{i,j}$ on $\mathbb{P}^9$ and identify them with the entries of a general 4×4 symmetric matrix Z. The change of variables:

$$z_{0,0} \mapsto z_{1,2},\ z_{0,1} \mapsto z_{0,1},\ z_{0,2} \mapsto \frac{z_{1,4} - z_{2,3}}{2},\ z_{0,3} \mapsto z_{0,2},\ z_{1,1} \mapsto z_{1,3},$$

$$z_{1,2} \mapsto z_{0,3},\ z_{1,3} \mapsto \frac{z_{1,4} + z_{2,3}}{2},\ z_{2,2} \mapsto z_{3,4},\ z_{2,3} \mapsto z_{0,4},\ z_{3,3} \mapsto z_{2,4}$$

transforms the five equations in Remark 3.2.10 into the standard Plücker equations cutting out $\mathbb{G}(1,4)$ in $\mathbb{P}^9$.

By Remark 3.2.14 we have $\mathcal{V}^3 \subset X_4$. We can compute the tangent cones of $S_2(\mathcal{V}^3) \subset X_4$ at a point representing a rank one matrix, and at a point representing a rank two matrix using the equations for X_4 in Remark 3.2.10 together with the equations cutting out $\mathbb{S}ec_2(\mathcal{V}^3)$. In the first case we get a cone with a 3-dimensional vertex over $\mathcal{V}_1^2$ which in particular is irreducible and reduced, and in the second case we get a 5-dimensional linear space. Finally, since by Proposition 3.2.13 Y_2 has dimension five we conclude that the equations in Remark 3.2.10 together with the equations cutting out $\mathbb{S}ec_2(\mathcal{V}^3)$ define Y_2 scheme-theoretically. □

Remark 3.2.17. The variety $X_4 \cong \mathbb{G}(1,4)$ has been studied in relation to moduli spaces of rank two vector bundles over a smooth quadric [OS94, Table I].

Proposition 3.2.18. *The tangent cone of X_{2r} at a point $p_k \in S_k(\mathcal{V}^{2r-1}) \backslash S_{k-1}(\mathcal{V}^{2r-1})$ for $k = 1, \dots, r-1$ is a cone with vertex of dimension $k(2r+1-k)-1$ over $X_{2(r-k)}$. Moreover, X_{2r} is smooth along $S_r(\mathcal{V}^{2r-1}) \backslash S_{r-2}(\mathcal{V}^{2r-1})$, and the equations in Remark 3.2.10 define X_{2r} scheme-theoretically.*

The tangent cone of $S_h(\mathcal{V}^{2r-1})$ at a point $p_k \in S_k(\mathcal{V}^{2r-1}) \backslash S_{k-1}(\mathcal{V}^{2r-1})$ for $k = 1, \dots, r-1$, $k < h$ is a cone with vertex of dimension $k(2r+1-k)-1$ over the variety $S_{h-k}(\mathcal{V}^{2(r-k)-1})$. Moreover, the equations in Remark 3.2.10 together with the equations cutting out $\mathcal{S}ec_h(\mathcal{V}^{2r-1})$ define $S_h(\mathcal{V}^{2r-1})$ scheme-theoretically.

In particular, X_{2r} is smooth along $S_r(\mathcal{V}^{2r-1}) \backslash S_{r-2}(\mathcal{V}^{2r-1})$ and $S_r(\mathcal{V}^{2r-1})$ is a divisor in X_{2r}.

Proof. Let $p_k = (p_{i,j})_{i,j=0,\dots,2r-1, i \leqslant j}$ be the point representing the standard matrix of rank k with $p_{i,i} = 1$ for $i = 0, \dots, k-1$ and $p_{i,j} = 0$ otherwise.

We proceed by induction on r. The base case $r = 2$ is in Proposition 3.2.16. We will use the equations in Remark 3.2.10 to compute $TC_{p_k} X_{2r}$. Consider the change of coordinates $z_{i,i} \mapsto z_{i,i} - z_{0,0}$ for $i = 1, \dots, k-1$, and set $z_{0,0} = 1$. Note that the lowest degree terms of the equations in Remark 3.2.10 after this change of coordinates are obtained by removing from $Z^t \Omega Z = \lambda \Omega$ the rows and columns indexed by $0, \dots, k-1, r, \dots, r+k-1$. Therefore, we get a cone with vertex of dimension $k(2r+1-k)-1$ over $X_{2(r-k)}$ which by induction hypothesis is irreducible and reduced since the equations in Remark 3.2.10 define $X_{2(r-k)}$ scheme-theoretically. Now, $k(2r+1-k) + \dim(X_{2(r-k)}) = \dim(X_{2r})$ yields that this is the tangent cone $TC_{p_k} X_{2r}$, and hence the equations in Remark 3.2.10 define X_{2r} scheme-theoretically. Note that at the points representing I_r and $I_{2r,2r}$ the equations in Remark 3.2.10 yield a linear subspace of the same dimension of X_{2r}.

Now, consider $S_h(\mathcal{V}^{2r-1})$. Note that $TC_{p_k} S_h(\mathcal{V}^{2r-1})$ is contained in $TC_{p_k} X_{2r} \cap TC_{p_k} \mathcal{S}ec_h(\mathcal{V}^{2r-1})$. By the previous computation of $TC_{p_k} X_{2r}$ and the computation of $TC_{p_k} \mathcal{S}ec_h(\mathcal{V}^{2r-1})$ in Proposition 3.1.9, we conclude that $TC_{p_k} X_{2r} \cap TC_{p_k} \mathcal{S}ec_h(\mathcal{V}^{2r-1})$ is a cone with vertex of dimension $k(2r+1-k)-1$ over the variety $S_h(\mathcal{V}^{2(r-k)-1}) = \mathcal{S}ec_{h-k}(\mathcal{V}^{2(r-k)-1}) \cap X_{2(r-k)}$. Again by induction this is an irreducible and reduced cone which by the computation of the dimension of $S_h(\mathcal{V}^{2r-1})$ in Proposition 3.2.13 must coincide with $TC_{p_k} S_h$. Hence the equations in Remark 3.2.10 together with the equations cutting out $\mathcal{S}ec_h(\mathcal{V}^{2r-1})$ define $S_h(\mathcal{V}^{2r-1})$ scheme-theoretically. $\square$

Now, we are ready to prove the main result of this section. We will denote by $S_h^{(i)}(\mathcal{V}^{2r-1})$ the strict transform of $S_h(\mathcal{V}^{2r-1})$ in $X_{2r}^{(i)}$.

Theorem 3.2.19. *For any $i = 1, \dots, r-1$ the strict transform $S_{i+1}^{(i)}(\mathcal{V}^{2r-1})$ of $S_{i+1}(\mathcal{V}^{2r-1})$ in $X_{2r}^{(i)}$ is smooth. Moreover, the variety $\mathcal{S}_{2r}$ is smooth and the exceptional divisors $E_1, \dots, E_{r-1} \subset \mathcal{S}_{2r}$ are smooth as well.*

The closures of the orbits of the $Sp(2r)$-action on $\mathcal{S}_{2r}$ induced by the action in (3.9) are given by all the possible intersections among $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ and $X_{2r}^{(i)}$ itself.

In particular, the variety $\mathcal{S}_{2r}$ with boundary divisors $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ is wonderful.

Proof. For every r in $X_{2r}^{(0)}$ we have $S_1^{(0)}(\mathcal{V}^{2r-1}) = \mathcal{V}^{2r-1}$ which is smooth. We will assume that $S_j^{(j-1)}(\mathcal{V}^{2r-1})$ is smooth for every r and for every $j < i$ and prove that also

$S_i^{(i-1)}(\mathcal{V}^{2r-1})$ in $X_{2r}^{(i-1)}$ is smooth. We have $S_i^{(i-1)}(\mathcal{V}^{2r-1}) = \mathbb{S}ec_i^{(i-1)}(\mathcal{V}^{2r-1}) \cap X_{2r}^{(i-1)}$, so we consider $S_i^{(i-1)}(\mathcal{V}^{2r-1})$ inside $\mathcal{Q}(2r-1)_{(i-1)}$. As remarked in Proposition 3.1.11, for every r and for every $i = 0, \dots, 2r-1$ the varieties $\mathcal{Q}(2r-1)_{(i-1)}$, $\mathbb{S}ec_i^{(i-1)}(\mathcal{V}^{2r-1})$, $E_1^q, \dots, E_{i-1}^q$ are smooth.

Now, $S_i^{(i-1)}(\mathcal{V}^{2r-1})$ is smooth away from $E_1^q, \dots, E_{i-1}^q$. Moreover, by Proposition 3.2.18 for every $k = 1, \dots, i-1$, $S_i^{(i-1)}(\mathcal{V}^{2r-1}) \cap E_k^q \to S_k^{(k-1)}(\mathcal{V}^{2r-1})$ is a fibration with fibers isomorphic to $S_{i-k}^{(i-k-1)}(\mathcal{V}^{2(r-k)-1})$ which is smooth by induction. Proposition 3.1.10 yields that $S_i^{(i-1)}(\mathcal{V}^{2r-1}) \cap E_k^q$ is smooth for $k = 1, \dots, i-1$. Now, since by Proposition 3.2.13 we have

$$\begin{aligned}\dim S_k^{(k-1)}(\mathcal{V}^{2r-1}) + \dim S_{i-k}^{(i-k-1)}(\mathcal{V}_2^{2(r-k)-1}) &= 2ri + i - i^2 - 2 \\ &= \dim S_i^{(i-1)}(\mathcal{V}^{2r-1}) - 1\end{aligned}$$

we get that $S_i^{(i-1)}(\mathcal{V}^{2r-1})$ is smooth as well.

By Proposition 3.2.18 for every r, in $X_{2r}^{(1)}$ we have that $E_1 \cap S_2^{(1)}(\mathcal{V}^{2r-1}) \to \mathcal{V}^{2r-1}$ is a fibration with fibers isomorphic to $\mathcal{V}^{2(r-1)-1}$ and then by Proposition 3.1.10 $E_1 \cap S_2^{(1)}(\mathcal{V}^{2r-1})$ is smooth of dimension $4r - 4 = \dim(S_2^{(1)}(\mathcal{V}^{2r-1})) - 1$. More generally, consider intersections of the form $S_{i+1}^{(i)}(\mathcal{V}^{2r-1}) \cap E_{j_1} \cap \dots \cap E_{j_t}$, for $1 \leqslant j_1 < \dots < j_t \leqslant i$. By Proposition 3.2.18, the restriction of the blow-down morphism

$$S_{i+1}^{(i)}(\mathcal{V}^{2r-1}) \cap E_{j_1} \cap \dots \cap E_{j_t} \to E_{j_1} \cap \dots \cap E_{j_{t-1}} \cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1})$$

has fibers isomorphic to $S_{i+1-j_t}^{(i-j_t)}(\mathcal{V}^{2(r-j_t)-1})$.

Now, by Proposition 3.1.10 $S_{i+1}^{(i)}(\mathcal{V}^{2r-1}) \cap E_{j_1} \cap \dots \cap E_{j_t}$ is smooth since we proved before that $S_{i+1-j_t}^{(i-j_t)}(\mathcal{V}^{2(r-j_t)-1})$ is smooth and $E_{j_1} \cap \dots \cap E_{j_{t-1}} \cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1})$ is smooth by induction. Moreover,

$$\begin{aligned}\dim(S_{i+1}^{(i)}(\mathcal{V}^{2r-1}) \cap E_{j_1} \cap \dots \cap E_{j_t}) = \dim(E_{j_1} \cap \dots \cap E_{j_{t-1}} \cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1})) \\ + \dim(S_{i+1-j_t}^{(i-j_t)}(\mathcal{V}^{2(r-j_t)-1}))\end{aligned}$$

and by induction $\dim(E_{j_1} \cap \dots \cap E_{j_{t-1}} \cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1})) = 2rj_t + j_t - j_t^2 - 1 - (t-1)$. This yields, using Proposition 3.2.13, that

$$\begin{aligned}\dim(S_{i+1}^{(i)}(\mathcal{V}^{2r-1}) \cap E_{j_1} \cap \dots \cap E_{j_t}) &= 2r(i+1) + (i+1) - (i+1)^2 - 1 - t \\ &= \dim(S_{i+1}^{(i)}(\mathcal{V}^{2r-1})) - t.\end{aligned} \tag{3.11}$$

Now, consider the variety $\mathcal{S}_{2r}$ as a subvariety of the variety $\mathcal{Q}(2r-1)_{r-1}$ in Construction 3.1.6. By Proposition 3.2.18 $\mathcal{S}_{2r}$ is smooth away from the exceptional divisors. Furthermore, the exceptional divisor E_i^q in Construction 3.1.6 intersects $\mathcal{S}_{2r}$ in the exceptional divisor E_i in Construction 3.2.11. By Proposition 3.2.18 $E_i \to S_i^{(i-1)}(\mathcal{V}^{2r-1})$ is a fibration with $\mathcal{S}_{2(r-i)}$ as fiber. Hence $E_i^q \cap \mathcal{S}_{2r}$ is a smooth divisor in $\mathcal{S}_{2r}$ and therefore $\mathcal{S}_{2r}$ is smooth.

Now, consider an intersection of the form $E_{j_1} \cap \dots \cap E_{j_t}$ and the fibration

$$E_{j_1} \cap \dots \cap E_{j_t} \to E_{j_1} \cap \dots E_{j_{t-1}} \cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1}).$$

By Proposition 3.2.18 this fibration has fibers isomorphic to $\mathcal{S}_{2(r-j_t)}$. By the previous part of the proof we have that

$$\dim(E_{j_1}\cap\dots E_{j_{t-1}}\cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1}))+\dim(\mathcal{S}_{2(r-j_t)})=r^2+r-t=\dim(\mathcal{S}_{2r})-t$$

and hence the intersection $E_{j_1}\cap\dots\cap E_{j_t}$ is transversal. Note also that considering the fibration

$$S_r^{(r-1)}(\mathcal{V}^{2r-1})\cap E_{j_1}\cap\dots\cap E_{j_t}\to E_{j_1}\cap\dots\cap E_{j_{t-1}}\cap S_{j_t}^{(j_t-1)}(\mathcal{V}^{2r-1})$$

and (3.11) we get that the intersection $S_r^{(r-1)}(\mathcal{V}^{2r-1})\cap E_{j_1}\cap\dots\cap E_{j_t}$ is transversal as well.

Finally, for the claim about the orbit closures it is enough to recall that the $Sp(2r)$-action on $\mathcal{S}_{2r}$ is the restriction of the $SL(2r)$-action on $\mathcal{Q}(2r-1)_{r-1}$ in (3.3) and to use the statement about the orbit closures in Proposition 3.1.11. □

Proposition 3.2.21. *We have that*

$$\operatorname{mult}_{S_r(\mathcal{V}^{2r-1})}S_{2r-1}(\mathcal{V}^{2r-1})=r.$$

Moreover, if $H_{X_{2r}}$ is the hyperplane section of X_{2r}, we have that $S_r(\mathcal{V}^{2r-1})\sim 2H_{X_{2r}}$.

Proof. We will compute the tangent cone of the variety $S_r(\mathcal{V}^{2r-1})$ at the point $p_r=(p_{i,j})_{i,j=0,\dots,2r-1}$, where $p_{i,i}=1$ for $i=0,\dots,r-1$ and $p_{i,j}=0$ otherwise.

Consider the change of coordinates $z_{i,i}\mapsto z_{i,i}-z_{0,0}$ ans set $z_{0,0}=1$. By Remark 3.2.10 the tangent space of X_{2r} at p_r is cut out by a set of linear equations and among these equations we have $\{z_{i,j}=0\}$ for $i,j=r,\dots,2r-2$, and $z_{i,i}=z_{i+1,i+1}$ for $i=r,\dots,2r-2$.

Now, the tangent cone of $\mathrm{Sec}_{2r-1}(\mathcal{V}_2^{2r-1})$ is cut out by the determinant of the bottom right $r\times r$ submatrix of the matrix Z in (3.1). Note that substituting the relations on the $z_{i,j}$ above in this determinant we get $z_{2r-1,2r-1}^r$.

By Proposition 3.2.15 $\mathrm{Sec}_{2r-1}(\mathcal{V}_2^{2r-1})$ and $\mathrm{Sec}_r(\mathcal{V}_2^{2r-1})$ cut out on X_{2r} the same divisor set-theoretically. The previous computation yields that $\mathrm{Sec}_{2r-1}(\mathcal{V}_2^{2r-1})$ cuts out $\mathrm{Sec}_r(\mathcal{V}_2^{2r-1})\cap X_{2r}$ on X_{2r} with multiplicity r.

Now, recall that by Remark 3.1.8 $\deg(\mathrm{Sec}_{2r-1}(\mathcal{V}^{2r-1}))=2r$. Let D be the divisor $\mathrm{Sec}_r(\mathcal{V}^{2r-1})\cap X_{2r}$. Finally $\mathrm{Sec}_{2r-1}(\mathcal{V}^{2r-1})\cap X_{2r}\sim 2rH_{X_{2r}}$ yields $D\sim 2H_{X_{2r}}$. □

Remark 3.2.22. In the case $r=2$ we worked out explicitly the quadratic polynomial cutting out $S_2(\mathcal{V}^3)$ in X_4 and we got that $S_2(\mathcal{V}^3)=X_4\cap\{z_{0,3}z_{1,2}+z_{1,3}^2-z_{0,1}z_{2,3}-z_{1,1}z_{3,3}=0\}$.

3.2.1 *Divisors on $\mathcal{S}_{2r}$*

In this section we will study the Picard rank and the cones of effective and nef divisors of the wonderful compactification $\mathcal{S}_{2r}$. We will need the following result.

Lemma 3.2.23. *Let $SO(2r)$ be the special orthogonal group. Then*

$$SO(2r)\cap Sp(2r)\cong GL(r).$$

In particular, $SO(2)\cong GL(1)\cong K^*$.

Proof. Consider the bilinear symmetric form given by the matrix $J = \begin{pmatrix} 0_{r,r} & I_{r,r} \\ I_{r,r} & 0_{r,r} \end{pmatrix}$.
Set $N = \begin{pmatrix} I_{r,r} & \xi I_{r,r} \\ \frac{1}{2} I_{r,r} & -\frac{\xi}{2} I_{r,r} \end{pmatrix}$, with $\xi^2 = -1$. Note that $N^t J N = I_{2r,2r}$ and $N^t \Omega N = -\xi\Omega$.
Therefore, we may prove the statement for the intersection $SO_J(2r) \cap Sp(2r)$, where $SO_J(2r)$ is the group of determinant one matrices which are orthogonal with respect to J.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2r)$ be a general $2r \times 2r$ invertible matrix, where A, B, C, D are $r \times r$ matrices. Now, $M \in SO_J(2r) \cap Sp(2r)$ if and only if

$$\begin{cases} A^t C = 0_{r,r}; \\ A^t D = I_{r,r}; \\ B^t C = 0_{r,r}; \\ B^t D = 0_{r,r}; \end{cases} \quad \text{that is} \quad \begin{cases} D = A^{-t}; \\ C = 0_{r,r}; \\ B = 0_{r,r}; \end{cases}$$

and hence

$$SO(2r) \cap Sp(2r) \cong SO_J(2r) \cap Sp(2r) = \left\{ \begin{pmatrix} A & 0_{r,r} \\ 0_{r,r} & A^{-t} \end{pmatrix} \text{ for } A \in GL(r) \right\} \cong GL(r).$$

For the last claim in the case $r = 1$ it is enough to note that $SO(2) \cap Sp(2) = SO(2)$. In fact every 2×2 matrix with determinant one is symplectic. $\square$

Proposition 3.2.24. *Let $O_{2r} \subset X_{2r}$ be the orbit of the identity. Then* $\mathrm{Pic}(O_{2r}) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The group $G = Sp(2r)$ is semi-simple and simply connected. If $H \subset G$ is the stabilizer of the identity then [ADHL15, Theorem 4.5.1.2] yields that $\mathrm{Pic}(G/H) \cong \mathbb{X}(H)$, where $\mathbb{X}(H)$ is the group of characters of H. We have that

$$H = \{M \in Sp(2r), MM^t = \lambda_M I_{2r,2r}, \text{for some } \lambda_M \in K^*\}.$$

Then, for a general element $M \in H$ we have

$$\begin{cases} MM^t = \lambda_M I_{2r,2r}; \\ M^t \Omega M = \Omega; \end{cases} \Rightarrow \lambda_M M^{-1} \Omega M = \Omega \Rightarrow \lambda_M \Omega M = M\Omega.$$

Let v be an eigenvector of Ω with eigenvalue μ. Then

$$\lambda_M \Omega M v = M\Omega v = M\mu v = \mu M v.$$

Setting $y = Mv$ we have $\Omega y = (\lambda_M^{-1}\mu) y$ and so y is an eigenvector of Ω with eigenvalue $\lambda_M^{-1}\mu$. The characteristic polynomial of Ω is $P_\Omega(\lambda) = (\lambda - \xi)^r (\lambda + \xi)^r$ where $\xi^2 = -1$. Therefore the only eigenvalues of Ω are ξ and $-\xi$. So

$$\begin{cases} \mu = \pm\xi; \\ \lambda_M^{-1}\mu = \pm\xi; \end{cases} \Rightarrow \lambda_M^{-1} = \pm 1 \Rightarrow \lambda_M = \pm 1$$

and there is a morphism of groups

$$\begin{aligned}\varphi\colon H &\longrightarrow \mathbb{Z}/2\mathbb{Z}\\ M &\longmapsto \lambda_M\end{aligned}$$

The morphism φ is surjective. Indeed we have $\varphi(I_{2r,2r}) = 1$, and if $S = \begin{pmatrix} 0_{r,r} & \xi I_{r,r} \\ \xi I_{r,r} & 0_{r,r}\end{pmatrix}$ then $S^t\Omega S = \Omega$, $SS^t = -I_{2r,2r}$, $S \in H$ and $\varphi(S) = -1$. This yields an exact sequence

$$1 \to \overline{H} \to H \to \mathbb{Z}/2\mathbb{Z} \to 1 \tag{3.12}$$

where $\overline{H} = \{M \in Sp(2r), MM^t = I_{2r,2r}\}$, and we can write $H = \overline{H} \cup S\overline{H}$.

As in Lemma 3.2.23, we consider the bilinear form $J = \begin{pmatrix} 0_{r,r} & I_{r,r} \\ I_{r,r} & 0_{r,r}\end{pmatrix}$, which is congruent to the bilinear form $I_{2r,2r}$ via the matrix $N = \begin{pmatrix} I_{r,r} & \xi I_{r,r} \\ \frac{1}{2}I_{r,r} & -\frac{\xi}{2}I_{r,r}\end{pmatrix}$, where $\xi^2 = -1$. Set $\overline{H}_J = \{M \in Sp(2r), MJM^t = J\}$ and $H_J = \{M \in Sp(2r), MJM^t = \lambda_M J, \text{for some } \lambda_M \in K^*\}$. There is an isomorphism

$$\begin{aligned}\alpha\colon H &\longrightarrow H_J\\ M &\longmapsto NMN^{-1}\end{aligned}$$

such that $\alpha(\overline{H}) = \overline{H}_J$, $\tilde{S} := \alpha(S) = \begin{pmatrix} 0 & -2I_{r,r} \\ \frac{1}{2}I_{r,r} & 0\end{pmatrix}$ and $H_J = \overline{H}_J \cup \tilde{S}\overline{H}_J$. Take $B \in H_J$ and consider $\alpha^{-1}(B) \in H$. By the first part of the proof there is a morphism of groups $H_J \to \mathbb{Z}/2\mathbb{Z}$ mapping B to $\lambda_{\alpha^{-1}(B)}$, and fitting in the following exact sequence

$$1 \to \overline{H}_J \to H_J \to \mathbb{Z}/2\mathbb{Z} \to 1$$

Since $H_J/\overline{H}_J$ is abelian the commutator $[H_J, H_J]$ of H_J is contained in $\overline{H}_J$. By the proof of Lemma 3.2.23 we have that an element $h \in \overline{H}_J$ is of the form $h = \begin{pmatrix} A & 0_{r,r} \\ 0_{r,r} & A^{-t}\end{pmatrix}$ for $A \in GL(r)$. Then $h^{-1} = \begin{pmatrix} A^{-1} & 0_{r,r} \\ 0_{r,r} & A^{t}\end{pmatrix}$ for $A \in GL(r)$. Furthermore $\tilde{S}^{-1} = \begin{pmatrix} 0_{r,r} & 2I_{r,r} \\ -\frac{1}{2}I_{r,r} & 0_{r,r}\end{pmatrix}$.

Therefore:

$$\begin{aligned}[\tilde{S}, h] &= \tilde{S}h\tilde{S}^{-1}h^{-1}\\ &= \begin{pmatrix} 0_{r,r} & -2I_{r,r} \\ \frac{1}{2}I_{r,r} & 0_{r,r}\end{pmatrix}\begin{pmatrix} A & 0_{r,r} \\ 0_{r,r} & A^{-t}\end{pmatrix}\begin{pmatrix} 0_{r,r} & 2I_{r,r} \\ -\frac{1}{2}I_{r,r} & 0_{r,r}\end{pmatrix}\begin{pmatrix} A^{-1} & 0_{r,r} \\ 0_{r,r} & A^{t}\end{pmatrix}\\ &= \begin{pmatrix} A^{-t}A^{-1} & 0_{r,r} \\ 0_{r,r} & AA^{t}\end{pmatrix}.\end{aligned}$$

Setting $B = A^{-t}A^{-1}$, we have $B^{-t} = (A^{-t}A^{-1})^{-t} = AA^t$ with $B \in GL(r)$ symmetric. So $[H_J, H_J]$ is the subgroup of $\overline{H}_J \cong GL(r)$ generated by symmetric matrices and since by [Bos86, Theorem 1] all $r \times r$ matrices can be written as product of symmetric matrices we get $[H_J, H_J] = \overline{H}_J$.

Then, $H/[H,H] \cong H_J/[H_J,H_J] \cong H_J/\overline{H}_J \cong H/\overline{H}$ and by the exact sequence (3.12) we have $H/\overline{H} \cong \mathbb{Z}/2\mathbb{Z}$.

Finally, by [Bur65, Lemma 22.2] $\mathbb{X}(H) \cong \mathbb{X}(H/[H,H])$, and hence $\mathrm{Pic}(G/H) \cong \mathbb{X}(H) \cong \mathbb{Z}/2\mathbb{Z}$. □

Now, we are ready to compute the Picard rank and the colors of the wonderful variety $\mathcal{S}_{2r}$.

Proposition 3.2.26. *The Picard rank of $\mathcal{S}_{2r}$ is $\rho(\mathcal{S}_{2r}) = r$.*

Proof. As before set $G = Sp(2r)$ and let H be the stabilizer of the identity. By Theorem 3.2.19 the variety $\mathcal{S}_{2r}$ is wonderful with boundary divisors $E_1, \dots, E_{r-1}$, $S_r^{(r-1)}(\mathcal{V}^{2r-1})$. By [Bri07, Proposition 2.2.1] there is an exact sequence

$$0 \to \mathbb{Z}^r \to \mathrm{Pic}(\mathcal{S}_{2r}) \to \mathrm{Pic}(G/H) \to 0$$

Hence, Proposition 3.2.24 yields that the Picard rank of $\mathcal{S}_{2r}$ is r. □

For $i = 1, \dots, r$ we define the divisors D_i as the strict transforms in $\mathcal{S}_{2r}$ of the divisor given by the intersection of

$$\det \begin{pmatrix} z_{0,0} & \cdots & z_{0,i-1} \\ \vdots & \ddots & \vdots \\ z_{0,i-1} & \cdots & z_{i-1,i-1} \end{pmatrix} = 0$$

with X_{2r}.

Proposition 3.2.27. *The set of boundary divisors of $\mathcal{S}_{2r}$ is $\{E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})\}$ while the set of colors of $\mathcal{S}_{2r}$ is $\{D_1, \dots, D_r\}$.*

Proof. The claim on the set of boundary divisors follows from Theorem 3.2.19. We compute the colors. We first prove that $D_r \subseteq \mathcal{S}_{2r}$ is stabilized by the Borel subgroup. Consider a matrix $Z = \begin{pmatrix} Z_{0,0} & Z_{0,1} \\ Z_{0,1} & Z_{1,1} \end{pmatrix}$ where the $Z_{i,j}$ are $r \times r$ matrices. Let $M = \begin{pmatrix} A & 0_{r,r} \\ B & A^{-t} \end{pmatrix} \in \mathscr{B}$, then

$$\begin{aligned} \bar{Z} &= M \cdot Z \cdot M^t \\ &= \begin{pmatrix} AZ_{0,0}A^t & AZ_{0,0}B^t + AZ_{0,1}A^{-1} \\ BZ_{0,0}A^t + A^{-t}Z_{0,1}A^t & BZ_{0,0}B^t + A^{-t}Z_{0,1}B^t + BZ_{0,1}A^{-1} + A^{-t}Z_{1,1}A^{-1} \end{pmatrix} \end{aligned}$$

and $\det(\bar{Z}_{0,0}) = \det(AZ_{0,0}A^t) = \det(A)^2 \det(Z_{0,0})$ where $\det(A) \neq 0$ since $A \in GL(r)$. Therefore, D_r is stabilized by the Borel subgroup.

We focus now on the block $\bar{Z}_{0,0}$ of the matrix $\bar{Z}$. We divide the matrices A and $Z_{0,0}$ respectively in blocks $A_{j,k}$, $W_{j,k}$ of matrices $j \times k$ as follows $A = \begin{pmatrix} A_{i,i} & A_{i,r-i} \\ A_{r-i,i} & A_{r-i,r-i} \end{pmatrix}$

and $Z_{0,0} = \begin{pmatrix} W_{i,i} & W_{i,r-i} \\ W_{r-i,i} & W_{r-i,r-i} \end{pmatrix}$. Recall that by Remark 3.2.4 the matrix A is lower triangular. We have $\bar{Z}_{0,0} = \begin{pmatrix} \bar{W}_{i,i} & \bar{W}_{i,r-i} \\ \bar{W}_{r-i,i} & \bar{W}_{r-i,r-i} \end{pmatrix}$ with $\bar{W}_{i,i} = A_{i,i}W_{i,i}A_{i,i}^t$. The divisor D_i is defined by $\det(W_{i,i}) = 0$ and since $\det(A) = \det(A_{i,i})\det(A_{r-i,r-i}) \neq 0$ we get that D_i is stabilized by $\mathscr{B}$ for $i = 1, \dots r$.

As noticed in [ADHL15, Remark 4.5.5.3], if $(X, \mathscr{G}, \mathscr{B}, x_0)$ is a spherical wonderful variety with colors $D_1, \dots, D_s$ the big cell $X \backslash (D_1 \cup \dots \cup D_s)$ is an affine space. Therefore, it admits only constant invertible global functions and $\mathrm{Pic}(X)$ is generated by $D_1, \dots, D_s$.

Therefore, in order to conclude that we found all the colors of $\mathcal{S}_{2r}$ it is enough to recall that by Proposition 3.2.26 $\mathcal{S}_{2r}$ has Picard rank r. $\square$

In the following we will denote by H the pull-back in $\mathcal{S}_{2r}$ of the hyperplane section of X_{2r}. By Proposition 3.2.26 $H, E_1 \dots, E_{r-1}$ generate $\mathrm{Pic}(\mathcal{S}_{2r})$.

Proposition 3.2.28. *The extremal rays of the effective cone* $\mathrm{Eff}(\mathcal{S}_{2r})$ *are generated by* $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ *and the extremal rays of the nef cone* $\mathrm{Nef}(\mathcal{S}_{2r})$ *are generated by* $D_1, \dots, D_r$.

Proof. By [ADHL15, Proposition 4.5.4.4] and Proposition 3.2.27 $\mathrm{Eff}(\mathcal{S}_{2r})$ is generated by $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$ and $D_1, \dots, D_r$.

Note that by Constructions 3.1.6 and 3.2.11 there as an inclusion $i \colon \mathcal{S}_{2r} \to \mathcal{Q}(2r-1)_{r-1}$ inducing an isomorphism of the Picard groups. By [Hue15, Section 2] the linear system on $\mathcal{Q}(2r-1)_{r-1}$ that restricts to the linear system of D_i on $\mathcal{S}_{2r}$ induces a birational morphism $\mathcal{Q}(2r-1)_{r-1} \to W_i$ whose exceptional locus is contained in the union of the exceptional divisors in Construction 3.1.6. Therefore, D_i induces a birational morphism $\mathcal{S}_{2r} \to Z_i$ and hence D_i lies in the interior of the effective cone of $\mathcal{S}_{2r}$ for any $i = 1, \dots, r$. This proves that the effective cone of $\mathcal{S}_{2r}$ is generated by $E_1, \dots, E_{r-1}, S_r^{(r-1)}(\mathcal{V}^{2r-1})$. Finally, by [Bri89, Section 2.6] $D_1, \dots, D_r$ generate the extremal rays of the nef cone. $\square$

In order to study the birational geometry of $\mathcal{S}_{2r}$ we will need the following result.

Proposition 3.2.29. *Let H_i^r be the divisor in $X_{2r} \subset \mathbb{P}^{N_+}$ cut out by the determinant of the $i \times i$ top left submatrix of the matrix Z in (3.1). The tangent cone of H_i^r at a point of $S_k(\mathcal{V}^{2r-1}) \backslash S_{k-1}(\mathcal{V}^{2r-1})$ for $i = 2, \dots, r$ and $k < i$ is a cone with vertex of dimension $k(2r+1-k)$ over H_{i-k}^{r-k}.*

Proof. For the proof it is enough to note that the tangent cone of H_i^r at the point $p_k = (p_{i,j})_{i,j=0,\dots,2r-1}$, where $p_{i,i} = 1$ for $i = 0, \dots, k-1$ and $p_{i,j} = 0$ otherwise, is cut out by

$$\det \begin{pmatrix} z_{k,k} & z_{k,k+1} & \dots & z_{k,i-1} \\ z_{k,k+1} & z_{k+1,k+1} & \dots & z_{k+1,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,i-1} & z_{k+1,i-1} & \dots & z_{i-1,i-1} \end{pmatrix} = 0$$

and by the equations for the tangent cone of X_{2r} in the proof of Proposition 3.2.18. $\square$

3.2.2 *Birational geometry of $\mathcal{S}_{2r}$*

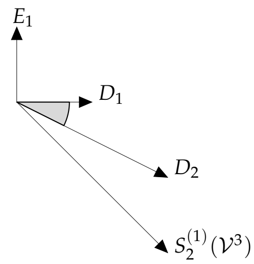
By the work of M. Brion [Bri93] we have that $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [Per14, Section 4]. In particular, by Theorem 3.2.19 the wonderful compactification $\mathcal{S}_{2r}$ is a Mori dream space.

Corollary 3.2.30. *The Cox ring of $\mathcal{S}_{2r}$ is generated by the sections of the divisors $D_1, \dots D_r$, $E_1, \dots, E_{r-1}$, $S_r^{(r-1)}(\mathcal{V}_r^{2r-1})$.*

Proof. This follows from Proposition 3.2.27 and Remark 2.2.7. $\square$

Our aim is to study the Mori chamber decomposition of the wonderful compactification $\mathcal{S}_{2r}$. Since $\mathcal{S}_2 \cong \mathbb{P}^2$ the first interesting case is for $r = 2$.

Proposition 3.2.31. *For the variety $\mathcal{S}_4$ we have that $\mathrm{Pic}(\mathcal{S}_4)$ is generated by D_1, E_1. Furthermore, $D_1 \sim H$, $D_2 \sim 2H - E_1$, $S_2^{(1)}(\mathcal{V}^3) \sim 2H - 2E_1$, and $\mathrm{Cox}(\mathcal{S}_4)$ is generated by the sections of $D_1, D_2, E_1, S_2^{(1)}(\mathcal{V}^3)$. The Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}_4)$ has three chambers as displayed in the following picture:*



and the movable cone coincides with the nef cone generated by D_1 and D_2.

Proof. Since $\mathcal{S}_4$ is the blow-up of a smooth variety along a smooth subvariety the relations $D_2 \sim 2H - E_1$, $S_2^{(1)}(\mathcal{V}^3) \sim 2H - 2E_1$ follow from Propositions 3.2.18, 3.2.21, 3.2.29 and Remark 3.1.29.

The statement on the generators of the Cox ring follows from Corollary 3.2.30. Furthermore, by Remarks 2.2.6 and 2.2.7 the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}_4)$ is a, possibly trivial, coarsening of the decomposition in the statement. On the other hand, by Proposition 3.2.28 we know that H and $2H - E_1$ generate $\mathrm{Nef}(\mathcal{S}_4)$ while E_1 and $2H - 2E_1$ generate $\mathrm{Eff}(\mathcal{S}_4)$. So no ray can be removed and the above decomposition coincides with the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}_4)$. $\square$

Next, we consider the case $r = 3$.

Lemma 3.2.32. *For the variety $\mathcal{S}_6$ the Picard group $\mathrm{Pic}(\mathcal{S}_6)$ is generated by H, E_1, E_2, and we have the following relations: $D_1 \sim H$, $D_2 \sim 2H - E_1$, $D_3 \sim 3H - 2E_1 - E_2$ and $S_3^{(2)}(\mathcal{V}^5) \sim 2H - 2E_1 - 2E_2$.*

Proof. Recall, that the first blow-up $f_1\colon X_6^{(1)} \to X_6$ in Construction 3.2.11 is the blow-up of X_6 along the Veronese variety $\mathcal{V}^5$ which by Proposition 3.2.18 is the singular locus of X_6. Hence, in this case we can not use Remark 3.1.29 to compute the discrepancies of the relevant divisors with respect to E_1. In order to do this we consider the line $L = \{z_{1,1} - z_{0,1} = z_{1,1} - z_{2,2} = z_{0,2} = z_{0,3} = z_{0,4} = z_{0,5} = z_{1,2} = z_{1,3} = z_{1,4} = z_{1,5} = z_{2,3} = z_{2,4} = z_{2,5} = z_{3,3} = z_{3,4} = z_{3,5} = z_{4,4} = z_{4,5} = z_{5,5} = 0\}$ and let $\widetilde{L}$ be its strict transform in $X_6^{(1)}$. Slightly abusing the notation we will denote by D_i also the strict transform in $X_6^{(1)}$ of the divisor H_i^3 in Proposition 3.2.29 for $i = 1,2,3$ and by H the pull-back of the hyperplane section to $X_6^{(1)}$. Clearly, $D_1 \sim H$.

Now, let us write $D_2 \sim 2H - aE_1$. Note that the line L intersects $\mathcal{V}^5$ just at the point $p = [1 : 0 \cdots : 0]$, and by Remark 3.2.10 and Proposition 3.2.18 $L \subset X_6$. By Proposition 3.2.18 the tangent cone of X_6 at p is a cone over $X_4 \cong \mathbb{G}(1,4)$ with 5-dimensional vertex and $\widetilde{L}$ intersects E_1 just at the point $q = [1 : 0 : 0 : 0 : 1 : 0 : \cdots : 0]$ of X_4. Hence $\widetilde{L} \cdot E_1 = 1$. The divisor H_2^3 intersects L in p and in another point not lying on $\mathcal{V}^5$. Moreover, by Proposition 3.2.29 the tangent cone of H_2^3 at p is a hyperplane section of X_4 not passing through q. Then $\widetilde{L} \cdot D_2 = 1$. By the projection formula we have

$$1 = \widetilde{L} \cdot D_2 = 2\widetilde{L} \cdot H - a\widetilde{L} \cdot E_1 = 2L \cdot H_1^3 - a = 2 - a$$

and hence $a = 1$. So we may write $D_2 \sim 2H - E_1$.

Now, write $D_3 \sim 3H - bE_1$. The divisor H_3^3 intersects L in p with multiplicity two and in another point not lying on $\mathcal{V}^5$. By Proposition 3.2.29 the tangent cone of H_3^3 at p is a quadratic section of X_4 not passing through q. Hence

$$1 = \widetilde{L} \cdot D_3 = 3\widetilde{L} \cdot H - a\widetilde{L} \cdot E_1 = 3L \cdot H_1^3 - a = 3 - a$$

and $a = 2$. Then $D_3 \sim 3H - 2E_1$.

We will denote by S_3 the strict transform of $S_3(\mathcal{V}^5)$ in $X_6^{(1)}$. Let $R \subset X_4 \cong \mathbb{G}(1,4)$ be a general line. Note that R is contracted by the blow-down morphism and hence

$$1 = R \cdot D_2 = 2R \cdot H - R \cdot E_1 = -R \cdot E_1$$

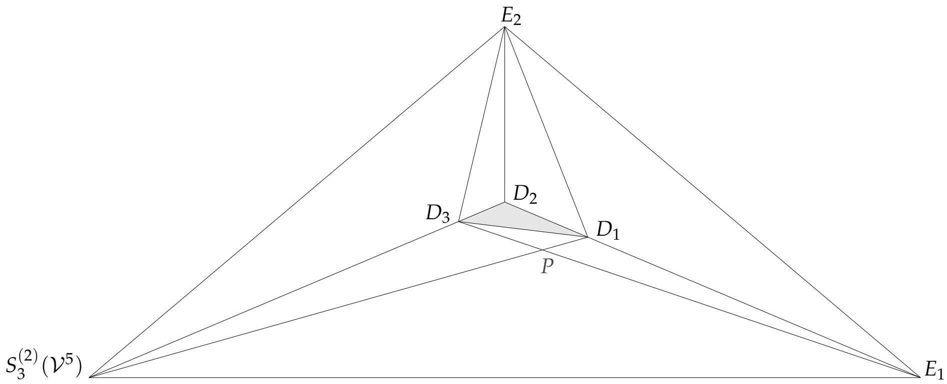
yields $R \cdot E_1 = -1$. By Proposition 3.2.21 we may write $S_3 \sim 2H - cE_1$ and since by Proposition 3.2.18 the tangent cone of $S_3(\mathcal{V}^5)$ at a point of $\mathcal{V}^5$ is a quadratic section of X_4 we have $R \cdot S_3 = 2$. This yields

$$2 = R \cdot S_3 = 2R \cdot H - cR \cdot E_1 = -cR \cdot E_1 = c$$

and $S_3 \sim 2H - 2E_1$.

Now, by Proposition 3.2.18 the morphism $f_2\colon \mathcal{S}_6 \to X_6^{(1)}$ in Construction 3.2.11 is the blow-up of a smooth variety along a smooth subvariety. So we can apply Remark 3.1.29 in order to compute the discrepancies of the divisors with respect to E_2. Finally, again by Proposition 3.2.18 we get the claim. $\square$

Theorem 3.2.33. *The sections of $D_1, D_2, D_3, E_1, E_2, S_3^{(2)}(\mathcal{V}^5)$ generate the Cox ring of $\mathcal{S}_6$. The Mori chamber decomposition of the effective cone of $\mathcal{S}_6$ has nine chambers as displayed in the following* 2*-dimensional section of* $\mathrm{Eff}(\mathcal{S}_6)$*:*



where $P \sim 3H - E_1 - E_2$ and $\mathrm{Mov}(\mathcal{S}_6)$ *is generated by D_1, D_2, D_2 and P.*

Proof. The computation of the movable cone follows from [ADHL15, Proposition 3.3.2.3], Proposition 3.2.27 and Remark 2.2.7, and the statement on the generators of $\mathrm{Cox}(\mathcal{S}_6)$ follows from Corollary 3.2.30.

Furthermore, by Lemma 3.2.32, Proposition 3.2.27 and Remarks 2.2.6, 2.2.7 the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}_6)$ is a, possibly trivial, coarsening of the decomposition in the statement.

Note that the stable base loci of a divisor in the interior of chamber delimited by $S_3^{(2)}(\mathcal{V}^5), P, E_1$; $S_3^{(2)}(\mathcal{V}^5), P, D_3$; $S_3^{(2)}(\mathcal{V}^5), D_3, E_2$; D_2, D_3, D_1, E_2; $E_1, D_1, E_2,$; P, D_1, E_1 are respectively given by $S_3^{(2)}(\mathcal{V}^5) \cup E_1$; $S_3^{(2)}(\mathcal{V}^5)$; $E_2 \cup S_3^{(2)}(\mathcal{V}^5)$; E_2; $E_1 \cup E_2$; E_1. Furthermore, since Mori chambers are convex the stable base locus chamber delimited by D_2, D_3, D_1, E_2 must be divided in two Mori chambers by the wall joining D_2 and E_2. Hence the decomposition in the statement gives the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}_6)$ outside of the movable cone.

Finally, note that the only modifications we could perform inside the movable cone are removing the wall joining D_1 and D_3 and adding a wall joining D_2 and P. However, both these modifications are not allowed since by Proposition 3.2.28 the chamber delimited by D_1, D_2, D_3 is the nef cone of $\mathcal{S}_6$. □

4 KONTSEVICH SPACES OF DEGREE TWO STABLE MAPS

4.1 CONICS IN $\mathbb{P}^n$

Let $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ be the Kontsevich space of conics in $\mathbb{P}^n$. We will denote by $\Delta \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$ the boundary divisor parametrizing maps with reducible domain, and by $\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$ the locus of maps of degree two onto a line.

It is well-known that $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics $\mathcal{Q}(2, 3)$, see for example [FP97, Section 0.4]. The following result generalizes this fact.

Proposition 4.1.1. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n, 2)$, parametrizing conics in $\mathbb{P}^n$, is isomorphic to the blow-up $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ of $\mathcal{S}ec_3(\mathcal{V}^n)$ along $\mathcal{V}^n$.*

Proof. Recall that by Definition 3.1.12 the space $\mathcal{Q}(n, 3)$, of complete rank three quadrics in $\mathbb{P}^n$, is the blow-up of $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ along the strict transform of $\mathcal{S}ec_2(\mathcal{V}^n)$.

First, we will define maps on the strata of $\mathcal{Q}(n, 3)$. Then we will show that these maps glue to a morphism $\mathcal{Q}(n, 3) \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$, and finally we will observe that this morphism yields an isomorphism $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n) \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$.

Consider the variety $\mathcal{Q}(n, 3)^o$ parametrizing quadrics of rank equal to three. Let $Q \subset \mathbb{P}^n$ be such a quadric, and consider its vertex V_Q. The dual V_Q^* is a 2-plane in $\mathbb{P}^{n*}$. Dually a line in Q corresponds to an $(n-2)$-planes in $\mathbb{P}^{n*}$ intersecting $V_Q^* \cong \mathbb{P}^2$ along a line. These lines sweep out a conic $C_Q \subset V_Q^* \subset \mathbb{P}^{n*}$. This yields a morphism

$$\begin{array}{rccc} \phi^o\colon & \mathcal{Q}(n,3)^o & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, 2). \\ & Q & \longmapsto & C_Q \end{array}$$

Note that conversely, given a smooth conic $C_Q \in \overline{M}_{0,0}(\mathbb{P}^n, 2)$ we can consider the cone swept by the duals of the tangents lines of C_Q and whose vertex is the dual of the plane spanned by C_Q. Hence ϕ^o is an isomorphism.

Now, let $Q = H_1 \cup H_2$ be a rank two quadric, with vertex $V_Q = H_1 \cap H_2 \cong \mathbb{P}^{n-2}$. Passing to the dual we get two points $p_i = H_i^* \in \mathbb{P}^{n*}$ for $i = 1, 2$ spanning the line $L_Q = V_Q^*$. We associate to Q the unique, up to automorphisms of $\mathbb{P}^1$, 2-to-1 map $f_Q\colon \mathbb{P}^1 \to L \subset \mathbb{P}^{n*}$ ramifying at the two points p_1 and p_2. We will show that this association extends the morphism ϕ^o to E_2^Q and that such extension contracts the fibers of $E_2^Q \to \widetilde{\mathcal{S}ec_2(\mathcal{V}^n)}$, where $\widetilde{\mathcal{S}ec_2(\mathcal{V}^n)} \subseteq \mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ is the strict transform of $\mathcal{S}ec_2(\mathcal{V}^n)$.

By Proposition 3.1.9 a point $p \in E_2^Q$ is the datum of a pair (Q, L^2) where Q is a quadric of rank two and L is a linear form. Up to the $SL(n+1)$-action in (3.3) we may assume that $Q = x_0x_1$. Consider a family of rank three quadrics of the following form

$$Q_{t,L} = \{x_0x_1 - tL(x_2, \ldots, x_n)^2 = 0\}$$

where $L = \alpha_2 x_2 + \dots + \alpha_n x_n$. Then

$$\phi^o(Q_{t,L}) = \begin{cases} tx_0x_1 - L(x_2, \dots, x_n)^2 = 0; \\ \alpha_{i+1}x_i - \alpha_i x_{i+1} = 0 \text{ for } i = 2\dots, n-1; \end{cases}$$

and hence

$$\lim_{t \mapsto 0} \phi^o(Q_{t,L}) = \begin{cases} L(x_2, \dots, x_n) = 0; \\ \alpha_{i+1}x_i - \alpha_i x_{i+1} = 0 \text{ for } i = 2\dots, n-1. \end{cases}$$

Note that the right hand side of this last equality is the line $R = \{x_2 = \dots = x_n = 0\}$. Therefore, we get that ϕ^o can be extended by mapping x_0x_1 to the 2-to-1 map $\mathbb{P}^1 \to R$ ramified on $[1 : 0 \cdots : 0], [0 : 1 \cdots : 0]$. Furthermore, since R does not depend on the linear form L such extension contracts the fibers $E_2^Q \to \widetilde{\mathcal{S}ec_2(\mathcal{V}^n)}$.

Now, consider the exceptional divisor E_1^Q. By Proposition 3.1.9 a point $p \in E_1^Q$ is the datum of a pair (H, Q_H) where H is a quadric in $\mathbb{P}^n$ of rank one and $Q_H \subset H$ is a quadric of rank two in the hyperplane H. In particular $Q_H = \Pi_1 \cup \Pi_2$ with Π_i hyperplanes in H. Passing to the dual this configuration corresponds to the point $p_H = H^*$ together with the two lines $L_i = \Pi_i^*$ intersecting in p_H. This yields a reducible conic $C_H = L_1 \cup L_2$. Take a family of the form

$$Q_{t,L} = \{x_0^2 - tL(x_1, \dots, x_n)R(x_1, \dots, x_n) = 0\}$$

where $L = \alpha_1 x_1 + \dots + \alpha_n x_n$, $R = \beta_1 x_1 + \dots + \beta_n x_n$ are linear forms. Then

$$\phi^o(Q_{t,L}) = \begin{cases} tx_0^2 - L(x_1, \dots, x_n)R(x_1, \dots, x_n) = 0; \\ x_i(\alpha_j\beta_k - \alpha_k\beta_j) - x_j(\alpha_i\beta_k - \alpha_k\beta_i) + x_k(\alpha_i\beta_j - \alpha_j\beta_i) = 0 \text{ for } 1 \leqslant i < j < k \leqslant n; \end{cases}$$

and hence

$$\lim_{t \mapsto 0} \phi^o(Q_{t,L}) = \begin{cases} L(x_1, \dots, x_n)R(x_1, \dots, x_n) = 0; \\ x_i(\alpha_j\beta_k - \alpha_k\beta_j) - x_j(\alpha_i\beta_k - \alpha_k\beta_i) + x_k(\alpha_i\beta_j - \alpha_j\beta_i) = 0 \text{ for } 1 \leqslant i < j < k \leqslant n. \end{cases}$$

Therefore, we get a reducible conic made of two lines intersecting in $[1 : 0 : \dots : 0]$. Summing-up we have a birational morphism

$$\phi \colon \mathcal{Q}(n, 3) \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$$

mapping E_1^Q to the boundary divisor $\Delta \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$ and E_2^Q to $\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$.

Furthermore, since ϕ contracts the fibers of $E_2^Q \to \widetilde{\mathcal{S}ec_2(\mathcal{V}^n)}$ it induces a birational morphism

$$\psi \colon \mathcal{S}ec_3^{(1)}(\mathcal{V}^n) \to \overline{M}_{0,0}(\mathbb{P}^n, 2)$$

mapping E_1^Q to $\Delta \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$ and $\widetilde{\mathcal{S}ec_2(\mathcal{V}^n)}$ to $\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$. Finally, since ψ does not contract any divisor and $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ is normal Zariski's main theorem [Mum99, Chapter 3, Section 9] yields that it is an isomorphism. □

In $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ we consider the following divisor classes:

- $\mathcal{H}$ of conics intersecting a fixed codimension two linear subspace of $\mathbb{P}^n$;

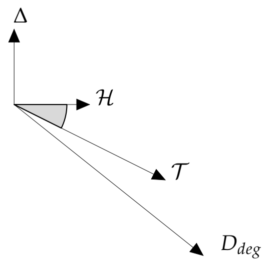
- $\mathcal{T}$ of conics which are tangent to a fixed hyperplane in $\mathbb{P}^n$;
- D_{deg} of conics spanning a plane that intersects a fixed linear subspace of dimension $n-3$ in $\mathbb{P}^n$.

As an application of Proposition 4.1.1 we have the following result.

Proposition 4.1.2. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n,2)$ of conics in $\mathbb{P}^n$ is a spherical variety with respect to the following $SL(n+1)$-action:*

$$\begin{array}{ccc} SL(n+1)\times \overline{M}_{0,0}(\mathbb{P}^n,2) & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^n,2). \\ (A,[C,\alpha]) & \longmapsto & [C,A\circ\alpha] \end{array} \tag{4.1}$$

The effective cone $\overline{M}_{0,0}(\mathbb{P}^n,2)$ is generated by Δ and D_{deg}, and the nef cone of $\overline{M}_{0,0}(\mathbb{P}^n,2)$ is generated by $\mathcal{H}$ and $\mathcal{T}$. Furthermore, the following



is the Mori chamber decomposition of $\mathrm{Eff}(\overline{M}_{0,0}(\mathbb{P}^n,2))$, where $\mathcal{T}\sim 2\mathcal{H}-\Delta$ and $D_{deg}\sim 3\mathcal{H}-2\Delta$.

Proof. The $SL(n+1)$-action on $\overline{M}_{0,0}(\mathbb{P}^n,2)$ in (4.1) corresponds to the $SL(n+1)$-action on $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ induced by (3.3) via the isomorphism in Proposition 4.1.1. Note that with respect to this action $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ is spherical but not wonderful. However, we can deduce its boundary divisors and colors from those of $\mathcal{Q}(n,3)$ via the blow-down $\mathcal{Q}(n,3)\to\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ of $E_2^{\mathcal{Q}}$. Since boundary divisors and colors of $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$ lifts to boundary divisors and colors of $\mathcal{Q}(n,3)$, by Proposition 3.1.27 we get that $E_1^{\mathcal{Q}}$ is the only boundary divisor of $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$, and that its colors are $D_1^{\mathcal{Q}},D_2^{\mathcal{Q}},D_3^{\mathcal{Q}}$, where we kept the same notation for divisors on $\mathcal{Q}(n,3)$ and $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$. Hence, arguing as in the proof of Proposition 3.1.32 we get that $D_1^{\mathcal{Q}},D_2^{\mathcal{Q}}$ generates the nef cone of $\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)$, $D_3^{\mathcal{Q}},E_1^{\mathcal{Q}}$ generates it effective cone, and the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}ec_3^{(1)}(\mathcal{V}^n))$ has three chambers delimited by $D_3^{\mathcal{Q}},D_2^{\mathcal{Q}}$; $D_2^{\mathcal{Q}},D_1^{\mathcal{Q}}$ and $D_1^{\mathcal{Q}},E_1^{\mathcal{Q}}$.

Now, by the proof of Proposition 4.1.1 we have that $E_1^{\mathcal{Q}}$ gets mapped to Δ by the isomorphism $\psi\colon\mathcal{S}ec_3^{(1)}(\mathcal{V}^n)\to\overline{M}_{0,0}(\mathbb{P}^n,2)$. Moreover, a straightforward computation shows that $\psi^*\mathcal{H}=D_1^{\mathcal{Q}}$, $\psi^*\mathcal{T}=D_2^{\mathcal{Q}}$ and $\psi^*D_{deg}=D_3^{\mathcal{Q}}$. Finally, the statement follows from Lemma 3.1.30, Proposition 4.1.1 and the description of the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{S}ec_3^{(1)}(\mathcal{V}^n))$ in the first part of the proof. $\square$

Remark 4.1.4. We sum up the birational models of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ in the following diagram:

$$\begin{tikzcd}
 & & \mathcal{Q}(n,3) \arrow[dll] \arrow[drr] & & \\
\mathbb{S}ec_3^{(1)}(\mathcal{V}^n) \cong \overline{M}_{0,0}(\mathbb{P}^n,2) \arrow[d, "\mathcal{H}"'] \arrow[drr, "\mathcal{T}"'] \arrow[rrrr, dashed, "\chi"'] & & & & \mathrm{Hilb}_2(\mathbb{P}^n) \arrow[dll] \arrow[d, "\widetilde{D}_{deg}"'] \\
\mathbb{S}ec_3(\mathcal{V}^n) & & \mathrm{Chow}_2(\mathbb{P}^n) & & \mathbb{G}(2,n)
\end{tikzcd}$$

where $\mathrm{Hilb}_2(\mathbb{P}^n)$ and $\mathrm{Chow}_2(\mathbb{P}^n)$ are respectively the Hilbert scheme and the Chow variety of conics in $\mathbb{P}^n$, χ is the flip of $\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^n, 2)$, $\mathbb{G}(2,n)$ is the Grassmannians of planes in $\mathbb{P}^n$, and $\widetilde{D}_{deg}$ is the strict transform of D_{deg} via χ. The morphism induced by $\widetilde{D}_{deg}$ associates to a conic in $\mathrm{Hilb}_2(\mathbb{P}^n)$ the unique plane of $\mathbb{P}^n$ in which it is contained. We would like to stress that the modular interpretation of the flip of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ as a Hilbert scheme was well-know, see for instance [Kie11, Section 3].

4.2 CONICS IN $\mathbb{P}^n \times \mathbb{P}^m$

Let $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ be the Kontsevich space parametrizing conics in $\mathbb{P}^n \times \mathbb{P}^m$. Denote by $\pi\colon \overline{M}_{0,1}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ the forgetful morphism, and by $ev\colon \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \to \mathbb{P}^n \times \mathbb{P}^m$ the evaluation morphism.

Let H_n and H_m be the hyperplane sections of $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, and $H_{n,m} \cong \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^n \times \mathbb{P}^m$. Consider the divisors

$$\mathcal{K}^n := \pi_* ev^* H_n^2,\ \mathcal{K}^m := \pi_* ev^* H_m^2,\ \mathcal{K}^{n,m} := \pi_* ev^* \mathcal{H}_{n,m}$$

and let Δ be the boundary divisor of maps with reducible domain.

By the proof of [Opr05, Lemma 1, Section 2.1], the Picard group of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is generated by $\Delta, \mathcal{K}^n, \mathcal{K}^m$. In particular, since $H_1^2 = 0$, the Picard rank of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is:

$$\rho(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))) = \begin{cases} 1 & \text{if } n = m = 1; \\ 2 & \text{if } n = 1, m \geqslant 2; \\ 3 & \text{if } n, m \geqslant 2. \end{cases} \tag{4.2}$$

Proposition 4.2.2. *The Kontsevich space $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$, of conics in $\mathbb{P}^n \times \mathbb{P}^m$, is isomorphic to the space $\mathcal{C}(n,m,2)$ of rank two complete collineations on $\mathbb{P}^n \times \mathbb{P}^m$.*

Proof. First consider the case $n = m = 1$. We have that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1)) \cong \mathbb{P}^3$. Indeed, we may embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ as a smooth quadric Q, and the conics in Q are in bijection with the hyperplanes in $\mathbb{P}^3$. Furthemore, $\mathcal{C}(1,1,2) \cong \mathbb{P}^3$ as well, and we may write down explicitly an isomorphism $\mathcal{C}(1,1,2) \to \overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ as follows. Write a point of $\mathcal{C}(1,1,2)$ as a 2×2 matrix Z, fix homogeneous coordinates $([x_0 : x_1], [y_0 : y_1])$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and associated to Z the conic $C_Z = \{(x_0, x_1) \cdot M \cdot (y_0, y_1)^t = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Now, let $Z \in \mathcal{S}ec_2(\mathcal{S}^{n,m}) \backslash \mathcal{S}^{n,m}$ be an $(n+1) \times (m+1)$ matrix of rank two. The image of Z yields a line L_Z in $\mathbb{P}^n$, and the dual of the kernel of Z gives a line R_Z in $\mathbb{P}^{m*}$. Hence, we get a morphism

$$\begin{array}{cccc} \gamma^o\colon & \mathcal{C}(n,m,2)^o & \longrightarrow & \mathbb{G}(1,n) \times \mathbb{G}(1,m). \\ & Z & \longmapsto & (L_Z, R_Z) \end{array}$$

The fiber of γ^o over (L_Z, R_Z) can be identified with the collineations on $L_Z \times R_Z$ which, by the first part of the proof, are in bijection with $\overline{M}_{0,0}(L_Z \times R_Z, (1,1)) \subseteq \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$. This yields an isomorphism

$$\begin{array}{cccc} \delta^o\colon & \mathcal{C}(n,m,2)^o & \longrightarrow & M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)) \\ & Z & \longmapsto & C_Z \end{array}$$

Next, we show that δ^o extends to an isomorphism on $\mathcal{C}(n,m,2)$ mapping $E_1^{\mathcal{C}}$ to Δ. In order to do this, since all the points of $\mathcal{S}^{n,m}$ are in the same $(SL(n+1) \times SL(m+1))$-orbit of (3.2), we may consider the following family of rank two $(n+1) \times (m+1)$ matrices

$$Z_t = \begin{pmatrix} M_t & 0_{2,m-1} \\ 0_{n-1,2} & 0_{n-1,m-1} \end{pmatrix}$$

where $M_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. Fix homogeneous coordinates $[x_0 : \cdots : x_n]$ on $\mathbb{P}^n$, and $[y_0 : \cdots : y_m]$ on $\mathbb{P}^m$. Then $L_{Z_t} = \{x_2 = \cdots = x_n\}$, $R_{Z_t} = \{y_2 = \cdots = y_m = 0\}$, $\delta^o(Z_t) = \{x_0y_0 + tx_1y_1 = 0\} \subset L_{Z_t} \times R_{Z_t} \subseteq \mathbb{P}^n \times \mathbb{P}^m$, and

$$\lim_{t \mapsto 0} \delta^o(Z_t) = \{x_0y_0 = 0\} \in \overline{M}_{0,0}(L_{Z_0} \times R_{Z_0}, (1,1)) \cap \Delta \subset \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)).$$

Hence, we got a morphism

$$\delta\colon \mathcal{C}(n,m,2) \to \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$$

mapping $E_1^{\mathcal{C}}$ to Δ. Moreover, by Proposition 3.1.22 and (4.2) we get that δ does not contract any divisor. Finally, since $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ is smooth we conclude, by Zariski's main theorem [Mum99, Chapter 3, Section 9], that δ is an isomorphism. □

Remark 4.2.3. Via the isomorphism

$$\delta\colon \mathcal{C}(n,m,2) \to \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$$

we have

$$\delta^*(\Delta) = E_1^{\mathcal{C}},\ \delta^*(\mathcal{K}^n) = H_1^{\mathcal{C}},\ \delta^*(\mathcal{K}^m) = H_2^{\mathcal{C}},\ \delta^*(\mathcal{K}^{n,m}) = D_1^{\mathcal{C}}.$$

These equalities together with Proposition 3.1.33 give that for $n = 1 < m$, the Mori chamber decomposition of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ has two chambers delimited by $\Delta, \mathcal{K}^{n,m}$ and $\mathcal{K}^{n,m}, \mathcal{K}^m$, while for $1 < n \leqslant m$ the Mori chamber decomposition of $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))$ has three chambers delimited by $\mathcal{K}^n, \mathcal{K}^m, \mathcal{K}^{n,m}$; $\mathcal{K}^n, \mathcal{K}^{n,m}, \Delta$ and $\mathcal{K}^m, \mathcal{K}^{n,m}, \Delta$.

4.3 CONICS IN $\mathbb{G}(k,n)$

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-planes in $\mathbb{P}^n$. Following [CC10, Section 2] we describe divisor classes on $\overline{M}_{0,0}(\mathbb{G}(k,n),2)$.

Fix projective subspaces $\Pi^{n-k}, \Pi^{n-k-2} \subset \mathbb{P}^n$ of dimension $n-k$ and $n-k-2$, and consider the Schubert cycles

$$\begin{aligned} \sigma_{1,1}^{k,n} &= \{W \in \mathbb{G}(k,n) \mid \dim(W \cap \Pi^{n-k}) \geqslant 1\}; \\ \sigma_{2}^{k,n} &= \{W \in \mathbb{G}(k,n) \mid \dim(W \cap \Pi^{n-k-2}) \geqslant 0\}. \end{aligned}$$

Consider $\pi\colon \overline{M}_{0,1}(\mathbb{G}(k,n),2) \to \overline{M}_{0,0}(\mathbb{G}(k,n),2)$ the forgetful morphism and the evaluation morphism $ev\colon \overline{M}_{0,0}(\mathbb{G}(k,n),2) \to \mathbb{G}(k,n)$. We define

$$H_{\sigma_{1,1}}^{k,n} = \pi_* ev^* \sigma_{1,1},\ H_{\sigma_2}^{k,n} = \pi_* ev^* \sigma_2.$$

Furthermore, we will denote by $T^{k,n}$ the class of the divisor of conics that are tangent to a fixed hyperplane section of $\mathbb{G}(k,n)$.

Let $D_{deg}^{k,n}$ the class of the divisor of maps $[C,\alpha] \in \overline{M}_{0,0}(\mathbb{G}(k,n),2)$ such that the projection of the span of the linear spaces parametrized by $\alpha(C)$ from a fixed subspace of dimension $n-k-3$ has dimension less than $k+2$.

Next we define the divisor class $D_{unb}^{k,n}$. A stable map $\alpha\colon \mathbb{P}^1 \to \mathbb{G}(k,n)$ induces a rank $k+1$ subbundle $\mathcal{E}_\alpha \subset \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{n+1}$. If $k=1$ we define $D_{unb}^{k,n}$ as the closure of the locus of maps $[\mathbb{P}^1,\alpha] \in \overline{M}_{0,0}(\mathbb{G}(k,n),2)$ such that $\mathcal{E}_\alpha \neq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. If $k \geqslant 2$ there is a trivial subbundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus k-1} \subset \mathcal{E}_\alpha$ which induces a $(k-2)$-dimensional subspace $H_\alpha \subset \mathbb{P}^n$. In this way we get a map

$$\begin{array}{rccc} \xi\colon & \overline{M}_{0,0}(\mathbb{G}(k,n),2) & \longrightarrow & \mathbb{G}(k-2,n). \\ & [\mathbb{P}^1,\alpha] & \mapsto & H_\alpha \end{array}$$

We define $D_{unb}^{k,n} = \xi^* \mathcal{O}_{\mathbb{G}(k-2,n)}(1)$ that is $D_{unb}^{k,n}$ is the closure of the locus of maps $[\mathbb{P}^1,\alpha] \in \overline{M}_{0,0}(\mathbb{G}(k,n),2)$ such that H_α intersects a fixed $(n-k+1)$-dimensional subspace of $\mathbb{P}^n$. Finally, we denote by $\Delta^{k,n}$ the boundary divisor parametrizing stable maps with reducible domain.

Proposition 4.3.1. *There is a finite* 2*-to-*1 *morphism*

$$\varphi\colon \overline{M}_{0,0}(\mathbb{G}(1,n),2) \longrightarrow \mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$$

mapping a stable map $[\mathbb{P}^1,\alpha] \in M_{0,0}(\mathbb{G}(1,n),2)$ *to the rank four quadric* $Q_C^* \subset \mathbb{P}^{n*}$*, where* $Q_C^* = \bigcup_{p \in Q_C} (T_p Q_C)^*$ *and* $Q_C = \bigcup_{[L] \in \alpha(\mathbb{P}^1)} L$.

Proof. By [CM17, Proposition 4.10, Theorem 5.1, Corollary 5.4] there is a birational morphism $f\colon \overline{M}_{0,0}(\mathbb{G}(1,n),2) \to \mathcal{T}_4^n$, contracting $D_{deg}^{1,n}$ and $\Delta^{1,n}$, where $\mathcal{T}_4^n$ is the double symmetric determinantal locus of rank at most four constructed in [HT15, Section 2.2]. By [HT15, Proposition 2.3] there is a finite 2-to-1 morphism $\rho\colon \mathcal{T}_4^n \to \mathbb{S}ec_4(\mathcal{V}^n)$ branched along $\mathbb{S}ec_3(\mathcal{V}^n)$.

Now, consider the morphism $\rho \circ f\colon \overline{M}_{0,0}(\mathbb{G}(1,n),2) \to \mathbb{S}ec_4(\mathcal{V}^n)$. By [Har77, Proposition 7.14] there is a morphism $\varphi\colon \overline{M}_{0,0}(\mathbb{G}(1,n),2) \to \mathbb{S}ec_4^{(2)}(\mathcal{V}^n)$ such that $\pi \circ \varphi = \rho \circ f$, where $\pi\colon \mathbb{S}ec_4^{(2)}(\mathcal{V}^n) \to \mathbb{S}ec_4(\mathcal{V}^n)$ is the blow-down.

Hence φ is 2-to-1 and by [HT15, Theorem 1.1] on $M_{0,0}(\mathbb{G}(1,n),2)$ it is defined by

$$\begin{array}{rccc} \varphi_{|M_{0,0}(\mathbb{G}(1,n),2)}: & M_{0,0}(\mathbb{G}(1,n),2) & \longrightarrow & \mathbb{S}ec_4^{(2)}(\mathcal{V}^n) \\ & [\mathbb{P}^1,\alpha] & \mapsto & Q_C^* \end{array}$$

where $Q_C^* = \bigcup_{p\in Q_C}(T_pQ_C)^* \subset \mathbb{P}^{n*}$, and $Q_C = \bigcup_{[L]\in\alpha(\mathbb{P}^1)} L$. Note that Q_C^* is indeed a quadric hypersurface of rank four, and since Q_C can be constructed from either of its two rulings $\varphi_{|M_{0,0}(\mathbb{G}(1,n),2)}$ is 2-to-1. $\square$

Remark 4.3.2. For $n=3$ the double cover in Proposition 4.3.1 had been constructed in [Hue15, Section 5].

Remark 4.3.3. As an application of Propositions 3.1.34, 4.3.1 we recover some results of [CC10]. Indeed, on $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ there is an $SL(n+1)$-action given by

$$\begin{array}{ccc} SL(n+1)\times\overline{M}_{0,0}(\mathbb{G}(1,n),2) & \longrightarrow & \overline{M}_{0,0}(\mathbb{G}(1,n),2) \\ (M,[C,\alpha]) & \longmapsto & [C,\wedge^2M\circ\alpha] \end{array}$$

inducing on $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ a structure of spherical variety.

Considering the subspace $H=\{x_4=\cdots=x_n=0\}\subset\mathbb{P}^n$ we get an embedding $i\colon \mathbb{G}(1,H)\hookrightarrow\mathbb{G}(1,n)$ which in turn induces an embedding $j\colon \overline{M}_{0,0}(\mathbb{G}(1,3),2)\to\overline{M}_{0,0}(\mathbb{G}(1,n),2)$.

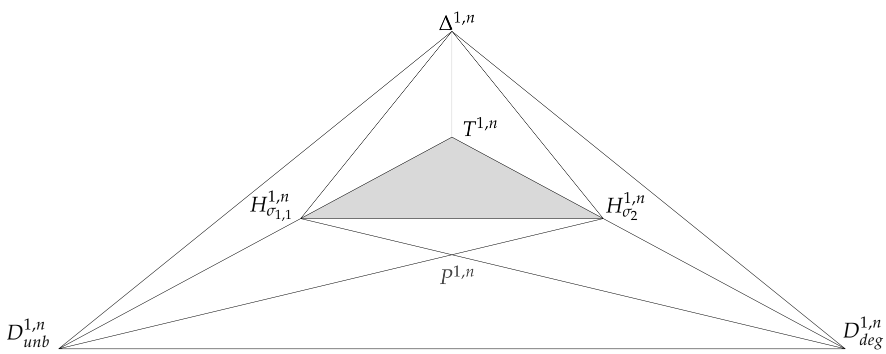
Furthermore, the pull-back map $j^*\colon \operatorname{Pic}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))\to\operatorname{Pic}(\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ is an isomorphism. Then, the study of the birational geometry of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ is reduced to that of $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$.

By Proposition 3.1.34 and the 2-to-1 morphism in Proposition 4.3.1 we get that the divisor classes $\Delta^{1,n}, D^{1,n}_{deg}, D^{1,n}_{unb}$ and the divisor classes $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$ are respectively the classes of the boundary divisors and the colors of the spherical variety $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$.

Furthermore, the divisors classes $D^{1,n}_{unb}, D^{1,n}_{deg}, \Delta^{1,n}$ generate the effective cone of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$. The Cox ring of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ is generated by the global sections of the divisors $\Delta^{1,n}, D^{1,n}_{deg}, D^{1,n}_{unb}$ and $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$.

The nef cone of $\overline{M}_{0,0}(\mathbb{G}(1,n),2)$ is generated by $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}$.

Moreover, the following is a 2-dimensional section of the Mori chamber decomposition of $\operatorname{Eff}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))$:



where $P^{1,n}\sim\frac{1}{4}(3H^{1,n}_{\sigma_{1,1}}+3H^{1,n}_{\sigma_2}-\Delta^{1,n})$, and $\operatorname{Mov}(\overline{M}_{0,0}(\mathbb{G}(1,n),2))$ is generated by $H^{1,n}_{\sigma_{1,1}}, H^{1,n}_{\sigma_2}, T^{1,n}, P^{1,n}$.

As remarked in [CC10], this decomposition holds even for $\overline{M}_{0,0}(\mathbb{G}(k,n),2)$.

Finally, we have the following result on the automorphisms of Kontsevich spaces of conics.

Corollary 4.3.4. *We have that*

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1))) \cong \begin{cases} PGL(n+1) \times PGL(m+1) & \text{if } n < m; \\ S_2 \ltimes (PGL(n+1) \times PGL(n+1)) & \text{if } n = m \geqslant 2; \end{cases}$$

and $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))) \cong PGL(4)$.

Furthermore, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n, 2)) \cong PGL(n+1)$ *for* $n \geqslant 3$, $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^2, 2)) \cong PGL(3) \rtimes S_2$, *and* $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1, 2)) \cong PGL(3)$.

Proof. The first claim on $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (1,1)))$ follows from Proposition 4.2.2 and Theorem 3.1.36. For the second claim notice that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1)) \cong \mathbb{P}^3$ since curves of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are in bijection with the hyperplane sections of a smooth quadric surface in $\mathbb{P}^3$.

The automorphism group of $\overline{M}_{0,0}(\mathbb{P}^n, 2)$ for $n \geqslant 3$ follows from Proposition 4.1.1 and Theorem 3.1.36. The automorphism group of $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ has been computed in [Mas20a, Remark 7.6]. Finally, to get the claim on $\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^1, 2))$ notice that $\overline{M}_{0,0}(\mathbb{P}^1, 2) \cong \mathbb{P}^2$. Indeed, a 2-to-1 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is determined by its branch locus, and so $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ mod out by the involution switching the factors which is $\mathbb{P}^2$. $\square$

Finally, we compute the automorphism group of $\overline{M}_{0,0}(\mathbb{G}(1,n), 2)$. Since the cases $n = 2$ has been covered in Corollary 4.3.4 we assume that $n \geqslant 3$.

Proposition 4.3.5. *The automorphism group of* $\overline{M}_{0,0}(\mathbb{G}(1,n), 2)$ *is given by*

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,n), 2)) \cong \begin{cases} S_2 \ltimes PGL(n+1) & \text{if } n > 3; \\ S_2 \ltimes (S_2 \ltimes PGL(n+1)) & \text{if } n = 3. \end{cases}$$

Proof. First, consider the case $n = 3$. An automorphism of $\overline{M}_{0,0}(\mathbb{G}(1,3), 2)$ must either preserve or switch the extremal rays $D^{1,3}_{unb}$ and $D^{1,3}_{deg}$. Indeed, there is an automorphism $\tau\colon \overline{M}_{0,0}(\mathbb{G}(1,3), 2) \to \overline{M}_{0,0}(\mathbb{G}(1,3), 2)$ switching them, namely the automorphism induced by the involution of $\mathbb{G}(1,3)$ given by projective duality. This yields a surjective morphism of groups

$$\begin{array}{cccc} \Psi\colon & \mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,3), 2)) & \longrightarrow & S_2 \\ & \varphi & \mapsto & \sigma_\varphi \end{array} \tag{4.3}$$

where σ_φ is the permutation of the extremal rays of $\mathrm{Eff}(\overline{M}_{0,0}(\mathbb{G}(1,3), 2))$ induced by φ. Now, assume that σ_φ is trivial. Then φ descends to an automorphism $\overline{\varphi}$ of the variety $\mathcal{T}^3_4$ in the proof of Proposition 4.3.1. By [HT15, Proposition 2.5 (3)] $\mathcal{T}^3_4$ is Fano and the morphism $\rho\colon \mathcal{T}^3_4 \to \mathcal{S}ec_4(\mathcal{V}^3) = \mathbb{P}^9$ in the proof of Proposition 4.3.1 is induced by a multiple of $-K_{\mathcal{T}^3_4}$. Hence, $\overline{\varphi}$ in turn descends to an automorphism of $\mathcal{S}ec_4(\mathcal{V}^3) = \mathbb{P}^9$ stabilizing the branch locus $\mathcal{S}ec_3(\mathcal{V}^3)$. Since the group of automorphisms of $\mathbb{P}^9$ stabilizing $\mathcal{S}ec_3(\mathcal{V}^3)$ is isomorphic to $PGL(4)$ we get an exact sequence

$$1 \to S_2 \to \mathrm{Aut}(\mathcal{T}^3_4) \to PGL(4) \to 1.$$

Note that $PGL(4)$ acts on $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ and hence on $\mathcal{T}_4^3$. So the last morphism in the sequence has a section, and $\mathrm{Aut}(\mathcal{T}_4^3) \cong PGL(4) \rtimes S_2$.

Now, the morphism Ψ in (4.3) yields the exact sequence

$$1 \to \mathrm{Aut}(\mathcal{T}_4) \to \mathrm{Aut}(\overline{M}_{0,0}(\mathbb{G}(1,3),2)) \to S_2 \to 1$$

and since the last morphism in this sequence has a section we get the statement.

When $n > 3$ it is enough to argue as in the case $n = 3$ noticing that in this case $D_{unb}^{1,n}$ and $D_{deg}^{1,n}$ can not be switched and applying Proposition 3.1.35. $\square$

4.4 CONICS IN $LG(r,2r)$

As in Section 4.3, let $\overline{M}_{0,0}(\mathbb{G}(k,n),2)$ be the moduli space of degree two stable maps to the Grassmannian $\mathbb{G}(k,n)$ with divisors $H_{\sigma_{1,1}}^{k,n}, H_{\sigma_2}^{k,n}, T^{k,n}, D_{deg}^{k,n}, D_{unb}^{k,n}$ and $\Delta^{k,n}$.

The Lagrangian Grassmannian $LG(r,2r)$ is the subvariety of $\mathbb{G}(r-1,2r-1)$ parametrizes r-dimensional subspaces of K^{2r} which are isotropic with respect to the standard symplectic form Ω in (3.7). By [Tev05, Section 2.1] $LG(r,2r)$ is an irreducible variety of dimension $\frac{r(r+1)}{2}$ and of Picard rank one. Moreover, the restriction of the Plücker embedding of $\mathbb{G}(r-1,2r-1)$ yields the minimal homogeneous embedding of $LG(r,2r)$.

In this section we will study the moduli space $\overline{M}_{0,0}(LG(r,2r),2)$ parametrizing conics in $LG(r,2r)$.

Let $\mathcal{E}$ be the universal quotient bundle on $\mathbb{G}(r-1,2r-1)$. The Lagrangian Grassmannian $LG(r,2r) \subset \mathbb{G}(r-1,2r-1)$ is the zero locus of a section of $\bigwedge^2 \mathcal{E}$ which has first Chern class $(r-1)c_1(\mathcal{O}_{\mathbb{G}(r-1,2r-1)}(1))$. Hence the canonical bundle of $LG(r,2r)$ is given by $\omega_{LG(r,2r)} \cong \mathcal{O}_{LG(r,2r)}(-r-1)$, and $\dim(\overline{M}_{0,0}(LG(r,2r),2)) = \frac{r^2+5r-2}{2}$.

Remark 4.4.1. We recall some facts about the cohomology of $LG(r,2r)$. For details we refer to [BKT03, Section 3]. Consider a flag $F^1 \subset F^2 \subset \cdots \subset F^r \subset K^{2r}$, where F^j are isotropic subspaces of K^{2r} of dimension j. Let $\mathcal{D}_r$ be the set of strict partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ with $0 < \lambda_l < \cdots < \lambda_1 \leqslant r$ and denote by $|\lambda| = \lambda_1 + \cdots + \lambda_l$ the weight of λ. For each $\lambda \in \mathcal{D}_r$ there is a codimension $|\lambda|$ Schubert variety $\Sigma_\lambda^r \subseteq LG(r,2r)$ defined by

$$\Sigma_\lambda^r := \{W \in LG(r,2r),\ \dim(W \cap F^{r+1-\lambda_i}) \geqslant i,\ i = 1, \dots, l\}.$$

The class of the Schubert variety Σ_λ^r in the cohomology ring $H^*(LG(r,2r),\mathbb{Z})$ will be denoted by σ_λ^r. We have that

$$H^*(LG(r,2r),\mathbb{Z}) = \bigoplus_{\lambda \in \mathcal{D}_r} \mathbb{Z} \cdot \sigma_\lambda^r$$

with the following relations:

$$(\sigma_i^r)^2 + 2\sum_{k=1}^{r-i} (-1)^k \sigma_{i+k}^r \sigma_{i-k}^r = 0 \tag{4.4}$$

where by convention $\sigma_0^r = 1$ and $\sigma_i^r = 0$ for $i < 0$.

Now, we define divisor classes on $\overline{M}_{0,0}(LG(r,2r),2)$. We denote by Δ^r, the boundary divisor parametrizing stable maps with reducible domain, this is the restriction to $\overline{M}_{0,0}(LG(r,2r),2)$ of the divisor $\Delta^{r-1,2r-1}$ on $\overline{M}_{0,0}(\mathbb{G}(r-1,2r-1),2)$.

Fix an isotropic subspace F^{r-1} of dimension $r-1$, and consider the divisor $H^r_{\sigma_2} = \pi_* ev^* \sigma^r_2$, where $\pi\colon \overline{M}_{0,1}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2)$ is the forgetful morphism, $ev\colon \overline{M}_{0,1}(LG(r,2r),2) \to LG(r,2r)$ is the evaluation morphism, and σ^r_2 is the Schubert cycle corresponding to the Schubert variety

$$\Sigma^r_2 := \{W \in LG(r,2r), \dim(W \cap F^{r-1}) \geqslant 1\}.$$

By Remark 4.4.1, in $LG(r,2r)$ the only Schubert cycle of codimension two is σ^r_2, so by [Opr05, Theorem 1] we get that Δ^r and $H^r_{\sigma_2}$ generate the Picard group of $\overline{M}_{0,0}(LG(r,2r),2)$. Furthermore, we have that both the divisors $H^{r-1,2r-1}_{\sigma_{1,1}}$ and $H^{r-1,2r-1}_{\sigma_2}$ of $\overline{M}_{0,0}(\mathbb{G}(r-1,2r-1),2)$ restrict to $H^r_{\sigma_2}$ on $\overline{M}_{0,0}(LG(r,2r),2)$. Then, also $D^{r-1,2r-1}_{deg}$ and $D^{r-1,2r-1}_{unb}$ restrict to the same divisor D^r_{unb} on $\overline{M}_{0,0}(LG(r,2r),2)$.

Finally, we will denote by T^r the restriction to $\overline{M}_{0,0}(LG(r,2r),2)$ of the divisor $T^{r-1,2r-1}$, this is the class of the divisor of conics that are tangent to a fixed hyperplane section of $LG(r,2r)$.

Proposition 4.4.3. *Consider the subspaces* $H = \{x_2 = \cdots = x_{r-1} = x_{r+2} = \cdots = x_{2r-1} = 0\}$ *and* $\Pi^{r-3} = \{x_0 = \cdots = x_{r+1} = 0\}$ *in* $\mathbb{P}^{2r-1}$. *There is an embedding*

$$\begin{array}{rccc} i\colon & LG(2,H) & \hookrightarrow & LG(r,2r) \\ & L & \mapsto & \langle L, \Pi^{r-3}\rangle \end{array}$$

which induces an embedding $j\colon \overline{M}_{0,0}(LG(2,4),2) \to \overline{M}_{0,0}(LG(r,2r),2)$. *Moreover, the pull-back map* $j^*\colon \operatorname{Pic}(\overline{M}_{0,0}(LG(r,2r),2)) \to \operatorname{Pic}(\overline{M}_{0,0}(LG(2,4),2))$ *is an isomorphism.*

Proof. Since Π^{r-3} is the projectivization of an isotropic subspace of K^{2r}, and disjoint from H, the map i is well-defined. By [Opr05, Theorem 1] the Picard group of $\overline{M}_{0,0}(LG(r,2r),2)$ is generated by Δ^r and $H^r_{\sigma_2}$.

Furthermore, we have that $i^*(\sigma^r_2) = \sigma^2_2$ and then $j^*(H^r_{\sigma^r_2}) = H^2_{\sigma^2_2}$. Finally, since $j^*(\Delta^r) = \Delta^2$ we conclude that the pull-back map is an isomorphism. $\square$

As noticed in Remark 4.3.2, the connection between $\overline{M}_{0,0}(\mathbb{G}(1,3),2)$ and the space of complete quadrics $\mathcal{Q}(3)$ due to [Hue15, Lemma 21] states that there is a finite morphism of degree two

$$\phi\colon \overline{M}_{0,0}(\mathbb{G}(1,3),2) \to \mathcal{Q}(3) \tag{4.5}$$

which maps a smooth conic $C \subset \mathbb{G}(1,3)$ to the quadric surface $\bigcup_{[L]\in C} L \subset \mathbb{P}^3$.

Lemma 4.4.5. *Let* $C_1, C_2 \subset \mathbb{G}(1,3)$ *be two smooth conics corresponding to the rulings* $\bigcup_{[L]\in C_1} L$ *and* $\bigcup_{[L]\in C_2} L$ *of a smooth quadric* $Q \subset \mathbb{P}^3$. *The following are equivalent:*

(a) C_1 *is contained in* $LG(2,4)$ *but* C_2 *is not;*

(b) the lines in the ruling $\bigcup_{[L]\in C_1} L$ *are Lagrangian while the general line in the ruling* $\bigcup_{[L]\in C_2} L$ *is not;*

(c) the matrix of Q has a scalar multiple that is symplectic.

Proof. The actions of $Sp(4)$ on $\overline{M}_{0,0}(LG(2,4),2)$ in (4.6) and on $\mathcal{S}_4$ in (3.9) are compatible. Therefore, it is enough to prove that the equivalence of the conditions in statement holds for a particular smooth quadric.

Consider the quadric $Q = \{x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0\} \subset \mathbb{P}^3$. If M_Q is the matrix of Q we have $M_Q^t \Omega M_Q = -\Omega$, and hence iM_Q is symplectic.

Now, one of the rulings of Q is given by the following lines

$$L_{s,t} = \langle (t,-s,-t,s),(s,t,s,t)\rangle$$

with $[s:t] \in \mathbb{P}^1$. Note that $L_{s,t}$ is Lagrangian with respect to Ω for all $[s:t] \in \mathbb{P}^1$.

Fix homogeneous coordinates $[Z_0 : \cdots : Z_5]$ on $\mathbb{P}^5$. The Lagrangian Grassmannian $LG(2,4)$ is cut out on the Grassmannian $\mathbb{G}(1,3)$ by the hyperplane $H = \{Z_1 + Z_4 = 0\}$. Via the Plücker embedding the ruling $L_{s,t}$ corresponds to the conic given by the image of the following morphism

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{G}(1,3) \\ (s,t) & \longmapsto & [t^2+s^2 : 2st : t^2-s^2 : -s^2+t^2 : -2st : -t^2-s^2] \end{array}$$

which therefore is contained in $H \cap \mathbb{G}(1,3) = LG(2,4)$. The other ruling of Q is given by

$$R_{u,v} = \langle (u,-v,u,v),(v,u,-v,u)\rangle$$

with $[u:v] \in \mathbb{P}^1$. The corresponding conic is given by the image of

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{G}(1,3) \\ (u,v) & \longmapsto & [u^2+v^2 : -2uv : u^2-v^2 : v^2-u^2 : -2uv, u^2+v^2] \end{array}$$

which is not contained in $H \cap \mathbb{G}(1,3) = LG(2,4)$. Hence, the general line in the ruling $R_{u,v}$ is not Lagrangian. $\square$

Lemma 4.4.6. *The following $Sp(4)$-action on $\overline{M}_{0,0}(LG(2,4),2)$*

$$\begin{array}{ccc} Sp(4) \times \overline{M}_{0,0}(LG(2,4),2) & \longrightarrow & \overline{M}_{0,0}(LG(2,4),2) \\ (M,[C,\alpha]) & \longmapsto & [C, \wedge^r M \circ \alpha] \end{array} \tag{4.6}$$

gives to $\overline{M}_{0,0}(LG(2,4),2)$ a structure of spherical variety.

Proof. By Lemma 4.4.5 a ruling of the quadric $Q = \{x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0\}$ yields a conic in $LG(2,4)$. Let $\mathscr{B} \subset Sp(4)$ be the Borel subgroup of the symplectic group in Remark 3.2.4. Note that $\dim(\mathscr{B}) = 6$. The stabilizer of Q in $\mathscr{B}$ is given by

$$\begin{pmatrix} A_{2,2} & 0_{2,2} \\ B_{2,2} & A_{2,2}^{-t} \end{pmatrix} \begin{pmatrix} I_{2,2} & 0_{2,2} \\ 0_{2,2} & -I_{2,2} \end{pmatrix} \begin{pmatrix} A_{2,2}^t & B_{2,2}^t \\ 0_{2,2} & A_{2,2}^{-1} \end{pmatrix} = \begin{pmatrix} A_{2,2}^t A_{2,2} & A_{2,2}B_{2,2}^t \\ B_{2,2}A_{2,2}^t & B_{2,2}B_{2,2}^t - A_{2,2}^{-t}A_{2,2}^{-1} \end{pmatrix}$$

So, we get $B_{2,2} = 0_{2,2}$ and $A_{2,2}^t A_{2,2} = I_{2,2}$. Then

$$Stab_{\mathscr{B}}(Q) = \left\{ M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & \frac{1}{b} \end{pmatrix} ;\ \text{with } a^2 = b^2 = 1 \right\}$$

and $\dim(Stab_{\mathscr{B}}(Q)) = 0$. $\square$

Proposition 4.4.8. *The restriction of the map in (4.5) to $\overline{M}_{0,0}(LG(2,4),2)$ yields an isomorphism*

$$\varphi\colon \overline{M}_{0,0}(LG(2,4),2) \to \mathcal{S}_4 \tag{4.7}$$

where $\mathcal{S}_4$ is the wonderful compactification of the space of symplectic quadrics of $\mathbb{P}^3$.

Proof. By Lemma 4.4.5 the restriction of the map in (4.5) to $\overline{M}_{0,0}(LG(2,4),2)$ yields a 1-to-1 morphism which is surjective since both $\overline{M}_{0,0}(LG(2,4),2)$ and $\mathcal{S}_4$ are 6-dimensional.

Finally, since $\mathcal{S}_4$ is smooth and $\overline{M}_{0,0}(LG(2,4),2)$ is normal Zariski's main theorem [Mum99, Chapter 3, Section 9] yields that the morphism in (4.7) is an isomorphism. □

Lemma 4.4.10. *The divisor classes Δ^2, D^2_{unb} and the divisor classes $H^2_{\sigma_2}, T^2$ are respectively the classes of the boundary divisors and the colors of the spherical variety $\overline{M}_{0,0}(LG(2,4),2)$.*

Proof. The actions (4.6) and (3.9) are equivariant with respect to the map φ in (4.7). So boundary divisors and colors of $\overline{M}_{0,0}(LG(2,4),2)$ are mapped by φ to boundary divisors and colors of $\mathcal{S}_4$ respectively. By Proposition 3.2.27, in $\mathcal{S}_4$ the colors are D_1, D_2 and the boundary divisors are $E_1, S_2^{(1)}(\mathcal{V}^3)$. Moreover, Δ^2, D^2_{unb} are stabilized by the $Sp(4)$-action in (4.6) and choosing the flag of isotropic linear subspaces $\{x_0 = x_1 = 0\} \subset \{x_0 = 0\}$ we see that $H^2_{\sigma_2}, T^2$ are stabilized by the action of the Borel subgroup of $Sp(4)$ in Remark 3.2.4. Moreover, it is straightforward to see that the inverse image via the morphism φ in (4.7) of $S_2^{(1)}(\mathcal{V}^3), E_1, D_1, D_2$ are divisors of classes $\Delta^2, D^2_{unb}, H^2_{\sigma_2}, T^2$. Now, assume to have another boundary divisor in $\overline{M}_{0,0}(LG(2,4),2)$. Then, φ maps this divisor to a boundary divisor of $\mathcal{S}_4$, but the only boundary divisors of $\mathcal{S}_4$ are $S_2^{(1)}(\mathcal{V}^3), E_1$. Then, the only boundary divisors of $\overline{M}_{0,0}(LG(2,4),2)$ are Δ^2, D^2_{unb}, and similarly the only colors of $\overline{M}_{0,0}(LG(2,4),2)$ are $H^2_{\sigma_2}, T^2$. □

We denote by $\overline{M}_{0,0}(LG(r,2r),2,1)$ the moduli space of weighted stable maps to $LG(r,2r)$. In this space degree one tails of a stable map are replaced by their attaching point. We refer to [MM07] for the construction of moduli of weighted stable maps.

Proposition 4.4.11. *The divisors Δ^r, D^r_{unb} generate the effective cone of $\overline{M}_{0,0}(LG(r,2r),2)$, and the divisors $H^r_{\sigma_2}, T^r$ generate the nef cone of $\overline{M}_{0,0}(LG(r,2r),2)$.*

The divisor $H^r_{\sigma_2}$ induces a birational morphism

$$f_{H^r_{\sigma_2}}\colon \overline{M}_{0,0}(LG(r,2r),2) \to \widetilde{Chow}(LG(r,2r),2)$$

which is an isomorphism away form the locus $Q^r(1)$ of double covers of a line in $LG(r,2r)$, and contracts $Q^r(1)$ so that the locus of double covers with the same image maps to a point, where $\widetilde{Chow}(LG(r,2r),2)$ is the normalization of the Chow variety of conics in $LG(r,2r)$.

The divisor T^r induces a morphism

$$f_{T^r}\colon \overline{M}_{0,0}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2,1)$$

which is an isomorphism away from Δ^r and contracts the locus of maps with reducible domain $[C_1 \cup C_2, \alpha]$ to $\alpha(C_1 \cap C_2)$. Hence, f_{T^r} contracts the divisor Δ^r onto $LG(r,2r) \subset \overline{M}_{0,0}(LG(r,2r),2,1)$.

Proof. By [ADHL15, Proposition 4.5.4.4] and Lemma 4.4.10 the effective cone of $\overline{M}_{0,0}(LG(2,4),2)$ is generated by $\Delta^2, D^2_{unb}, H^2_{\sigma_2}, T^2$. Consider the isomorphism φ in (4.7). We have

$$\varphi^* E_1 = D^2_{unb},\ \varphi^* S_2^{(1)}(\mathcal{V}^3) = \Delta^2,\ \varphi^* D_1 = H^2_{\sigma_2},\ \varphi^* D_2 = T^2.$$

Now, the relations among the boundary divisors and the colors of $\mathcal{S}_4$ in Proposition 3.2.31 yield the following relations in the Picard group of $\overline{M}_{0,0}(LG(2,4),2)$:

$$H^2_{\sigma_2} \sim \frac{\Delta^2 + 2D^2_{unb}}{2},\ T^2 \sim \Delta^2 + D^2_{unb} \tag{4.8}$$

and the statement in the case $r = 2$ follows from Propositions 3.2.28 and 4.4.8.

Now, consider the case $r > 2$. Since T^r is the pull-back of $T^{r-1,2r-1}$ via the embedding $\overline{M}_{0,0}(LG(r,2r),2) \hookrightarrow \overline{M}_{0,0}(\mathbb{G}(r-1,2r-1),2)$ [CC10, Theorem 3.8] yields that T^r induces a morphism

$$f_{T^r}\colon \overline{M}_{0,0}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2,1)$$

which is an isomorphism away from Δ^r and contracts the locus of maps with reducible domain $[C_1 \cup C_2, \alpha]$ to $\alpha(C_1 \cap C_2)$. Hence, f_{T^r} contracts the divisor Δ^r onto $LG(r,2r) \subset \overline{M}_{0,0}(LG(r,2r),2,1)$. So Δ^r generates an extremal ray of the effective cone, and T^r generates an extremal ray of the nef cone.

Similarly, [CC10, Proposition 3.7] yields the morphism $f_{H^r_{\sigma_2}}\colon \overline{M}_{0,0}(LG(r,2r),2) \to \widetilde{Chow}(LG(r,2r),2)$, and hence $H^r_{\sigma_2}$ generates the other extremal ray of the nef cone.

Now, following the proof of [CC10, Lemma 3.4] we define the class of a curve Γ in $\overline{M}_{0,0}(LG(r,2r),2)$ whose deformations cover the whole of $\overline{M}_{0,0}(LG(r,2r),2)$. Consider a general hyperplane section Z of $LG(2,4) \subset \mathbb{P}^4$, and a general line in this hyperplane section. The planes containing the line cut out a pencil of conics on $Z \subset LG(2,4)$. Hence we get a rational curve $C \subset \overline{M}_{0,0}(LG(2,4),2)$ parametrizing these conics. Let Γ be the image of C via the embedding in Proposition 4.4.3. Then $H^r_{\sigma_2} \cdot \Gamma = 1$, and $\Delta^r \cdot \Gamma = 2$ since there are two reducible conics in a general pencil of conics in the quadric surface Z. Now, by (4.8) we get that $D^r_{unb} \cdot \Gamma = 0$, and by [BDPP13, Theorem 2.2] we conclude that D^r_{unb} generates the other extremal ray of the effective cone. □

Remark 4.4.13. Note that $Q^r(1)$ is a divisor in $\overline{M}_{0,0}(LG(r,2r),2)$ if and only if $r = 2$. By Proposition 4.4.8 we have $\overline{M}_{0,0}(LG(2,4),2) \cong \mathcal{S}_4$ which by Proposition 3.2.16 is the blow-up of $\mathbb{G}(1,4)$ along the Veronese $\mathcal{V}^3$. In this case

$$f_{H^2_{\sigma_2}}\colon \overline{M}_{0,0}(LG(2,4),2) \to \widetilde{Chow}(LG(2,4),2)$$

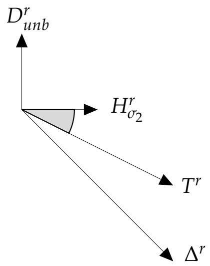
is nothing but the blow-down morphism $\mathcal{S}_4 \to \mathbb{G}(1,4)$. Indeed, since $LG(2,4) \subset \mathbb{P}^4$ is a quadric hypersurface and hence does not contain any plane we have that all planes in $\mathbb{P}^4$ cut out a conic on $LG(2,4)$. Hence, we may identify the Chow variety of conics in $LG(2,4)$ with $\mathbb{G}(2,4) \cong \mathbb{G}(1,4)$.

Furthermore, by Proposition 3.2.31 the morphism

$$f_{T^2}\colon \overline{M}_{0,0}(LG(2,4),2) \to \overline{M}_{0,0}(LG(2,4),2,1)$$

is induced by the strict transform of the restriction to $\mathbb{G}(1,4)$ of the linear system of quadrics in $\mathbb{P}^9$ containing $\mathcal{V}^3$. In this way we realize $\overline{M}_{0,0}(LG(2,4),2,1)$ as a 6-fold of degree 40 in $\mathbb{P}^{14}$ which is singular along a 3-fold isomorphic to $LG(2,4)$.

Theorem 4.4.14. *The Mori chamber decomposition of* $\mathrm{Eff}(\overline{M}_{0,0}(LG(r,2r),2))$ *has three chambers as displayed in the following picture:*



where $H^r_{\sigma_2} \sim \frac{1}{2}(\Delta^r + 2D^r_{unb})$ *and* $T^r \sim \Delta^r + D^r_{unb}$. *Furthermore,* $\mathrm{Mov}(\overline{M}_{0,0}(LG(r,2r),2))$ *is generated by* T^r *and* D^r_{unb} *if* $r > 2$, *while* $\mathrm{Mov}(\overline{M}_{0,0}(LG(2,4),2))$ *is generated by* T^2 *and* $H^2_{\sigma_2}$. *The Cox ring* $\mathrm{Cox}(\overline{M}_{0,0}(LG(2,4),2))$ *is generated by the sections of* $\Delta^2, D^2_{unb}, H^2_{\sigma_2}, T^2$.

The birational model X_r *corresponding to the chamber delimited by* $H^r_{\sigma_2}$ *and* D^r_{unb} *is a fibration* $X_r \to SG(r-2,2r)$ *with fibers isomorphic to* $\mathbb{G}(2,4)$, *where* $SG(r-2,2r)$ *is the symplectic Grassmannian parametrizing isotropic subspaces of dimension* $r-2$. *Finally,* D^r_{unb} *contracts* $\overline{M}_{0,0}(LG(r,2r),2)$ *onto* $SG(r-2,2r)$.

Proof. First consider the case $r = 2$. The statement on the generators of the Cox ring follows from Proposition 4.4.11 and Remark 2.2.7. Furthermore, by Remarks 2.2.6 and 2.2.7 the Mori chamber decomposition of $\mathrm{Eff}(\overline{M}_{0,0}(LG(2,4),2))$ is a, possibly trivial, coarsening of the decomposition in the statement. Since by Proposition 4.4.11 the effective cone $\mathrm{Eff}(\overline{M}_{0,0}(LG(2,4),2))$ is generated by Δ^2 and D^2_{unb}, and $H^2_{\sigma_2}, T^2$ generate $\mathrm{Nef}(\overline{M}_{0,0}(LG(2,4),2))$ no ray can be removed, and the Mori chamber decomposition is as in the statement. The relations $H^r_{\sigma_2} \sim \frac{1}{2}(\Delta^r + 2D^r_{unb})$ and $T^r \sim \Delta^r + D^r_{unb}$ follow from the proof of Proposition 4.4.3 and (4.8).

Now, consider the case $r > 2$. By Proposition 4.4.11 the wall-crossing of T^r induces a divisorial contraction, and a divisor inside the chamber delimited by T^r and $H^r_{\sigma_2}$ is ample. By Proposition 4.4.11 the wall-crossing of $H^r_{\sigma_2}$ yields a birational contraction whose exceptional locus is the variety $Q^r(1)$ of double covers of a line in $LG(r,2r)$.

Next, we will construct the birational model of $\overline{M}_{0,0}(LG(r,2r),2)$ corresponding to the chamber delimited by $H^r_{\sigma_2}$ and D^r_{unb}. Let $H \subset \mathbb{P}^{2r-1}$ be an $(r+1)$-plane containing an isotropic $(r-1)$-plane $\Pi \subset \mathbb{P}^{2r-1}$. Then $\Pi = \Pi^\perp \supset H^\perp$. So $H^\perp \subset H$. Now, the $(r+1)$-planes containing their orthogonal are in bijection with the $(r-3)$-planes of $\mathbb{P}^{2r-1}$ that are isotropic. The variety parametrizing such $(r-3)$-planes is the symplectic Grassmannian $SG(r-2,2r)$. Let $\mathcal{U}_r$ be the universal bundle on $SG(r-2,2r)$, $\mathcal{U}_r^\perp \subset \mathcal{U}_r$ its orthogonal, and $\mathcal{Q}_r = \mathcal{U}_r/\mathcal{U}_r^\perp$ the quotient bundle. Then $\mathcal{Q}_r$ has rank four, and we may consider the relative Lagrangian Grassmannian $LG(2,\mathcal{Q}_r) \to SG(r-2,2r)$, and the relative Hilbert scheme $\mathrm{Hilb}_2(LG(2,\mathcal{Q}_r)) \to SG(r-2,2r)$. Note that since $LG(2,4)$ does not contain planes the fibers of $\mathrm{Hilb}_2(LG(2,\mathcal{Q}_r)) \to SG(r-2,2r)$ are isomorphic to $\mathbb{G}(2,4)$.

Indeed, we can associate to a plane in $\mathbb{P}^4$ the conic it cuts out on $LG(2,4)$. Set $X_r := \mathrm{Hilb}_2(LG(2,\mathcal{Q}_r)) \to SG(r-2,2r)$. Note that

$$\begin{aligned}\dim(X_r) &= \dim(SG(r-2,2r)) + 6\\ &= 2r^2 - 4r - \frac{3(r-2)^2 - r + 2}{2} + 6\\ &= \frac{r^2+5r-2}{2}\\ &= \dim(\overline{M}_{0,0}(LG(r,2r),2))\end{aligned}$$

and there is a birational transformation $\overline{M}_{0,0}(LG(r,2r),2) \dashrightarrow X_r$ inducing an isomorphism between the complement of $Q^r(1)$ in $\overline{M}_{0,0}(LG(r,2r),2)$ and the complement of the locus of double lines in X_r. Since $r > 2$ both these loci are in codimension greater that one. Furthermore, $H^r_{\sigma_2}$ induces a morphism on X_r associating to a conic the reduced curve on which it is supported. Hence, this morphism is birational and contracts the locus of double lines. Finally D^r_{unb} induces on X_r the fibration $X_r \to SG(r-2,2r)$. Indeed, this fibration yields the rational fibration $\overline{M}_{0,0}(LG(r,2r),2) \dashrightarrow SG(r-2,2r)$ associating to a stable map that is not 2-to-1 onto a line the orthogonal of the $(r+1)$-plane in $\mathbb{P}^{2r-1}$ generated by the $(r-1)$-planes parametrized by the image of the map. Hence, the cone generated by $H^r_{\sigma_2}$ and D^r_{unb} is the nef cone of X_r.

Finally, the claim about the movable cones follows from Remark 4.4.13 since $H^2_{\sigma_2}$ induces a divisorial contraction, while for $r > 2$ the divisor $H^2_{\sigma_2}$ yields a small contraction and D^r_{unb} induces a non trivial fibration. □

We now study the positivity of the anti-canonical divisor of $\overline{M}_{0,0}(LG(r,2r),2)$.

Proposition 4.4.15. *Let $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ be the smooth Deligne-Mumford stack of degree two stable maps to $LG(r,2r)$, $\overline{H}^r_{\sigma_2}, \overline{T}^r, \overline{\Delta}^r, \overline{D}^r_{unb}$ the divisors on $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ corresponding to $H^r_{\sigma_2}, T^r, \Delta^r, D^r_{unb}$ respectively.*

The anti-canonical divisor of the stack $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ is given by

$$-K_{\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)} = 5\overline{H}^r_{\sigma_2} + \frac{r-7}{2}\overline{D}^r_{unb}$$

for $r > 2$, while $-K_{\overline{\mathcal{M}}_{0,0}(LG(2,4),2)} = 5\overline{H}^2_{\sigma_2} - 5\overline{D}^2_{unb}$. Furthermore, the anti-canonical divisor of $\overline{M}_{0,0}(LG(2,4),2)$ is given by

$$-K_{\overline{M}_{0,0}(LG(r,2r),2)} = 5H^r_{\sigma_2} + \frac{r-7}{2}D^r_{unb}$$

for $r > 2$, while for $r = 2$ we have that

$$-K_{\overline{M}_{0,0}(LG(2,4),2)} = 5H^2_{\sigma_2} - 2D^2_{unb}.$$

Proof. We will compute the canonical divisor of $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ using the formula in [JS17, Theorem 1.1]. So, we need the Chern classes $c_1(T_{LG(r,2r)}), c_2(T_{LG(r,2r)})$, where $T_{LG(r,2r)}$ is the tangent bundle of $LG(r,2r)$. Recall that $T_{LG(r,2r)} \cong \mathrm{Sym}^2(S^\vee)$, where S is the universal bundle.

Let us pretend that $S^\vee = L_1 \oplus \cdots \oplus L_r$ splits as direct sum of line bundles. We will then use Whitney's formula along with the splitting principle to compute the Chern classes of $\mathrm{Sym}^2(S^\vee)$. Set $c_1(L_i) = \alpha_i$ for $i = 1, \ldots, r$. Then

$$c(S^\vee) = \prod_{i=1}^{r}(1+\alpha_i)$$

and hence

$$\begin{aligned} c_1(S^\vee) &= \alpha_1 + \cdots + \alpha_r, \\ c_2(S^\vee) &= \alpha_1\alpha_2 + \cdots + \alpha_1\alpha_r + \alpha_2\alpha_3 + \cdots + \alpha_{r-1}\alpha_r. \end{aligned} \tag{4.9}$$

Furthermore

$$\mathrm{Sym}^2(S^\vee) = L_1^{\otimes 2} \oplus (L_1 \otimes L_2) \oplus \cdots \oplus (L_1 \otimes L_r) \oplus L_2^{\otimes 2} \oplus \cdots \oplus L_r^{\otimes 2}$$

yields

$$\begin{aligned} c(\mathrm{Sym}^2(S^\vee)) &= (1+2\alpha_1)(1+\alpha_1+\alpha_2)\ldots(1+\alpha_1+\alpha_r)(1+2\alpha_2)\ldots(1+2\alpha_r) \\ &= 1 + (r+1)\sum_{i=1}^{r}\alpha_i + \frac{r^2+r-2}{2}\sum_{i=1}^{r}\alpha_i^2 + (r^2+2r)(\alpha_1\alpha_2 + \cdots + \alpha_{r-1}\alpha_r) + \ldots \\ &= 1 + (r+1)\sum_{i=1}^{r}\alpha_i + \frac{r^2+r-2}{2}(\sum_{i=1}^{r}\alpha_i)^2 + (r+2)(\alpha_1\alpha_2 + \cdots + \alpha_{r-1}\alpha_r) + \ldots \\ &= 1 + (r+1)c_1(S^\vee) + \frac{r^2+r-2}{2}c_1(S^\vee)^2 + (r+2)c_2(S^\vee) + \ldots \end{aligned}$$

where in the last equality we plugged-in the formulas in (4.9). Recall that $c_1(S^\vee) = \sigma_1^r$, $c_2(S^\vee) = \sigma_2^r$ and that by (4.4) we have $(\sigma_1^r)^2 = 2\sigma_2^r$. Hence

$$c_1(T_{LG(r,2r)}) = (r+1)\sigma_1^r, \quad c_2(T_{LG(r,2r)}) = (r^2+2r)\sigma_2^r.$$

Now, plugging-in these formulas in [JS17, Theorem 1.1] we get

$$K_{\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)} = -\frac{2r+6}{4}\overline{H}_{\sigma_2}^r + \frac{r-7}{4}\overline{\Delta}^r.$$

Let $\pi\colon \overline{\mathcal{M}}_{0,0}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2)$ be the canonical morphism from $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ to its coarse moduli space. Note that $\pi\colon \overline{\mathcal{M}}_{0,0}(LG(r,2r),2) \to \overline{M}_{0,0}(LG(r,2r),2)$ is an isomorphism in codimension one for all $r > 2$, while for $r = 2$ it is ramified on the divisor D_{unb}^2. When $r = 2$ the stack has non trivial inertia along the divisor $\overline{D}_{unb}^2$ since a general stable map in $\overline{D}_{unb}^2$ has automorphism group $\mathbb{Z}/2\mathbb{Z}$. Taking this into account we get that $\pi^* D_{unb}^2 = 2\overline{D}_{unb}^2$, and hence Theorem 4.4.14 yields $\overline{\Delta}^r = 2\overline{H}_{\sigma_2}^r - 2\overline{D}_{unb}^r$ if $r > 2$, and $\overline{\Delta}^2 = 2\overline{H}_{\sigma_2}^2 - 4\overline{D}_{unb}^2$. So, in terms of $\overline{H}_{\sigma_2}^r$ and $\overline{D}_{unb}^r$ the canonical divisor of the stack is given by

$$K_{\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)} = -5\overline{H}_{\sigma_2}^r - \frac{r-7}{2}\overline{D}_{unb}^r$$

if $r > 2$, and $K_{\overline{\mathcal{M}}_{0,0}(LG(2,4),2)} = -5\overline{H}_{\sigma_2}^2 + 5\overline{D}_{unb}^2$. Furthermore, when $r > 2$ the formula above gives the expression of the canonical divisor of $\overline{M}_{0,0}(LG(r,2r),2)$

in the statement since $\overline{M}_{0,0}(LG(r,2r),2)$ and $\overline{\mathcal{M}}_{0,0}(LG(r,2r),2)$ are isomorphic in codimension one for $r > 2$.

However, when $r = 2$ we have that

$$K_{\overline{\mathcal{M}}_{0,0}(LG(2,4),2)} = \pi^* K_{\overline{M}_{0,0}(LG(2,4),2)} + \overline{D}^2_{unb}.$$

Let us write $K_{\overline{M}_{0,0}(LG(2,4),2)} = -5H^2_{\sigma_2} + aD^2_{unb}$. Recalling that $\pi^* D^2_{unb} = 2\overline{D}^2_{unb}$ we get

$$-5\overline{H}^2_{\sigma_2} + 5\overline{D}^2_{unb} = K_{\overline{\mathcal{M}}_{0,0}(LG(2,4),2)} = \pi^*(-5H^2_{\sigma_2} + aD^2_{unb}) + \overline{D}^2_{unb} = -5\overline{H}^2_{\sigma_2} + (2a+1)\overline{D}^2_{unb}.$$

Hence, $a = 2$ and $K_{\overline{M}_{0,0}(LG(2,4),2)} = -5H^2_{\sigma_2} + 2D^2_{unb}$. □

Remark 4.4.17. Since $\omega_{\mathbb{G}(1,4)} = \mathcal{O}_{\mathbb{G}(1,4)}(-5)$ and $\operatorname{codim}_{\mathbb{G}(1,4)}(\mathcal{V}^3) = 3$ the formula $K_{\overline{M}_{0,0}(LG(2,4),2)} = -5H^2_{\sigma_2} + 2D^2_{unb}$ can also be deduced from the description of $\overline{M}_{0,0}(LG(2,4),2)$ as the blow-up of $\mathbb{G}(1,4)$ along $\mathcal{V}^3$ in Proposition 4.4.8.

Corollary 4.4.18. *The moduli space $\overline{M}_{0,0}(LG(r,2r),2)$ is Fano for $2 \leqslant r \leqslant 6$, weak Fano for $r = 7$, and $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ is not ample for $r \geqslant 8$.*

Proof. By Propositions 4.4.14 and 4.4.15 we have that $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ is a multiple of $H^r_{\sigma_2}$ if $r = 7$. Moreover, $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ lies in the interior of $\operatorname{Nef}(\overline{M}_{0,0}(LG(r,2r),2))$ for $2 \leqslant r \leqslant 6$, while for $r \geqslant 8$ we have that $-K_{\overline{M}_{0,0}(LG(r,2r),2)}$ lies in the interior of the cone generated by $H^r_{\sigma_2}$ and D^r_{unb}. □

Finally, the following result on automorphisms of $\overline{M}_{0,0}(LG(2,4),2)$ is at hand.

Corollary 4.4.19. *The automorphism group of $\overline{M}_{0,0}(LG(2,4),2)$ is given by*

$$\operatorname{PsAut}(\overline{M}_{0,0}(LG(2,4),2)) \cong \operatorname{Aut}(\overline{M}_{0,0}(LG(2,4),2)) \cong PSp(4)$$

where $PSp(4)$ is the projective symplectic group, and $\operatorname{PsAut}(\overline{M}_{0,0}(LG(2,4),2))$ is the group of birational self-maps of $\overline{M}_{0,0}(LG(2,4),2)$ inducing automorphisms in codimension one.

Proof. By Propositions 3.2.16, 4.4.8 we have that $\overline{M}_{0,0}(LG(2,4),2)$ is isomorphic to the blow-up of $\mathbb{G}(1,4)$ along the Veronese $\mathcal{V}^3$. Let $\varphi \in \operatorname{Aut}(\overline{M}_{0,0}(LG(2,4),2))$ be an automorphism. Then either ϕ preserves the two extremal rays of the effective cone $\operatorname{Eff}(\overline{M}_{0,0}(LG(2,4),2))$ in Theorem 4.4.14 or it swaps them. In the second case ϕ must swap also the extremal rays of $\operatorname{Nef}(\overline{M}_{0,0}(LG(2,4),2))$ but this is not possible since for instance T^2 has more sections than $H^2_{\sigma_2}$. Therefore, ϕ stabilizes the exceptional divisor D^2_{unb} of the blow-up and then it induces an automorphism $\overline{\phi}$ of $\mathbb{G}(1,4)$ that stabilizes $\mathcal{V}^3$.

Now, the automorphism group of $\mathbb{G}(1,4)$ is isomorphic to $PGL(5)$ and all these automorphisms are induced by automorphisms of the ambient projective space $\mathbb{P}^9$ [Cow89, Theorem 1.1]. The restriction of $\overline{\phi}$ to $\mathcal{V}^3$ yields an automorphism $\overline{\phi}_{|\mathcal{V}^3}$ of $\mathbb{P}^3$. Since $\overline{\phi}$ is an automorphism of $\mathbb{G}(1,4)$, which we interpret as the closure of the space of symplectic and symmetric matrices modulo scalar, the restriction $\overline{\phi}_{|\mathcal{V}^3} \in PGL(4)$

must map symplectic matrices to symplectic matrices. Hence, $\overline{\phi}_{|\mathcal{V}^3} \in PSp(4)$. So, we get a morphism of groups

$$\begin{array}{rccc} \chi\colon & \mathrm{Aut}(\overline{M}_{0,0}(LG(2,4),2)) & \to & PSp(4) \\ & \phi & \mapsto & \overline{\phi}_{|\mathcal{V}^3} \end{array}$$

which is surjective. Now, if $\overline{\phi}_{|\mathcal{V}^3}$ is the identity it must be the restriction of the identity automorphism of the ambient projective space $\mathbb{P}^9$ in which both $\mathcal{V}^3$ and $\mathbb{G}(1,4)$ are embedded. Since $\mathbb{G}(1,4)$ and $\overline{M}_{0,0}(LG(2,4),2)$ are birational we get that $\overline{\phi}_{|\mathcal{V}^3}$ must come from the identity of $\mathrm{Aut}(\overline{M}_{0,0}(LG(2,4),2))$, and hence χ is an isomorphism. Finally, since by Proposition 4.4.8 and Corollary 4.4.18 $\overline{M}_{0,0}(LG(2,4),2)$ is a smooth Fano variety the result on $\mathrm{PsAut}(\overline{M}_{0,0}(LG(2,4),2))$ follows from [Mas20a, Proposition 7.2]. □

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