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# A FOCAL SUBGROUP THEOREM FOR OUTER COMMUTATOR WORDS

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**ABSTRACT.** Let  $G$  be a finite group of order  $p^a m$ , where  $p$  is a prime and  $m$  is not divisible by  $p$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $w$  is an outer commutator word, we prove that  $P \cap w(G)$  is generated by the intersection of  $P$  with the set of  $m$ th powers of all values of  $w$  in  $G$ .

Let  $G$  be a finite group and  $P$  a Sylow  $p$ -subgroup of  $G$ . The Focal Subgroup Theorem states that  $P \cap G'$  is generated by the set of commutators  $\{[g, z] \mid g \in G, z \in P, [g, z] \in P\}$ . This was proved by Higman in 1953 [5]. Nowadays the proof of the theorem can be found in many standard books on group theory (for example, Rose's book [7] or Gorenstein's [3]).

One immediate corollary is that  $P \cap G'$  can be generated by commutators lying in  $P$ . Of course,  $G'$  is the verbal subgroup of  $G$  corresponding to the group word  $[x, y] = x^{-1}y^{-1}xy$ . It is natural to ask the question on generation of Sylow subgroups for other words. More specifically, if  $w$  is a group word we write  $G_w$  for the set of values of  $w$  in  $G$  and  $w(G)$  for the subgroup generated by  $G_w$  (which is called the *verbal subgroup* of  $w$  in  $G$ ), and one is tempted to ask the following question.

**Question.** *Given a Sylow  $p$ -subgroup  $P$  of a finite group  $G$ , is it true that  $P \cap w(G)$  can be generated by  $w$ -values lying in  $P$ , i.e., that  $P \cap w(G) = \langle P \cap G_w \rangle$ ?*

However considering the case where  $G$  is the non-abelian group of order six,  $w = x^3$  and  $p = 3$  we quickly see that the answer to the above question is negative. Therefore we concentrate on the case where  $w$  is a *commutator* word. Recall that a group word is commutator if the sum of the exponents of any indeterminate involved in it is zero. Thus, we deal with the question whether  $P \cap w(G)$  can be generated by  $w$ -values whenever  $w$  is a commutator word.

The main result of this paper is a contribution towards a positive answer to this question: we prove that if  $w$  is an outer commutator word, then  $P \cap w(G)$  can be generated by the *powers of values of  $w$*  which lie in  $P$ . More precisely, we have the following result.

**Theorem A.** *Let  $G$  be a finite group of order  $p^a m$ , where  $p$  is a prime and  $m$  is not divisible by  $p$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $w$  is*

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an outer commutator word, then  $P \cap w(G)$  is generated by  $m$ th powers of  $w$ -values, i.e.,  $P \cap w(G) = \langle P \cap G_{w^m} \rangle$ .

Recall that an *outer commutator word* is a word which is obtained by nesting commutators, but using always *different indeterminates*. Thus the word  $[[x_1, x_2], [x_3, x_4, x_5], x_6]$  is an outer commutator while the Engel word  $[x_1, x_2, x_2, x_2]$  is not. An important family of outer commutator words are the simple commutators  $\gamma_i$ , given by

$$\gamma_1 = x_1, \quad \gamma_i = [\gamma_{i-1}, x_i] = [x_1, \dots, x_i], \quad \text{for } i \geq 2.$$

The corresponding verbal subgroups  $\gamma_i(G)$  are the terms of the lower central series of  $G$ . Another distinguished sequence of outer commutator words are the *derived words*  $\delta_i$ , on  $2^i$  indeterminates, which are defined recursively by

$$\delta_0 = x_1, \quad \delta_i = [\delta_{i-1}(x_1, \dots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})], \quad \text{for } i \geq 1.$$

Then  $\delta_i(G) = G^{(i)}$ , the  $i$ th derived subgroup of  $G$ .

Some of the ideas behind the proof of Theorem A were anticipated already in [4] where somewhat similar arguments, due to Guralnick, led to a result on generation of a Sylow  $p$ -subgroup of  $G'$  for a finite group  $G$  admitting a coprime group of automorphisms. Later the arguments were refined in [1]. In both papers [4] and [1] the results on generation of Sylow subgroups were used to reduce a problem about finite groups to the case of nilpotent groups. It is hoped that also our Theorem A will play a similar role in the subsequent projects.

Another important tool used in the proof of Theorem A is the famous result of Liebeck, O'Brien, Shalev and Tiep [6] that every element of a non-abelian simple group is a commutator. The result proved Ore's conjecture thus solving a long-standing problem. In turn, the proof in [6] uses the classification of finite simple groups as well as many other sophisticated tools.

## 1. PRELIMINARIES

If  $X$  and  $Y$  are two subsets of a group  $G$ , and  $N$  is a normal subgroup of  $G$ , it is not always the case that  $XN \cap YN = (X \cap Y)N$ , i.e., that  $\overline{X \cap Y} = \overline{X} \cap \overline{Y}$  in the quotient group  $\overline{G} = G/N$ . In our first lemma we have a situation in which this property holds, and which will be of importance in the sequel.

**Lemma 1.1.** *Let  $G$  be a finite group, and let  $N$  be a normal subgroup of  $G$ . If  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $X$  is a normal subset of  $G$  consisting of  $p$ -elements, then  $XN \cap PN = (X \cap P)N$ . In other words, if we use the bar notation in  $G/N$ , we have  $\overline{X} \cap \overline{P} = \overline{X \cap P}$ .*

*Proof.* We only need to worry about the inclusion  $\overline{X} \cap \overline{P} \subseteq \overline{X \cap P}$ . So, given an element  $g \in XN \cap PN$ , we prove that  $g \in xN$  for some  $x \in X \cap P$ . Since we have  $g \in XN$ , we may assume without loss of generality that  $g \in X$ , and in particular  $g$  is a  $p$ -element. Since also  $g \in PN$ , there exists  $z \in P$  such that  $gN = zN$ .

Put  $H = \langle g \rangle N = \langle z \rangle N$ , and observe that  $H' \leq N$ . Since  $P \cap N$  is a Sylow  $p$ -subgroup of  $N$  and  $z \in P$ , it follows that  $P \cap H = \langle z \rangle (P \cap N)$  is a Sylow  $p$ -subgroup of  $H$ . Now,  $g$  is a  $p$ -element of  $H$ , and so we get  $g^h \in P \cap H$  for

some  $h \in H$ . If we put  $x = g^h$  then  $x \in X \cap P$ , since  $X$  is a normal subset of  $G$ , and  $g = x^{h^{-1}} = x[x, h^{-1}] \in xH' \subseteq xN$ , as desired.  $\square$

The next lemma will be fundamental in the proof of Theorem A, since it will allow us to go up a series from 1 to  $w(G)$  in which all quotients of two consecutive terms are verbal subgroups of a word all of whose values are also  $w$ -values.

**Lemma 1.2.** *Let  $G$  be a finite group, and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Assume that  $N \leq L$  are two normal subgroups of  $G$ , and use the bar notation in the quotient group  $G/N$ . Let  $X$  be a normal subset of  $G$  consisting of  $p$ -elements such that  $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$ . Then  $P \cap L = \langle P \cap X, P \cap N \rangle$ .*

*Proof.* By Lemma 1.1, we have  $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$ , and this implies that  $PN \cap L = \langle P \cap X \rangle N$ . By intersecting with  $P$ , we get

$$P \cap L = P \cap (PN \cap L) = P \cap \langle P \cap X \rangle N = \langle P \cap X \rangle (P \cap N),$$

where the last equality follows from Dedekind's law. This proves the result.  $\square$

We will also need the following lemma, of a different nature.

**Lemma 1.3.** *Let  $G$  be a finite group, and let  $N$  be a minimal normal subgroup of  $G$ . If  $N$  does not contain any non-trivial elements of  $G_{\delta_i}$ , where  $i \geq 1$ , then  $[N, G^{(i-1)}] = 1$ .*

*Proof.* We argue by induction on  $i$ . If  $i = 1$  then, since  $N$  is normal in  $G$  and does not contain any non-trivial commutators of elements of  $G$ , it follows that  $[n, g] = 1$  for every  $n \in N$  and  $g \in G$ . Thus  $[N, G] = 1$ , as desired.

Assume now that  $i \geq 2$ . The fact that  $N$  is a minimal normal subgroup of  $G$  implies that the subgroup  $\langle N \cap G_{\delta_{i-1}} \rangle$  must be either equal to  $N$  or the trivial subgroup. In the former case, we have  $N = \langle N \cap G_{\delta_{i-1}} \rangle$  and so  $[N, G^{(i-1)}]$  is generated by elements of the form  $[a, b]$  where  $a \in N \cap G_{\delta_{i-1}}$  and  $b \in G_{\delta_{i-1}}$ . In particular, each commutator  $[a, b]$  belongs to  $N \cap G_{\delta_i}$  and must be 1 by the hypothesis. Hence  $[N, G^{(i-1)}] = 1$ . If  $N \cap G_{\delta_{i-1}} = 1$ , then it follows from the induction hypothesis that  $[N, G^{(i-2)}] = 1$ , and the result holds.  $\square$

We conclude this preliminary section by showing that Theorem A holds for every word under the assumption that the verbal subgroup  $w(G)$  is nilpotent.

**Theorem 1.4.** *Let  $G$  be a finite group of order  $p^a m$ , where  $p$  is a prime and  $m$  is not divisible by  $p$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $w$  is any word such that  $w(G)$  is nilpotent, then*

$$P \cap w(G) = \langle P \cap G_{w^m} \rangle.$$

*Proof.* By Bezout's identity, there exist two integers  $\lambda$  and  $\mu$  such that  $1 = \lambda p^a + \mu m$ . If we put  $u = w^{p^a}$  and  $v = w^m$ , then for every  $g \in G_w$  we have

$$g = (g^{p^a})^\lambda \cdot (g^m)^\mu \in \langle G_u \rangle \cdot \langle G_v \rangle.$$

Hence

$$(1) \quad w(G) = \langle G_u, G_v \rangle.$$

Note that all elements of  $G_u$  have  $p'$ -order, and all elements of  $G_v$  have  $p$ -power order. Since  $w(G)$  is nilpotent, it follows that  $\langle G_u \rangle$  is a  $p'$ -subgroup of  $w(G)$ ,  $\langle G_v \rangle$  is a  $p$ -subgroup, and  $G_u$  and  $G_v$  commute elementwise. As a consequence of this and (1), we get

$$(2) \quad w(G) = \langle G_u \rangle \times \langle G_v \rangle,$$

and  $\langle G_u \rangle$  and  $\langle G_v \rangle$  are a Hall  $p'$ -subgroup and a Sylow  $p$ -subgroup of  $w(G)$ , respectively. We conclude that  $P \cap w(G) = \langle G_v \rangle$ , which proves the theorem.  $\square$

## 2. THE PROOF OF THEOREM A

The first step in the proof of Theorem A is to verify it for  $\delta_i$ , which is done in Theorem 2.3 below. For this, we will rely on the result by Liebeck, O'Brien, Shalev and Tiep [6] that proved Ore's conjecture, according to which every element of a non-abelian simple group is a commutator, and *a fortiori*, also a value of  $\delta_i$  for every  $i$ . We will also need the following result of Gaschütz (see page 191 of [8]).

**Theorem 2.1.** *Let  $G$  be a finite group, and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $N$  is a normal abelian  $p$ -subgroup of  $G$ , then  $N$  is complemented in  $G$  if and only if  $N$  is complemented in  $P$ .*

In the proof of Theorem A for both  $\delta_i$  and an arbitrary outer commutator word, we will argue by induction. Then it will be important to take into account the following remark.

**Remark 2.2.** Let  $G$  be a group of order  $p^a m$  for which we want to prove Theorem A in the case of a given word  $w$ . Assume that  $K$  is a group whose order  $p^{a^*} m^*$  is a divisor of  $p^a m$  (for example, a subgroup or a quotient of  $G$ ), and let  $P^*$  be a Sylow  $p$ -subgroup of  $K$ . If Theorem A is known to hold for  $K$  and  $w$ , then we have  $P^* \cap w(K) = \langle P^* \cap K_{w^{m^*}} \rangle$ . Since  $m/m^*$  is a positive integer which is coprime to  $p$ , it follows that  $P^* \cap w(K) = \langle (P^* \cap K_{w^{m^*}})^{m/m^*} \rangle$ , and so also that  $P^* \cap w(K) = \langle P^* \cap K_{w^m} \rangle$ . In other words, in the statement of Theorem A for  $K$ , we can replace the power word  $w^{m^*}$  corresponding to the order of  $K$  with the word  $w^m$ , which corresponds to the order of  $G$ .

We can now proceed to the proof of Theorem A for  $\delta_i$ .

**Theorem 2.3.** *Let  $G$  be a finite group of order  $p^a m$ , where  $p$  is a prime and  $m$  is not divisible by  $p$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then, for every  $i \geq 0$ , we have*

$$P \cap G^{(i)} = \langle P \cap G_{\delta_i^m} \rangle.$$

*Proof.* We argue by induction on the order of  $G$ . The result is obvious if either  $i = 0$  or  $G^{(i)} = 1$ , so we assume that  $i \geq 1$  and  $G^{(i)} \neq 1$ .

Let  $N \neq 1$  be a normal subgroup of  $G$  which is contained in  $G^{(i)}$ . Then the result holds in  $\overline{G} = G/N$ , and we have  $\overline{P} \cap \overline{G}^{(i)} = \langle \overline{P} \cap \overline{G}_{\delta_i^m} \rangle$ . By applying Lemma 1.2, we get

$$(3) \quad P \cap G^{(i)} = \langle P \cap G_{\delta_i^m}, P \cap N \rangle.$$

Now we assume that  $N$  is a minimal normal subgroup of  $G$ , and we consider three different cases, depending on the structure of  $N$ .

(i)  $N$  is a direct product of isomorphic non-abelian simple groups.

By the positive solution to Ore's conjecture, we have  $N = N_{\delta_i}$ . Hence we get  $P \cap N \subseteq N_{\delta_i}$ , and since the map  $z \mapsto z^m$  is a bijection in  $P \cap N$ , it follows that  $P \cap N \subseteq P \cap N_{\delta_i^m}$ . Now the result is immediate from (3).

(ii)  $N \cong C_q \times \cdots \times C_q$ , where  $q$  is a prime different from  $p$ .

In this case,  $P \cap N = 1$  and the result obviously holds.

(iii)  $N \cong C_p \times \cdots \times C_p$ .

In this case, we have  $N \leq P$  and so  $P \cap N = N$ . Since  $\langle N \cap G_{\delta_i} \rangle$  is a normal subgroup of  $G$  and  $N$  is a minimal normal subgroup, it follows that either  $\langle N \cap G_{\delta_i} \rangle = N$  or  $N \cap G_{\delta_i} = 1$ . In the former case, we have  $N = \langle (N \cap G_{\delta_i})^m \rangle$ , since  $N$  is a finite  $p$ -group, and so  $N = \langle N \cap G_{\delta_i^m} \rangle$  and the theorem follows again from (3). So we are necessarily in the latter case, and then by Lemma 1.3, we have  $[N, G^{(i-1)}] = 1$ .

If  $G$  is not perfect, then the theorem holds by induction in  $G'$ , and so  $P \cap G^{(i+1)} = P \cap (G')^{(i)}$  can be generated by values of  $\delta_i^m$  lying in  $P$ . If  $G^{(i+1)} \neq 1$  then we can use (3) with  $G^{(i+1)}$  in the place of  $N$ , and we are done. On the other hand, if  $G^{(i+1)} = 1$  then  $G^{(i)}$  is abelian, and the result is a consequence of Theorem 1.4.

Thus we may assume that  $G$  is perfect. Then  $P \cap G^{(i)} = P$ . Also  $[N, G] = [N, G^{(i-1)}] = 1$ , and  $N$  is central in  $G$ . Being a minimal normal subgroup of  $G$ , this implies that  $|N| = p$ . If  $N \leq \Phi(P)$  then it follows from (3) that  $P = \langle P \cap G_{\delta_i^m} \rangle$ , as desired. Hence we may assume that  $N$  is not contained in a maximal subgroup  $M$  of  $P$ . Since  $|N| = p$ , it follows that  $M$  is a complement of  $N$  in  $P$ . By Theorem 2.1, it follows that  $N$  has also a complement in  $G$ , say  $H$ . Since  $N \leq Z(G)$ , we conclude that  $G = H \times N$ , a contradiction with the fact that  $G$  is perfect. This completes the proof.  $\square$

We will deal with arbitrary outer commutator words using some concepts from the paper [2], where outer commutator words are represented by binary rooted trees in the following way: indeterminates are represented by an isolated vertex, and if  $w = [u, v]$  is the commutator of two outer commutator words  $u$  and  $v$ , then the tree  $T_w$  of  $w$  is obtained by drawing the trees  $T_u$  and  $T_v$ , and a new vertex (which will be the root of the new tree) which is then connected to the roots of  $T_u$  and  $T_v$ . For example, the following are the trees for the words  $\gamma_4$  and  $\delta_3$  (we also label every vertex with the outer commutator word which is represented by the tree appearing on top of that vertex):

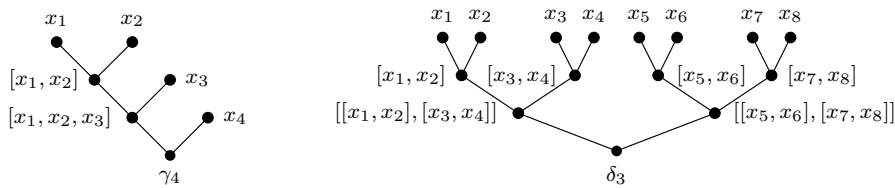


FIGURE 1. The trees of the words  $\gamma_4$  and  $\delta_3$ .

Each of these trees has a visual height, which coincides with the largest distance from the root to another vertex of the tree. Observe that the full binary tree of height  $i$  corresponds to the derived word  $\delta_i$ . The following two concepts, also introduced in [2], will be essential in our proof of Theorem A.

**Definition 2.4.** Let  $u$  and  $w$  be two outer commutator words. We say that  $u$  is an *extension* of  $w$  if the tree of  $u$  is an upward extension of the tree of  $w$ . If  $u \neq w$ , we say that  $u$  is a *proper extension* of  $w$ .

An important remark is that, if  $u$  is an extension of  $w$ , then  $G_u \subseteq G_w$ .

**Definition 2.5.** If  $w$  is an outer commutator word whose tree has height  $i$ , the *defect* of  $w$  is the number of vertices that need to be added to the tree of  $w$  in order to get the tree of  $\delta_i$ . In other words, if the tree of  $w$  has  $V$  vertices, the defect of  $w$  is  $2^{i+1} - 1 - V$ .

Thus the only words of defect 0 are the derived words. Our proof of Theorem A also depends on the following result, which is implicit in the proof of Theorem B of [2].

**Theorem 2.6.** Let  $w = [u, v]$  be an outer commutator word of height  $i$ , different from  $\delta_i$ . Then at least one of the subgroups  $[w(G), u(G)]$  and  $[w(G), v(G)]$  is contained in a product of verbal subgroups corresponding to words which are proper extensions of  $w$  of height  $i$ .

Let us now give the proof of Theorem A.

*Proof of Theorem A.* We argue by induction on the defect of the word  $w$ . If the defect is 0, then  $w$  is a derived word, and the result is true by Theorem 2.3. Hence we may assume that the defect is positive. If the height of  $w$  is  $i$ , let  $\Phi = \{\varphi_1, \dots, \varphi_r\}$  be the set of all outer commutator words of height  $i$  which are a proper extension of  $w$ . Since every word in the set  $\Phi$  has smaller defect than  $w$ , the theorem holds for all  $\varphi_i$ .

Put  $N_0 = 1$ ,  $N_i = \varphi_1(G) \dots \varphi_i(G)$  for  $1 \leq i \leq r$ , and  $N = N_r$ . Let us write  $w = [u, v]$ , where  $u$  and  $v$  are outer commutator words. Since  $[w(G), w(G)]$  is contained in both  $[w(G), u(G)]$  and  $[w(G), v(G)]$ , it follows from Theorem 2.6 that  $[w(G), w(G)] \leq N$ . Thus if  $\overline{G} = G/N$ , the verbal subgroup  $w(\overline{G})$  is abelian, and so Theorem A holds in  $\overline{G}$ , according to Theorem 1.4. Hence  $\overline{P} \cap w(\overline{G}) = \langle \overline{P} \cap \overline{G}_{w^m} \rangle$ , and by applying Lemma 1.2, we get  $P \cap w(G) = \langle P \cap G_{w^m}, P \cap N \rangle$ .

Consequently, it suffices to show that  $P \cap N$  can be generated by values of  $w^m$ . We see this by proving that  $P \cap N_i = \langle P \cap N_i \cap G_{w^m} \rangle$  for every  $i = 0, \dots, r$ , by induction on  $i$ . There is nothing to prove if  $i = 0$ , so we assume that  $i \geq 1$ . If  $\overline{G} = G/N_{i-1}$ , we have  $\overline{N}_i = \varphi_i(\overline{G})$ . Since the theorem is true for  $\varphi_i$ , it follows that  $\overline{P} \cap \overline{N}_i = \langle \overline{P} \cap \overline{G}_{\varphi_i^m} \rangle$ . By Lemma 1.2, we get

$$P \cap N_i = \langle P \cap G_{\varphi_i^m}, P \cap N_{i-1} \rangle.$$

Observe that, since  $\varphi_i$  is an extension of  $w$ , we have  $G_{\varphi_i^m} \subseteq G_{w^m}$ . Since also  $G_{\varphi_i^m} \subseteq \varphi_i(G) \leq N_i$ , we can further say that  $G_{\varphi_i^m} \subseteq N_i \cap G_{w^m}$ . Hence

$$P \cap N_i = \langle P \cap N_i \cap G_{w^m}, P \cap N_{i-1} \rangle,$$

and the result follows from the induction hypothesis.  $\square$

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