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GENUS, THICKNESS AND CROSSING NUMBER OF GRAPHS ENCODING THE GENERATING PROPERTIES OF FINITE GROUPS

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Abstract. Assume that G is a finite group and let a and b be non-negative integers. We define an undirected graph $\Gamma_{a,b}(G)$ whose vertices correspond to the elements of $G^a \cup G^b$ and in which two tuples $(x_1, \dots, x_a)$ and $(y_1, \dots, y_b)$ are adjacent if and only $\langle x_1, \dots, x_a, y_1, \dots, y_b \rangle = G$. Our aim is to estimate the genus, the thickness and the crossing number of the graph $\Gamma_{a,b}(G)$ when a and b are positive integers.

1. Introduction

Generating sets of a finite group may be quite complicated. If a group G is d-generated, the question of which sets of d elements of G generate G is nontrivial. The simplest interesting case is when G is 2-generated. One tool developed to study generators of 2-generated finite groups is the generating graph $\Gamma(G)$ of G; this is the graph which has the elements of G as vertices and an edge between two elements g_1 and g_2 if G is generated by g_1 and g_2. Note that the generating graph may be defined for any group, but it only has edges if G is 2-generated. A wider family of graphs which encode the generating property of G when G is an arbitrary finite group was introduced and investigated in [1]. The definition of these graphs is the following. Assume that G is a finite group and let a and b be non-negative integers. We define an undirected graph $\Gamma_{a,b}(G)$ whose vertices correspond to the elements of $G^a \cup G^b$ and in which two tuples $(x_1, \dots, x_a)$ and $(y_1, \dots, y_b)$ are adjacent if and only $\langle x_1, \dots, x_a, y_1, \dots, y_b \rangle = G$. Notice that $\Gamma_{1,1}(G)$ is the generating graph of G, so these graphs can be viewed as a natural generalization of the generating graph.

Let Δ be a graph. The *genus* $\gamma(\Delta)$ of Δ is the minimum integer g such that there exists an embedding of Δ into the orientable surface S_g of genus g (or in other words the minimum number g of handles which must be added to a sphere so that Δ can be embedded on the resulting surface). The *thickness* $\theta(\Delta)$ of Δ is the minimum number of planar graphs into which the edges of Δ can be partitioned. The *crossing number* $\mathrm{cr}(\Delta)$ of Δ is the minimum number of crossings in any simple drawing of $\Delta(G)$. In this paper we investigate genus, thickness and crossing number of the graphs $\Gamma_{a,b}(G)$, when $1 \le a \le b$ and $a + b \ge d(G)$, where $d(G)$ is the smallest cardinality of a generating set of G. Notice that the case $a = 0$ is not interesting: the graph $\Gamma_{0,b}(G)$ is a star with an internal node corresponding to the empty set and with $\phi_G(b)$ leaves, being $\phi_G(b)$ be the number of the generating b-uples of G. Our main result is the following:

Key words and phrases. generating graph; genus; thickness; crossing numbers.

Theorem 1. *Assume that G is a nontrivial d-generated finite group and that a, b are positive integer with $a+b \geq d$. Then*

$$\gamma(\Gamma_{a,b}(G)) \geq \frac{|G|^b}{6}\left(\frac{\sqrt{|G|}}{16}-3\right),$$
$$\theta(\Gamma_{a,b}(G)) \geq \frac{\sqrt{|G|}}{48},$$
$$\mathrm{cr}(\Gamma_{a,b}(G)) \geq \frac{|G|^{d+\frac{1}{2}}}{29}\left(\frac{1}{2^{11}}-\frac{70}{|G|^{3/2}}\right).$$

In order to estimate $\gamma(\Gamma_{a,b}(G))$, $\theta(\Gamma_{a,b}(G))$ and $\mathrm{cr}(\Gamma_{a,b}(G))$, it is important to obtain a lower bound for the ratio $e(\Gamma_{a,b}(G))/v(\Gamma_{a,b}(G))$ between the number of edges and the number of vertices of the graph $\Gamma_{a,b}(G)$. We will see in Section 3, that this is essentially related to the estimation of the ratio $\phi_G(d)/|G|^{d-1}$ for a d-generated finite group. Our main result in this direction is the following.

Theorem 2. *If G is a d-generated finite group, then*

$$\frac{\phi_G(d)}{|G|^{d-1}} \geq \frac{\sqrt{|G|}}{2}.$$

We think that this is a result of independent interest. For example it implies the following corollary.

Corollary 3. *Let G be a finite group and let $d = d(G)$. Denote by $\rho(G)$ the number of elements g in G such that $G = \langle g, x_1, \dots, x_{d-1}\rangle$, for some $x_1, \dots, x_{d-1} \in G$. We have*

$$\rho(G) \geq \frac{|G|^{1-\frac{1}{2d}}}{2^{\frac{1}{d}}}$$

Recall that a graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends. In [12] a classification of the 2-generated finite groups with planar generating graph is given. We generalize this result as follows.

Theorem 4. *Let G be a nontrival finite group and let a and b be two positive integers with $a \leq b$ and $a+b \geq d(G)$. Then $\Gamma_{a,b}(G)$ is planar if and only if one of the following occurs:*

(1) $G \in \{C_3, C_4, C_6, C_2 \times C_2, D_3, D_4, Q_8, C_4 \times C_2, D_6\}$ and $(a, b) = (1, 1)$.
(2) $G \cong C_2$ and either $a = 1$ or $(a, b) = (2, 2)$.

2. Proof of Theorem 2

Let G be a d-generated finite group and let $\phi_G(d)$ denote the number of the generating d-uples $(g_1, \dots, g_d) \in G^d$ with $\langle g_1, \dots, g_d\rangle = G$. Clearly $P_G(d) = \phi_G(d)/|G|^d$ coincides with the probability that d randomly chosen elements from G generate G.

Definition 5. *For a d-generated finite group G, set*

$$\alpha(G, d) := \frac{\phi_G(d)}{|G|^{d-1}} = P_G(d)|G|.$$

Let N be a normal subgroup of a finite group G and choose $g_1, \dots, g_k \in G$ with the property that $G = \langle g_1, \dots, g_k \rangle N$. By a result of Gaschütz [8] the cardinality of the set

$$\Phi_N(g_1, \dots, g_k) = \{(n_1, \dots, n_k) \in N \mid \langle g_1 n_1, \dots, g_k n_k \rangle = G\}$$

does not depend on the choice of $g_1, \dots, g_k$. Let

$$P_{G,N}(k) = \frac{|\Phi_N(g_1, \dots, g_k)|}{|N|^k}.$$

Notice that if $k \geq d(G/N)$, then $P_{G,N}(k) = P_G(k)/P_{G/N}(k)$.

Definition 6. *Let N be a normal subgroup of a d-generated finite group G. Set*

$$\alpha(G, N, d) := \frac{\alpha(G, d)}{\alpha(G/N, d))} = P_{G,N}(d)|N|.$$

Lemma 7. *Assume that N is a minimal abelian normal subgroup of a d-generated finite group G. We have $|N| = p^a$, where p is a prime and a is a positive integer. Let c be the number of complements of N in G. Then*

$$\alpha(G, N, d) = \frac{p^{d\cdot a} - c}{p^{(d-1)\cdot a}} \geq p^a - p^{a-1} = p^{a-1}(p-1).$$

In particular

(1) $\alpha(G, N, d) = 1$ if and only if $|N| = 2$, N has a complement in G and G/N admits C_2^{d-1} as an epimorphic image.

(2) $\alpha(G, N, d) \geq 3/2$ if $|N| = 2$, N has a complement in G and C_2^{d-1} is not an epimorphic image of G/N.

(3) $\alpha(G, N, d) \geq 2$ in all the remaining cases.

Proof. By [9, Satz 2], $P_{G,N}(d) = 1 - c/p^{d\cdot a}$, hence $\alpha(G, N, d) = \frac{p^{d\cdot a} - c}{p^{(d-1)\cdot a}}$. If $c \neq 0$, then c is the order of the group $\mathrm{Der}(G/N, N)$ of derivations from G/N to N; in particular c is a power of p. Moreover, since G is d-generated, it must be $c < p^{d\cdot a}$ and consequently

$$\alpha(G, N, d) = \frac{p^{d\cdot a} - c}{p^{(d-1)\cdot a}} \geq \frac{p^{d\cdot a} - p^{d\cdot a - 1}}{p^{(d-1)\cdot a}} = p^a - p^{a-1}.$$

In particular we can have $\alpha(G, N) < 2$ only if $|N| = 2$ and $c \neq 0$. Let H be a complement of N in G and let $K = H'H^2$. We have $c = |\mathrm{Der}(H, N)| = |\mathrm{Hom}(H/K, N)|$. Since G is d-generated, we have $H/K \cong C_2^t$ with $t < d$. We have $c = 2^t$ and $\alpha(G, N, d) = 2 - 2^{t-d+1}$. □

If a group G acts on a group A via automorphisms, then we say that A is a G-group. If G does not stabilise any nontrivial proper subgroup of A, then A is called an irreducible G-group. Two G-groups A and B are said to be G-isomorphic, or $A \cong_G B$, if there exists a group isomorphism $\phi : A \to B$ such that $\phi(g(a)) = g(\phi(a))$ for all $a \in A, g \in G$. Following [11], we say that two G-groups A and B are G-equivalent and we put $A \equiv_G B$, if there are isomorphisms $\phi : A \to B$ and $\Phi : A \rtimes G \to B \rtimes G$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\phi"] & A \rtimes G \arrow[r] \arrow[d, "\Phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & B \arrow[r] & B \rtimes G \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Note that two G-isomorphic G-groups are G-equivalent. In the abelian case, the converse is true: if A_1 and A_2 are abelian and G-equivalent, then A_1 and A_2 are also G-isomorphic. It is known (see for example [11, Proposition 1.4]) that two chief factors A_1 and A_2 of G are G-equivalent if and only if either they are G-isomorphic, or there exists a maximal subgroup M of G such that $G/\operatorname{Core}_G(M)$ has two minimal normal subgroups, N_1 and N_2, G-isomorphic to A_1 and A_2 respectively. Let $A = X/Y$ be a chief factor of G. We say that $A = X/Y$ is a Frattini chief factor if X/Y is contained in the Frattini subgroup of G/Y; this is equivalent to saying that A is abelian and there is no complement to A in G. The number of non-Frattini chief factors G-equivalent to A in any chief series of G does not depend on the series, and so this number is well-defined: we will denote it by $\delta_A(G)$.

The following numerical results will be useful.

Lemma 8. [7, 9.15 p. 54] *Let $n > 0$, then*

$$\sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot e^{-n}\cdot e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot e^{-n}\cdot e^{\frac{1}{12n}}.$$

Corollary 9. *If $t < n$, then*

$$\frac{n!}{(n-t)!} \ge \frac{9}{10}\frac{n^t}{e^t}.$$

Proof.

$$\begin{aligned}\frac{n!}{(n-t)!} &\ge \frac{n^{n+\frac{1}{2}}}{e^n}\frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}}\cdot\frac{e^{\frac{1}{12n+1}}}{e^{\frac{1}{12(n-t)}}} \ge \frac{n^{n+\frac{1}{2}}}{e^n}\frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}}\cdot\frac{1}{e^{\frac{1}{12}}} \ge\\ &\ge \frac{n^{n+\frac{1}{2}}}{e^n}\frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}}\cdot\frac{9}{10} \ge \frac{9}{10}\frac{n^{n+\frac{1}{2}}}{(n-t)^{n-t+\frac{1}{2}}}\cdot\frac{1}{e^t} \ge\\ &\ge \frac{9}{10}\frac{n^{n+\frac{1}{2}}}{n^{n-t+\frac{1}{2}}}\cdot\frac{1}{e^t} = \frac{9}{10}\frac{n^t}{e^t}. \quad \square\end{aligned}$$

Proposition 10. *Let G be a finite group and let B be a non-abelian chief factor of G. Denote by $t = \delta_G(B)$ the number of factors G-equivalent to B in a given chief series of G. More precisely let $X_1/Y_1, X_2/Y_2, \ldots, X_t/Y_t$, with $Y_t \le X_t \le \cdots \le Y_1 \le X_1$, be the factors G-equivalent to B in a given chief series of G. For $1 \le i \le t$, let $\alpha_i = \alpha(G/Y_i, X_i/Y_i, d)$. We have*

$$\prod_{1\le i\le t}\alpha_i \ge \frac{9}{10}\left(\frac{53|B|}{90e}\right)^t.$$

Proof. Let $L = G/C_G(B)$ be the monolithic primitive group associated to B and assume $L = \langle l_1, \ldots, l_d\rangle$. Moreover define $\Gamma := C_{\mathrm{Aut}(B)}(L/B)|$, $\gamma = |\Gamma|$, $\Phi := \Phi_B(l_1, \ldots, l_d)$. By [5, Proposition 16], for $1 \le i \le t$, we have

$$\alpha_i = \frac{|\Phi|}{|B|^{d-1}} - \frac{(i-1)\gamma}{|B|^{d-1}}.$$

Let $\rho = |\Phi|/\gamma$ (notice that ρ is an integer) and let $\tau = |B|^{d-1}/\gamma$. It follows from [6, Theorem 1.1] that $\rho/\tau \ge \frac{53}{90}|B|$. In view of Corollary 9 we have

$$\prod_{1\le i\le t}\alpha_i = \frac{\rho(\rho-1)\cdots(\rho-(t-1))}{\tau^t} \ge \frac{9}{10\cdot e^t}\left(\frac{\rho}{\tau}\right)^t \ge \frac{9}{10}\left(\frac{53|B|}{90e}\right)^t. \quad \square$$

Next we deal with the proof of Theorem 2.

Proof of Theorem 2. Let $X_t \leq X_{t-1} \leq \cdots \leq X_1 = G$ be a chief series of G and for $1 \leq i \leq t-1$, let $\alpha_i = \alpha(G/X_{i+1}, X_i/X_{i+1}, d)$. Since $d(G) = d$, it must be $\delta_G(C_2) \leq d$ and this implies in particular that there exists at most a unique index j^* such that X_{j^*}/X_{j^*+1} has order 2, is complemented in G/X_{j^*+1} and the quotient G/X_{j^*} admits C_2^{d-1} as an epimorphic image. If $|X_i/X_{i+1}| = 2$ and $i \neq j^*$, then, by Lemma 7, $\alpha_i \geq 3/2 \geq \sqrt{2} = \sqrt{|X_i/X_{i+1}|}$. If X_i/X_{i+1} is abelian and $|X_i/X_{i+1}| = p_i^{n_i} > 2$, then, again by Lemma 7, $\alpha_i \geq p_i^{n_i-1}(p_i-1) \geq \sqrt{p_i^{n_i}} = \sqrt{|X_i/X_{i+1}|}$. Now assume that B is a non-abelian chief factor of G and let

$$I_B = \{1 \leq k \leq t-1 \mid X_k/X_{k+1} \equiv_G B\}.$$

By Proposition 10, noticing that $\delta_B(G) = |I_B|$ and $|B| \geq 6\sqrt{|B|}$ since $|B| \geq 60$, we have

$$\prod_{k \in I_B} \alpha_k \geq \frac{9}{10}\left(\frac{53|B|}{90e}\right)^{\delta_B(G)} \geq \left(\frac{53|B|}{100e}\right)^{\delta_B(G)} \geq$$

$$\geq \left(\frac{|B|}{6}\right)^{\delta_B(G)} \geq \left(\sqrt{|B|}\right)^{\delta_B(G)} = \prod_{k \in I_B} \sqrt{|X_k/X_{k+1}|}.$$

The result follows since $\alpha(G,d) = \prod_{1 \leq i \leq t-1} \alpha_i$ and $|G| = \prod_{1 \leq i \leq t-1} |X_i/X_{i+1}|$. $\square$

We close this section with the proof of Corollary 3.

Proof of Corollary 3. By Theorem 2,

$$\rho(G)^d \geq \phi_G(d) = \alpha(G,d)|G|^{d-1} \geq \frac{|G|^{\frac{1}{2}}|G|^{d-1}}{2} = \frac{|G|^{d-\frac{1}{2}}}{2}. \quad \square$$

3. Proof of Theorem 1

Before proving Theorem 1, we recall some general results in graph theory concerning lower bounds for the genus, the thickness and the crossing number of a simple graph Δ.

Proposition 11. [10, 7.2.4 - F35] *If Δ is a simple graph with e edges and v vertices, then*

$$\gamma(\Delta) \geq 1 - \frac{v}{2} + \frac{e}{6} \geq \frac{v}{6}\left(\frac{e}{v} - 3\right).$$

Proposition 12. [3, 10.3.6 (a)]. *If Δ is a simple graph with e edges and $v \geq 3$ vertices, then*

$$\theta(\Delta) \geq \frac{e}{3v-6}.$$

Proposition 13. [2, Theorem 6] *If Δ is a simple graph with e edges and v vertice, then*

$$\mathrm{cr}(\Delta) \geq \frac{e^3}{29v^2} - \frac{35}{29}v.$$

Assume that G is a finite group and let a and b be positive integers. Let $d = a + b \geq d(G)$. If $a \neq b$ then $\Gamma_{a,b}(G)$ is a bipartite graphs with two parts, one corresponding to the elements of G^a and the other to the elements of G^b. In particular $\Gamma_{a,b}(G)$ has $|G|^a + |G|^b$ vertices and there exists a bijective correspondence between the set of the generating d-uples of G and the set of the edges of $\Gamma_{a,b}(G)$: indeed if $\langle g_1,\dots,g_d\rangle = G$, then $(g_1,\dots,g_a)$ and $(g_{a+1},\dots,g_d)$ are adjacent vertices of the graph. Hence the number of edges of $\Gamma_{a,b}(G)$ is $\phi_G(d)$. The situation is different if $a=b$. In that case $\Gamma_{a,a}(G)$ has $|G|^a$ vertices, $\phi_G(a)$ loops and other $(\phi_G(d)-\phi_G(a))/2$ edges connecting two different vertices (in other words if e is the the number of edges, excluding the loops, and l is the number of loops, then $2e+l=\phi_G(d)$); indeed the two elements $(g_1,\dots,g_a,g_{a+1},\dots,g_d)$ and $(g_{a+1},\dots,g_d,g_1,\dots,g_a)$ give rise to the same edge in $\Gamma_{a,a}(G)$. Summarizing, let ν and η be, respectively, the number of vertices and edges of $\Gamma_{a,b}(G)$, excluding the loops. We have

$$|G|^b \le \nu \le |G|^a + |G|^b \le 2|G|^{d-1}.$$

Moreover $\eta = \phi_G(a+b)$ if $a \ne b$, $\eta = (\phi_G(2a)-\phi_G(a))/2$ if $a=b$. If $\phi_G(a)\ne 0$, then $\phi_G(2a) \ge \phi_G(a)|G|^a$, so $\phi_G(a) \le \phi_G(2a)/|G|^a$. So if $|G|\ge 2$, then $\eta \ge \phi_G(d)/4$. By applying Theorem 2 and Propositions 11,12 and 13 respectively it follows that if $G \ne 1$, then we have the following inequalities.

$$\gamma(\Gamma_{a,b}(G)) \ge \frac{\nu}{6}\left(\frac{\eta}{\nu}-3\right) \ge \frac{|G|^b}{6}\left(\frac{\phi_G(d)}{8|G|^{d-1}}-3\right) \ge \frac{|G|^b}{6}\left(\frac{\sqrt{|G|}}{16}-3\right).$$

$$\theta(\Gamma_{a,b}(G)) \ge \frac{\eta}{3\nu} \ge \frac{\phi_G(d)}{24|G|^{d-1}} \ge \frac{\sqrt{|G|}}{48}.$$

$$\begin{aligned}\mathrm{cr}(\Gamma_{a,b}(G)) &\ge \frac{\eta^3}{29\cdot\nu^2} - \frac{35}{29}\cdot\nu \ge \frac{(\phi_G(d))^3}{29\cdot 4^3\cdot 4\cdot(|G|^{d-1})^2} - \frac{70\cdot|G|^{d-1}}{29}\\ &\ge \frac{\phi_G(d)|G|}{29\cdot 4^5} - \frac{70\cdot|G|^{d-1}}{29} \ge \frac{|G|^{d+\frac{1}{2}}}{29\cdot 2^{11}} - \frac{70\cdot|G|^{d-1}}{29}.\end{aligned}$$

This concludes the proof of Theorem 1.

4. Proof of Theorem 4

The main goal of this section is to prove Theorem 4. We star with two preliminary results.

Proposition 14. [4, Lemma 9.23]. *A simple bipartite planar graph on v vertices, whose every connected component contains at least three vertices, can have not more than $2v-4$ edges.*

Lemma 15. *Let G be a finite group and let $b \ge d(G)$. Consider the set $W = \{(x_1,\dots,x_b)\in G^b \mid \langle x_1,\dots,x_b\rangle = G\}$. If G is not cyclic, then $|W|\ge 3$.*

Proof. Assume $d=d(G)$ and $G=\langle g_1,\dots,g_d\rangle$. Then $(g_1,g_2,g_3,\dots,g_d,1,\dots,1)$, $(g_1g_2,g_2,g_3,\dots,g_d,1,\dots,1)$ and $(g_1,g_1g_2,g_3,\dots,g_d,1,\dots,1)$ are three different elements of W. □

We are now ready to embark on the proof of Theorem 4.

Proof of Theorem 4. Let a and b be positive integers with $a+b\ge d(G)$. We want to discuss when $\Gamma_{a,b}(G)$ is planar. We assume $a+b\ge d(G)$ and $a\le b$. If $a=0$, then $\Gamma_{a,b}(G)$ is a star, so it is planar. We may exclude from our discussion the case $a=b=1$, since the result in this case follows from the main result in [12] (notice that the cyclic group C_5 appears in the statement of [12, Theorem 1.1] but not in the statement of Theorem 4: this is because in [12] the identity element is not included in the vertex-set of $\Gamma_{1,1}(G)$).

First assume that $G=\langle g\rangle$ is cyclic.

- If $a \geq 3$, take

$$\alpha_1 = (1, 1, g, 1, \ldots, 1), \alpha_2 = (1, g, g, 1, \ldots, 1), \alpha_3 = (1, g, 1, 1, \ldots, 1) \in G^a,$$
$$\beta_1 = (g, 1, g, 1, \ldots, 1), \beta_2 = (g, g, g, 1, \ldots, 1), \beta_3 = (g, g, 1, 1, \ldots, 1) \in G^b.$$

- If $a = 2$ and $|G| \neq 2$, take

$$\alpha_1 = (1, g), \alpha_2 = (g, 1), \alpha_3 = (g, g) \in G^2,$$
$$\beta_1 = (1, g^2, 1, \ldots, 1), \beta_2 = (g^2, 1, \ldots, 1), \beta_3 = (g^2, g^2, 1, \ldots, 1) \in G^b.$$

- If $a = 2$ and $|G| = 2$ and $b \geq 3$, take

$$\alpha_1 = (1, g), \alpha_2 = (g, 1), \alpha_3 = (g, g) \in G^2,$$
$$\beta_1 = (1, g, g, 1 \ldots, 1), \beta_2 = (g, 1, g, 1 \ldots, 1), \beta_3 = (g, g, g, 1 \ldots, 1) \in G^b.$$

In all these cases, since α_i and β_j are adjacent for every $1 \leq i, j \leq 3$, $\Gamma_{a,b}(G)$ contains $K_{3,3}$, so it is not planar. If $a = b = 2$ and $|G| = 2$, then $\Gamma_{2,2}(G) \cong K_4$ is planar. If $a = 1$ and $|G| > 2$, then we may consider the subgraph of $\Gamma_{1,b}(G)$ induced by the following vertices: $(1), (g), (g^2), (g, x, \ldots, x) \in G^b$ for $x \in G$. This subgraph is bipartite with $3 + |G|$ vertices and $3|G|$ egdes. Since $3|G| > 2(3 + |G|) - 4$, it follows from Proposition 14, that this graph is not planar. On the other hand, if $a = 1$ and $|G| = 2$, then it can be easily seen that the graph $\Gamma_{1,b}(G)$ is planar.

Now assume that G is not cyclic. Let $d = d(G)$ and $G = \langle g_1, \ldots, g_d \rangle$.

First assume that $a \geq 2$. If $a + b = d$, then set

$$\begin{aligned}
\alpha_1 &= (g_1, g_2, g_3, \ldots, g_a) \in G^a,\\
\alpha_2 &= (g_1, g_1g_2, g_3, \ldots, g_a) \in G^a,\\
\alpha_3 &= (g_1g_2, g_2, g_3, \ldots, g_a) \in G^a,\\
\beta_1 &= (g_{a+1}, g_{a+2}, g_{a+3}, \ldots, g_b) \in G^b,\\
\beta_2 &= (g_{a+1}g_{a+2}, g_{a+2}, g_{a+3}, \ldots, g_b) \in G^b,\\
\beta_3 &= (g_{a+1}, g_{a+1}g_{a+2}, g_{a+3}, \ldots, g_b) \in G^b.
\end{aligned}$$

If $a + b > d$, choose three different elements x, y, z of G and set

$$\begin{aligned}
\alpha_1 &= (g_1, g_2, g_3, \ldots, g_a) \in G^a,\\
\alpha_2 &= (g_1, g_1g_2, g_3, \ldots, g_a) \in G^a,\\
\alpha_3 &= (g_1g_2, g_2, g_3, \ldots, g_a) \in G^a,\\
\beta_1 &= (g_{a+1}, g_{a+2}, g_{a+3}, \ldots, g_b, x, \ldots, x) \in G^b,\\
\beta_2 &= (g_{a+1}, g_{a+2}, g_{a+3}, \ldots, g_b, y, \ldots, y) \in G^b,\\
\beta_3 &= (g_{a+1}, g_{a+2}, g_{a+3}, \ldots, g_b, z, \ldots, z) \in G^b.
\end{aligned}$$

In both cases, since α_i and β_j are adjacent for every $1 \leq i, j \leq 3$, $\Gamma_{a,b}(G)$ contains $K_{3,3}$, so it is not planar.

Assume $a = 1$ and $a + b > d$. Let $W = \{(x_1, \ldots, x_b) \in G^b \mid \langle x_1, \ldots, x_b \rangle = G\}$ and let x, y, z be three different elements of G. We may consider the subgraph of $\Gamma_{1,b}(G)$ induced by following vertices: $(x), (y), (z), w \in W$. This subgraph is bipartite with $3 + |W|$ vertices and $3|W|$ egdes. Since, by Lemma 15, $|W| \geq 3$, it follows $3|W| > 2(3+|W|)-4$, and consequently, by Proposition 14, this graph is not planar.

Finally assume $a = 1$ and $a+b = d$. Let $H = \langle g_2, \dots, g_b \rangle$. If H is cyclic, then $d(G) \leq 2$, in contradiction with $d(G) = 1+b$ and $b > 1$. Let x, y, z be three different elements of H and $W = \{(x_1, \dots, x_b) \in G^b \mid \langle x_1, \dots, x_b \rangle = H\}$. We may consider the subgraph of $\Gamma_{1,b}(G)$ induced by following vertices: $(g_1x), (g_1y), (g_1z), w \in W$. It is bipartite with $3+|W|$ vertices and $3|W|$ egdes. Since H is not cyclic, we have $|W| \geq 3$ by Lemma 15. It follows $3|W| > 2(3+|W|)-4$, and consequently, by Proposition 14, this graph is not planar. $\square$

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