

This is a pre print version of the following article:

On the rank of a finite group of odd order with an involutory automorphism / Acciarri, C; Shumyatsky, P. - In: MONATSHEFTE FÜR MATHEMATIK. - ISSN 0026-9255. - 194:3(2021), pp. 461-469. [10.1007/s00605-020-01479-4]

Terms of use:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

18/12/2025 19:08

On the rank of a finite group of odd order with an involutory automorphism

Cristina Acciarri and Pavel Shumyatsky

ABSTRACT. Let G be a finite group of odd order admitting an involutory automorphism ϕ , and let $G_{-\phi}$ be the set of elements of G transformed by ϕ into their inverses. Note that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. Suppose that each subgroup generated by a subset of $G_{-\phi}$ can be generated by at most r elements. We show that the rank of $[G, \phi]$ is r -bounded.

1. Introduction

Let G be a finite group of odd order admitting an involutory automorphism ϕ . Here the term “involutory automorphism” means an automorphism ϕ such that $\phi^2 = 1$. We let $G_{-\phi}$ stand for the set $\{g \in G \mid g^\phi = g^{-1}\}$ and G_ϕ for the centralizer of ϕ , that is, the subgroup of fixed points of ϕ . As usual we denote by $[G, \phi]$ the subgroup generated by all elements of G that can be written as $g^{-1}g^\phi$ for a suitable $g \in G$. It is well known that $[G, \phi]$ is normal in G and ϕ induces the trivial automorphism of $G/[G, \phi]$. Observe that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. This is because an automorphism of order at most two of a group of odd order is nontrivial if and only if $G_{-\phi} \neq \{1\}$ (cf Lemma 2.1(i) in the next section). The following theorem was proved in [10, Theorem B].

THEOREM 1.1. *Let G a finite group of odd order admitting an involutory automorphism ϕ such that the rank of G_ϕ is at most r . Then the rank of $[G, \phi]'$ is r -bounded.*

2010 *Mathematics Subject Classification.* 20D45.

Key words and phrases. Finite groups, automorphisms.

This research was supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and Fundação de Apoio à Pesquisa do Distrito Federal (FAPDF), Brazil.

Recall that a rank of a finite group G is the least number r such that each subgroup of G can be generated by at most r elements. Throughout this manuscript we use the term “ $(a, b, c \dots)$ -bounded” to mean “bounded from above by some function depending only on the parameters $a, b, c \dots$ ”.

Since in a finite group of odd order with an involutory automorphism ϕ there is a kind of (very vague) duality between G_ϕ and $G_{-\phi}$, in this paper we address the question whether a rank condition imposed on the set $G_{-\phi}$ has an impact on the structure of G . We emphasize that $G_{-\phi}$ in general is not a subgroup of G and therefore the usual concept of rank does not apply to $G_{-\phi}$. Instead we impose the condition that each subgroup of G generated by a subset of $G_{-\phi}$ can be generated by at most r elements. Our main result is as follows.

THEOREM 1.2. *Let G be a group of odd order admitting an involutory automorphism ϕ and suppose that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements. Then $[G, \phi]$ has r -bounded rank.*

It is noteworthy that in the literature there are several papers dealing with finite groups admitting a (not necessarily involutory) automorphism whose fixed-point subgroup is of rank r (see for example [6, 5]). In particular, [5] contains a result similar to the above Theorem 1.1. Thus, it seems plausible that some analogues of Theorem 1.2 are valid for the case where the order of ϕ is bigger than two.

2. Nilpotent groups with involutory automorphisms

We start with a collection of well-known facts about involutory automorphisms of groups of odd order (see for example [3, Lemma 4.1, Chap. 10]).

LEMMA 2.1. *Let G be a finite group of odd order admitting an involutory automorphism ϕ . The following conditions hold:*

- (i) $G = G_\phi G_{-\phi} = G_{-\phi} G_\phi$ and $|G_{-\phi}| = [G : G_\phi]$;
- (ii) The subgroup generated by $G_{-\phi}$ is exactly $[G, \phi]$;
- (iii) If N is any ϕ -invariant normal subgroup of G we have $(G/N)_\phi = G_\phi N/N$, and $(G/N)_{-\phi} = \{gN \mid g \in G_{-\phi}\}$;
- (iv) If N is any ϕ -invariant normal subgroup of G such that $N = N_{-\phi}$ or $N = N_\phi$, then $[G, \phi]$ centralizes N ;
- (v) The subgroup G_ϕ normalizes $G_{-\phi}$.

It is well known that a maximal abelian normal subgroup of a nilpotent group coincides with its centralizer. We will require the following related result.

LEMMA 2.2. *Let G be a nilpotent group of odd order, ϕ an involutory automorphism of G , and A a maximal ϕ -invariant abelian normal subgroup of G . Then $A = C_G(A)$.*

PROOF. Let $C = C_G(A)$ and assume that the result is false, that is, $A < C$. Then C/A is a nontrivial ϕ -invariant normal subgroup of G/A . The nilpotency of G/A implies that $C/A \cap Z(G/A) \neq 1$.

Let U be the full inverse image of $C/A \cap Z(G/A)$ in G . Since $C/A \cap Z(G/A) \neq 1$, the subgroup A is properly contained in U . From Lemma 2.1(i) we know that $U = U_\phi U_{-\phi}$. Thus, either $U_\phi \not\leq A$ or $U_{-\phi} \not\leq A$. In any case we can choose $u \in U \setminus A$ satisfying either $u^\phi = u$ or $u^\phi = u^{-1}$. Take $H = A\langle u \rangle$ and note that $A < H$. Furthermore, H is a ϕ -invariant abelian normal subgroup of G . This yields a contradiction. \square

Note that the previous lemma fails if ϕ is allowed to be a coprime automorphism of arbitrary order. For example, the quaternion group of order 8 admits an automorphism α of order 3 and the maximal α -invariant abelian normal subgroup is central.

LEMMA 2.3. *Let p be an odd prime and G a p -group admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let M be a ϕ -invariant normal subgroup of G and assume that $|M_{-\phi}| = p^n$ for some nonnegative integer n . Then $M \leq Z_{2n+1}(G)$.*

PROOF. If $n = 0$, then the result follows from Lemma 2.1(iv), so assume that $n \geq 1$ and use induction on n .

Let $N = M \cap Z_2(G)$. If $N \not\leq Z(G)$, then Lemma 2.1(iv) implies that $N_{-\phi} \neq 1$, in which case we have $|(M/N)_{-\phi}| < |M_{-\phi}| = p^n$. By induction $M/N \leq Z_{2n-1}(G/N)$, whence $M \leq Z_{2n+1}(G)$. If $N \leq Z(G)$, then it turns out that $M \cap Z(G) = M \cap Z_i(G)$ for any $i \geq 2$ and so, obviously, $M \leq Z(G)$. This concludes the proof. \square

We now fix some notation and hypotheses that will be used throughout this section.

HYPOTHESIS 2.4. *Let p be an odd prime, r a positive integer and G a finite p -group with an involutory automorphism ϕ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements.*

LEMMA 2.5. *Assume Hypothesis 2.4 and suppose that G is of exponent p . There exists a number $l = l(r)$, depending on r only, such that the rank $r(G)$ of G is at most l .*

PROOF. Let A be a maximal ϕ -invariant abelian normal subgroup of G . The subgroup $\langle A_{-\phi} \rangle$ is an r -generated abelian subgroup of exponent p and so $|A_{-\phi}| \leq p^r$. Lemma 2.3 implies that $A \leq Z_{2r+1}(G)$. Since

$\gamma_{2r+1}(G)$ commutes with $Z_{2r+1}(G)$, we deduce that $\gamma_{2r+1}(G)$ centralizes A . Furthermore, by Lemma 2.2, $A = C_G(A)$. Thus $\gamma_{2r+1}(G) \leq A$, that is, the quotient group G/A is nilpotent of class $2r$. We deduce that G has r -bounded nilpotency class as well. Since $G = \langle G_{-\phi} \rangle$ is r -generated by hypothesis, it follows that the rank $r(G)$ of G is r -bounded, as desired. \square

The following result from [10, Lemma 2.2] is also useful.

LEMMA 2.6. *Let G be a group of prime exponent p and rank r_0 . Then there exists a number $s = s(r_0)$, depending only on r_0 , such that $|G| \leq p^s$.*

LEMMA 2.7. *Let G be a group satisfying Hypothesis 2.4. There exists a number $\lambda = \lambda(r)$, depending only on r , such that $\gamma_{2\lambda+1}(G)$ is powerful.*

PROOF. Let $s(r_0)$ be as in Lemma 2.6 and let $l(r)$ be as in Lemma 2.5. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = s(l(r))$. In order to show that $N' \leq N^p$, we assume that N is of exponent p and prove that N is abelian.

Note that the subgroup $\langle N_{-\phi} \rangle$ is of exponent p . By Lemma 2.5 the rank of $\langle N_{-\phi} \rangle$ is at most $l(r)$. It follows from Lemma 2.6 that $|N_{-\phi}| \leq p^{s(l(r))} = p^\lambda$. Now Lemma 2.3 yields $N \leq Z_{2\lambda+1}(G)$. By using the well-known fact that $[\gamma_i(G), Z_i(G)] = 1$, for any positive integer i and any group G , we conclude that N is abelian, as required. \square

LEMMA 2.8. *Assume Hypothesis 2.4. For any $i \geq 1$, there exists a number $m_i = m_i(i, r)$, depending only on i and r , such that $\gamma_i(G)$ is an m_i -generated group.*

PROOF. Let $N = \gamma_i(G)$. In view of the Burnside Basis Theorem [9, 5.3.2], we can pass to the quotient $G/\Phi(N)$ and assume that N is elementary abelian. Now $\langle N_{-\phi} \rangle$ is an elementary abelian r -generated group, so $|\langle N_{-\phi} \rangle| \leq p^r$. Thus, by Lemma 2.3, we have $N \leq Z_{2r+1}(G)$ and deduce that G has nilpotency class bounded only in terms of i and r . Since $G = \langle G_{-\phi} \rangle$ is r -generated, we conclude that $r(G)$ is (i, r) -bounded as well. Therefore N is m_i -generated for some (i, r) -bounded number m_i . This concludes the proof. \square

PROPOSITION 2.9. *Under Hypothesis 2.4 the rank of G is r -bounded.*

PROOF. Let $s(r_0)$ be as in Lemma 2.6 and $l(r)$ as in Lemma 2.5. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = \lambda(r) = s(l(r))$. Let d be the minimal number such that N is d -generated. Lemma 2.8 tells us that d is an r -bounded integer and N is powerful by Lemma 2.7. It follows from

[1, Theorem 2.9] that $r(N) \leq d$, and so the rank of N is r -bounded. Since the nilpotency class of G/N is r -bounded (recall that λ depends only on r) and $G = \langle G_{-\phi} \rangle$ is r -generated, we conclude that $r(G/N)$ is r -bounded as well. Now $r(G) \leq r(G/N) + r(N)$ and the result follows. \square

3. Main results

Throughout this section the Feit-Thompson Theorem [2] is used without explicit references and p stands for a fixed odd prime. Given a finite soluble group G , we denote by $r_p(G)$ and $l_p(G)$ the rank of a Sylow p -subgroup and the p -length of G , respectively. Recall that $l_p(G)$ is by definition the number of p -factors (that is, factors that are p -groups) of the lower p -series of G given by:

$$1 \leq O_{p'}(G) \leq O_{p',p}(G) \leq O_{p',p,p'}(G) \leq \cdots$$

We aim to establish the following generalisation of Proposition 2.9.

THEOREM 3.1. *Let G be a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let r be a positive integer and assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements, then $r_p(G)$ is r -bounded.*

We start with an extension of Lemma 2.3.

LEMMA 3.2. *Let G be a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Let M be a ϕ -invariant normal subgroup of G and assume that $|M_{-\phi}| \leq p^n$, for some nonnegative integer n . Then $M \leq Z_{2n+1}(O_p(G))$.*

PROOF. The proof can be reproduced word-by-word following that of Lemma 2.5. We argue by induction on n , being Lemma 2.1(iv) the case $n = 0$. Let $n \geq 1$. If $M \not\leq Z(O_p(G))$, then by Lemma 2.1(iv) we have $N_{-\phi} \neq 1$, where $N = M \cap Z_2(O_p(G))$. This implies that $|(M/N)_{-\phi}| < |M_{-\phi}|$. Thus we can pass to the quotient G/N and use the inductive hypothesis. The result follows. \square

For the sake of simplicity we fix the following hypothesis that we will use in the next arguments.

HYPOTHESIS 3.3. *Let r be a positive integer and G a group of odd order admitting an involutory automorphism ϕ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by r elements.*

As usual, we denote by $F(G)$ the Fitting subgroup of a group G . Write $F_0(G) = 1$, $F_1(G) = F(G)$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. If G is soluble, then the least number h such that $F_h(G) = G$ is called the Fitting height of G .

One key step forward to the proof of Theorem 3.1 consists in showing that there exists an r -bounded number f such that the f th term of the derived series of G is nilpotent. For our purpose we will require the following result which is an immediate corollary of Hartley-Isaacs Theorem B in [4].

PROPOSITION 3.4. *Let H be a finite group of odd order admitting an involutory automorphism ϕ such that $H = [H, \phi]$. Let k be a field with characteristic different from 2 and V a simple $k\langle\phi\rangle H$ -module. Suppose that $\dim V_{-\phi} = r$. There exists an r -bounded number $\delta = \delta(r)$ such that $\dim V \leq \delta$.*

In the proof of the next proposition we will use the well-known theorem of Zassenhaus (see [11, Satz 7] or [8, Theorem 3.23]) stating that for any $n \geq 1$ there exists a number $j = j(n)$, depending only on n , such that, whenever k is a field, the derived length of any soluble subgroup of $GL(n, k)$ is at most j .

PROPOSITION 3.5. *Assume Hypothesis 3.3. There exists a number $f = f(r)$, depending only on r , such that the f th term $G^{(f)}$ of the derived series of G is nilpotent.*

PROOF. Let $\delta = \delta(r)$ be as in Proposition 3.4 and $f = j(\delta)$ the number given by the Zassenhaus theorem.

Suppose that the proposition is false and let G be a group of minimal possible order for which Hypothesis 3.3 holds while $G^{(f)}$ is not nilpotent. Then G has a unique minimal ϕ -invariant normal subgroup M . Indeed, suppose that G has two minimal ϕ -invariant normal subgroups, say M_1 and M_2 . Then $M_1 \cap M_2 = 1$, being both elementary abelian p -groups for some prime p . Since $|G/M_1| < |G|$, the minimality of G implies that $(G/M_1)^{(f)}$ is nilpotent. For a symmetric argument $(G/M_2)^{(f)}$ is nilpotent too. This yields a contradiction since $G^{(f)}$ can be embedded into a subgroup of $G/M_1 \times G/M_2$ which is nilpotent, being isomorphic to the direct product of $(G/M_1)^{(f)}$ and $(G/M_2)^{(f)}$.

We claim that $M = C_G(M)$. Since M is a p -subgroup, for some prime p and it is unique, the Fitting subgroup $F = F(G)$ is a p -subgroup too. If $\Phi(F)$ is nontrivial, then we immediately get a contradiction because $F(G/\Phi(F)) = F/\Phi(F)$ and, again by the minimality of G , we know that $(G/\Phi(F))^{(f)}$ is nilpotent, so in particular $G^{(f)} \leq F$.

Assume now that $\Phi(F) = 1$ and so F is elementary abelian. If $M = F$, then $M = C_G(M)$, since the Fitting subgroup of a soluble group contains its own centralizer (see, for example, [3, Theorem 1.3, Chap. 6]). Thus we can assume that $M < F$. By hypotheses, on one hand, we know that $G^{(f)} \leq F_2(G)$ (to clarify, for the minimality of G the quotient $(G/F)^{(f)}$ is nilpotent, so it is contained in $F(G/F)$) and, on the other hand, that $(G/M)^{(f)}$ is nilpotent (again by the minimality of G). Now let T be a ϕ -invariant Hall p' -subgroup of $G^{(f)}$. It follows that both FT and MT are ϕ -invariant normal subgroups of G . Indeed, FT/F is normal in G/F , since $(G/F)^{(f)}$ is nilpotent and, similarly, MT/M is normal in G/M , being $(G/M)^{(f)}$ nilpotent as well.

Suppose first that $C_F(T) \neq 1$. Note that $C_F(T) = Z(FT)$, since F is abelian. Thus $C_F(T)$ is a ϕ -invariant normal subgroup of G , because FT is normal and ϕ -invariant. Hence $M \leq C_F(T)$. This implies that T centralizes M and so $MT = T \times M$. Recall that $T \leq F_2(G)$ and $T \cap F = 1$. It follows that T is nilpotent. Then $T \times M$ is normal nilpotent and $T \leq F$, a contradiction.

Thus, $C_F(T) = 1$. On the other hand, we see that $[F, T] \leq M$, since the nilpotent p' -subgroup MT/M and the p -subgroup F/M are both contained in $F(G/M)$ and commute, being $F(G/M)$ nilpotent. Now we have $M < F$ and $F = [F, T] \times C_F(T)$, so it should be $C_F(T) \neq 1$, a contradiction. Thus $M = C_G(M)$, as claimed above.

Then G/M acts faithfully and irreducibly on M . Moreover $\langle M_{-\phi} \rangle$ is r -generated and elementary abelian, so $|\langle M_{-\phi} \rangle| \leq p^r$. Now we can view M as a $G/M\langle\phi\rangle$ -module over the field with p elements. By Proposition 3.4 we have $\dim(M) \leq \delta(r)$. Applying the theorem of Zassenhaus the derived length of G/M is at most $f = f(\delta(r))$. Then $G^{(f)} \leq F$, which concludes the proof. \square

As a by-product of the previous result we obtain a bound for the p -length of G .

COROLLARY 3.6. *Assume Hypothesis 3.3. Then $l_p(G)$ is r -bounded, for any $p \in \pi(G)$.*

PROOF. By Proposition 3.5 we know that $G^{(f)}$ is nilpotent for some r -bounded number f . This implies that the Fitting height $h(G) \leq f$. The result easily follows since it can be shown, by induction on the Fitting height $h(K)$, that $l_p(K) \leq h(K)$ for any finite soluble group K and for any prime $p \in \pi(K)$. \square

The next result will be useful for a reduction argument inside the proof of Theorem 3.1.

LEMMA 3.7. *Let G be a group of odd order admitting an involutory automorphism ϕ . Assume that $G = PB$, where P is a ϕ -invariant normal elementary abelian p -subgroup and B is a cyclic subgroup such that $B = B_{-\phi}$. If $r(P_{-\phi}) = r$, then the rank of $[P, B]$ is at most $2r$.*

PROOF. Let $B = \langle b \rangle$, where b is a generator of B . Let $C = P_\phi$ and $C_0 = C \cap C^b$. Then it follows from Lemma 2.1(i) that

$$[P : C_0] \leq [P : C][P : C^b] \leq p^{2r},$$

since $r(P_{-\phi}) = r$. We claim that $C_0 \leq C_G(b)$. Indeed, choose $x \in C$ such that $x^b \in C$. Then, we have $x^b = (x^b)^\phi = x^{b^{-1}}$ and so x commutes with b^2 . Since b has odd order, it follows that $C_0 \leq C_G(b)$, as claimed. Thus $C_0 \leq Z(G)$. Choose now a_1, \dots, a_{2r} elements that generate P modulo C_0 . By using linearity in P and the fact that C_0 is central in G , we deduce that $[P, b]$ is generated by $[a_1, b], \dots, [a_{2r}, b]$. Hence the result. \square

We are ready to embark on the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Recall that G is a group satisfying Hypothesis 3.3 and we want to show that $r_p(G)$ is r -bounded for any fixed prime $p \in \pi(G)$.

First, we show that G is generated by r -boundedly many elements from $G_{-\phi}$. If G is a p -group, then the claim follows from the Burnside Basis Theorem since $G = \langle G_{-\phi} \rangle$ is r -generated. In the case where G is nilpotent, we have $[G, \phi] = [P_1, \phi] \times \dots \times [P_s, \phi]$, where $\{P_1, \dots, P_s\}$ are the Sylow subgroups of G , so the result easily follows from the case of p -groups. Assume now that G is not nilpotent. Let $h = h(G) \geq 2$. Since we know from the proof of Corollary 3.6 that h is r -bounded, it is sufficient to show that G is generated by (h, r) -boundedly many elements from $G_{-\phi}$. We argue by induction on h . Let $F = F(G)$. By induction there are boundedly many elements $a_1, \dots, a_d \in G_{-\phi}$ such that $G = F\langle a_1, \dots, a_d \rangle$. Let $D = \langle F_{-\phi}, a_1, \dots, a_d \rangle$. Note that D has an r -bounded number of generators from $G_{-\phi}$. Let N be the normal closure of $\langle F_{-\phi} \rangle$ in G . Then N is precisely $\langle F_{-\phi} \rangle^D$ because F normalizes $\langle F_{-\phi} \rangle$ by Lemma 2.1(v). Thus $N \leq D$. Recall that by Lemma 2.1(i) we have $F = F_\phi F_{-\phi}$. Hence the image of F in G/N is contained in $(G/N)_\phi$ and, therefore, it is central by Lemma 2.1(iv). Since $G = FD$, it follows that D/N becomes normal in G/N and, therefore, D is normal in G (because $N \leq D$). Now ϕ acts trivially on the quotient G/D , that is $[G, \phi] \leq D$. Since $G = [G, \phi]$, we have $G = D$. This concludes the proof that G can be generated by r -boundedly many elements from $G_{-\phi}$.

If G is a p -group, then the theorem follows immediately from Proposition 2.9. Assume that G is not a p -group and use induction on $l = l_p(G)$ that is r -bounded by Corollary 3.6. So it is sufficient to show that $r_p(G)$ is (l, r) -bounded. By induction assume that there exists r_1 , depending only on l and r , such that $r_p(K) \leq r_1$ for any ϕ -invariant quotient K of G having $l_p(K)$ at most $l - 1$.

Since $l = l_p(G/O_{p'}(G))$, we can assume that $O_{p'}(G) = 1$. Take $P = O_p(G)$. Note that

$$r_p(G) \leq r(P) + r_p(G/P).$$

Since $l_p(G/[P, G]) \leq l - 1$, by induction the rank $r_p(G/[P, G]) \leq r_1$. Then it is sufficient to bound the rank of P .

Let us show first that P has an r -bounded number of generators. Passing to the quotient $G/\Phi(P)$, we can assume that P is elementary abelian. As showed above, we know that G can be generated by $t = t(r)$ elements from $G_{-\phi}$, say d_1, \dots, d_t . Note that $[P, G] = [P, d_1][P, d_2] \dots [P, d_t]$. In view of Lemma 3.7 each $[P, d_i]$ has rank at most $2r$. Therefore the rank of the image of $[P, G]$ in $G/\Phi(P)$ is at most $2rt$ and by induction on l , $r_p(G/[P, G])$ is r -bounded, so P has an r -bounded number of generators, as claimed.

Next, we claim that for any $i \geq 2$ there exists a number $m_i = m_i(i, r)$, depending only on i and r , such that $V = \gamma_i(P)$ has m_i -bounded number of generators. We can pass to the quotient $G/\Phi(V)$ and assume that V is elementary abelian. Now $\langle V_{-\phi} \rangle$ is an elementary abelian r -generated group, so $|\langle V_{-\phi} \rangle| \leq p^r$. Thus, by Lemma 3.2, we have $V \leq Z_{2r+1}(P)$ and deduce that the nilpotency class of $P/\Phi(V)$ is bounded only in terms of i and r . Since P has an r -bounded number of generators, we conclude that $r(P/\Phi(V))$ is (i, r) -bounded as well. Therefore V is m_i -generated for some (i, r) -bounded number m_i , as claimed.

Let $s(r_0)$ be as in Lemma 2.6 and let $l(r)$ be as in Lemma 2.5. Take $M = \gamma_{2\lambda+1}(P)$, where $\lambda = s(l(r))$. We want to prove that M is powerful. In order to show that $M' \leq M^p$, we assume that M is of exponent p and prove that M is abelian. Note that the subgroup $\langle M_{-\phi} \rangle$ is of exponent p . By Lemma 2.5 the rank of $\langle M_{-\phi} \rangle$ is at most $l(r)$. It follows from Lemma 2.6 that $|M_{-\phi}| \leq p^{s(l(r))} = p^\lambda$. Now Lemma 3.2 yields that $M \leq Z_{2\lambda+1}(P)$. Since $[\gamma_i(P), Z_i(P)] = 1$, for any positive integer i , we conclude that M is abelian, as required.

Let now d_0 be the minimal number such that M is d_0 -generated. It was shown above that d_0 is an r -bounded integer. Since M is powerful, it follows from [1, Theorem 2.9] that $r(M) \leq d_0$, and so the rank of M is r -bounded. Since the nilpotency class of P/M is r -bounded and

P has an r -bounded number of generators, we conclude that $r(P/M)$ is r -bounded as well. Now $r(P) \leq r(P/M) + r(M)$ and the result follows. \square

It is now easy to give the proof of our main result, Theorem 1.2, which states that if G is a group satisfying Hypothesis 3.3, then the rank of G is r -bounded.

PROOF OF THEOREM 1.2. Without loss of generality we can assume that $G = [G, \phi]$. By a result of Kovács [7] for any soluble group H we have $r(H) \leq \max\{r_p(H) \mid p \in \pi(H)\} + 1$. Therefore it is enough to check that $r_p(G)$ is bounded in terms of r only for any $p \in \pi(G)$. This is immediate from Theorem 3.1. \square

References

- [1] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic pro- p groups*. Cambridge 1991.
- [2] W. Feit and J. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 773–1029.
- [3] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.
- [4] B. Hartley and I. M. Isaacs, On characters and fixed points of coprime operator groups. *J. Algebra* **131** (1990), 342–358.
- [5] E. I. Khukhro, Groups with an automorphism of prime order that is almost regular in the sense of rank, *J. London Math. Soc.*(2) **77** (2008), 130–148.
- [6] E. I. Khukhro and V. D. Mazurov, Finite groups with an automorphism of prime order whose centralizer has small rank, *J. Algebra* **301**(2006), 474–492.
- [7] L. G. Kovács, On finite soluble groups. *Math. Z.* **103**(1967), 37–39.
- [8] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups. part 1*, Springer-Verlag, 1972.
- [9] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York (1996).
- [10] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, *Arch. Math.* **71** (1998), 425–432.
- [11] H. Zassenhaus, Beweis eines Satzes über diskrete Gruppen, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 289–312.

CRISTINA ACCIARRI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRASILIA, BRASILIA-DF, 70910-900 BRAZIL

Email address: acciarricristina@yahoo.it

PAVEL SHUMYATSKY: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRASILIA, BRASILIA-DF, 70910-900 BRAZIL

Email address: pavel@unb.br