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# GENUS, THICKNESS AND CROSSING NUMBER OF GRAPHS ENCODING THE GENERATING PROPERTIES OF FINITE GROUPS

CRISTINA ACCIARRI AND ANDREA LUCCHINI

ABSTRACT. Assume that  $G$  is a finite group and let  $a$  and  $b$  be non-negative integers. We define an undirected graph  $\Gamma_{a,b}(G)$  whose vertices correspond to the elements of  $G^a \cup G^b$  and in which two tuples  $(x_1, \dots, x_a)$  and  $(y_1, \dots, y_b)$  are adjacent if and only  $\langle x_1, \dots, x_a, y_1, \dots, y_b \rangle = G$ . Our aim is to estimate the genus, the thickness and the crossing number of the graph  $\Gamma_{a,b}(G)$  when  $a$  and  $b$  are positive integers.

## 1. INTRODUCTION

Generating sets of a finite group may be quite complicated. If a group  $G$  is  $d$ -generated, the question of which sets of  $d$  elements of  $G$  generate  $G$  is nontrivial. The simplest interesting case is when  $G$  is 2-generated. One tool developed to study generators of 2-generated finite groups is the generating graph  $\Gamma(G)$  of  $G$ ; this is the graph which has the elements of  $G$  as vertices and an edge between two elements  $g_1$  and  $g_2$  if  $G$  is generated by  $g_1$  and  $g_2$ . Note that the generating graph may be defined for any group, but it only has edges if  $G$  is 2-generated. A wider family of graphs which encode the generating property of  $G$  when  $G$  is an arbitrary finite group was introduced and investigated in [1]. The definition of these graphs is the following. Assume that  $G$  is a finite group and let  $a$  and  $b$  be non-negative integers. We define an undirected graph  $\Gamma_{a,b}(G)$  whose vertices correspond to the elements of  $G^a \cup G^b$  and in which two tuples  $(x_1, \dots, x_a)$  and  $(y_1, \dots, y_b)$  are adjacent if and only  $\langle x_1, \dots, x_a, y_1, \dots, y_b \rangle = G$ . Notice that  $\Gamma_{1,1}(G)$  is the generating graph of  $G$ , so these graphs can be viewed as a natural generalization of the generating graph.

Let  $\Delta$  be a graph. The *genus*  $\gamma(\Delta)$  of  $\Delta$  is the minimum integer  $g$  such that there exists an embedding of  $\Delta$  into the orientable surface  $S_g$  of genus  $g$  (or in other words the minimum number  $g$  of handles which must be added to a sphere so that  $\Delta$  can be embedded on the resulting surface). The *thickness*  $\theta(\Delta)$  of  $\Delta$  is the minimum number of planar graphs into which the edges of  $\Delta$  can be partitioned. The *crossing number*  $cr(\Delta)$  of  $\Delta$  is the minimum number of crossings in any simple drawing of  $\Delta(G)$ . In this paper we investigate genus, thickness and crossing number of the graphs  $\Gamma_{a,b}(G)$ , when  $1 \leq a \leq b$  and  $a+b \geq d(G)$ , where  $d(G)$  is the smallest cardinality of a generating set of  $G$ . Notice that the case  $a=0$  is not interesting: the graph  $\Gamma_{0,b}(G)$  is a star with an internal node corresponding to the empty set and with  $\phi_G(b)$  leaves, being  $\phi_G(b)$  be the number of the generating  $b$ -uples of  $G$ . Our main result is the following:

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*Key words and phrases.* generating graph; genus; thickness; crossing numbers.

**Theorem 1.** *Assume that  $G$  is a nontrivial  $d$ -generated finite group and that  $a, b$  are positive integer with  $a + b \geq d$ . Then*

$$\begin{aligned}\gamma(\Gamma_{a,b}(G)) &\geq \frac{|G|^b}{6} \left( \frac{\sqrt{|G|}}{16} - 3 \right), \\ \theta(\Gamma_{a,b}(G)) &\geq \frac{\sqrt{|G|}}{48}, \\ \text{cr}(\Gamma_{a,b}(G)) &\geq \frac{|G|^{d+\frac{1}{2}}}{29} \left( \frac{1}{2^{11}} - \frac{70}{|G|^{3/2}} \right).\end{aligned}$$

In order to estimate  $\gamma(\Gamma_{a,b}(G))$ ,  $\theta(\Gamma_{a,b}(G))$  and  $\text{cr}(\Gamma_{a,b}(G))$ , it is important to obtain a lower bound for the ratio  $e(\Gamma_{a,b}(G))/v(\Gamma_{a,b}(G))$  between the number of edges and the number of vertices of the graph  $\Gamma_{a,b}(G)$ . We will see in Section 3, that this is essentially related to the estimation of the ratio  $\phi_G(d)/|G|^{d-1}$  for a  $d$ -generated finite group. Our main result in this direction is the following.

**Theorem 2.** *If  $G$  is a  $d$ -generated finite group, then*

$$\frac{\phi_G(d)}{|G|^{d-1}} \geq \frac{\sqrt{|G|}}{2}.$$

We think that this is a result of independent interest. For example it implies the following corollary.

**Corollary 3.** *Let  $G$  be a finite group and let  $d = d(G)$ . Denote by  $\rho(G)$  the number of elements  $g$  in  $G$  such that  $G = \langle g, x_1, \dots, x_{d-1} \rangle$ , for some  $x_1, \dots, x_{d-1} \in G$ . We have*

$$\rho(G) \geq \frac{|G|^{1-\frac{1}{2d}}}{2^{\frac{1}{d}}}$$

Recall that a graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends. In [12] a classification of the 2-generated finite groups with planar generating graph is given. We generalize this result as follows.

**Theorem 4.** *Let  $G$  be a nontrivial finite group and let  $a$  and  $b$  be two positive integers with  $a \leq b$  and  $a + b \geq d(G)$ . Then  $\Gamma_{a,b}(G)$  is planar if and only if one of the following occurs:*

- (1)  $G \in \{C_3, C_4, C_6, C_2 \times C_2, D_3, D_4, Q_8, C_4 \times C_2, D_6\}$  and  $(a, b) = (1, 1)$ .
- (2)  $G \cong C_2$  and either  $a = 1$  or  $(a, b) = (2, 2)$ .

## 2. PROOF OF THEOREM 2

Let  $G$  be a  $d$ -generated finite group and let  $\phi_G(d)$  denote the number of the generating  $d$ -uples  $(g_1, \dots, g_d) \in G^d$  with  $\langle g_1, \dots, g_d \rangle = G$ . Clearly  $P_G(d) = \phi_G(d)/|G|^d$  coincides with the probability that  $d$  randomly chosen elements from  $G$  generate  $G$ .

**Definition 5.** *For a  $d$ -generated finite group  $G$ , set*

$$\alpha(G, d) := \frac{\phi_G(d)}{|G|^{d-1}} = P_G(d)|G|.$$

Let  $N$  be a normal subgroup of a finite group  $G$  and choose  $g_1, \dots, g_k \in G$  with the property that  $G = \langle g_1, \dots, g_k \rangle N$ . By a result of Gaschütz [8] the cardinality of the set

$$\Phi_N(g_1, \dots, g_k) = \{(n_1, \dots, n_k) \in N \mid \langle g_1 n_1, \dots, g_k n_k \rangle = G\}$$

does not depend on the choice of  $g_1, \dots, g_k$ . Let

$$P_{G,N}(k) = \frac{|\Phi_N(g_1, \dots, g_k)|}{|N|^k}.$$

Notice that if  $k \geq d(G/N)$ , then  $P_{G,N}(k) = P_G(k)/P_{G/N}(k)$ .

**Definition 6.** Let  $N$  be a normal subgroup of a  $d$ -generated finite group  $G$ . Set

$$\alpha(G, N, d) := \frac{\alpha(G, d)}{\alpha(G/N, d)} = P_{G,N}(d)|N|.$$

**Lemma 7.** Assume that  $N$  is a minimal abelian normal subgroup of a  $d$ -generated finite group  $G$ . We have  $|N| = p^a$ , where  $p$  is a prime and  $a$  is a positive integer. Let  $c$  be the number of complements of  $N$  in  $G$ . Then

$$\alpha(G, N, d) = \frac{p^{d \cdot a} - c}{p^{(d-1) \cdot a}} \geq p^a - p^{a-1} = p^{a-1}(p-1).$$

In particular

- (1)  $\alpha(G, N, d) = 1$  if and only if  $|N| = 2$ ,  $N$  has a complement in  $G$  and  $G/N$  admits  $C_2^{d-1}$  as an epimorphic image.
- (2)  $\alpha(G, N, d) \geq 3/2$  if  $|N| = 2$ ,  $N$  has a complement in  $G$  and  $C_2^{d-1}$  is not an epimorphic image of  $G/N$ .
- (3)  $\alpha(G, N, d) \geq 2$  in all the remaining cases.

*Proof.* By [9, Satz 2],  $P_{G,N}(d) = 1 - c/p^{d \cdot a}$ , hence  $\alpha(G, N, d) = \frac{p^{d \cdot a} - c}{p^{(d-1) \cdot a}}$ . If  $c \neq 0$ , then  $c$  is the order of the group  $\text{Der}(G/N, N)$  of derivations from  $G/N$  to  $N$ ; in particular  $c$  is a power of  $p$ . Moreover, since  $G$  is  $d$ -generated, it must be  $c < p^{d \cdot a}$  and consequently

$$\alpha(G, N, d) = \frac{p^{d \cdot a} - c}{p^{(d-1) \cdot a}} \geq \frac{p^{d \cdot a} - p^{d \cdot a - 1}}{p^{(d-1) \cdot a}} = p^a - p^{a-1}.$$

In particular we can have  $\alpha(G, N) < 2$  only if  $|N| = 2$  and  $c \neq 0$ . Let  $H$  be a complement of  $N$  in  $G$  and let  $K = H'H^2$ . We have  $c = |\text{Der}(H, N)| = |\text{Hom}(H/K, N)|$ . Since  $G$  is  $d$ -generated, we have  $H/K \cong C_2^t$  with  $t < d$ . We have  $c = 2^t$  and  $\alpha(G, N, d) = 2 - 2^{t-d+1}$ .  $\square$

If a group  $G$  acts on a group  $A$  via automorphisms, then we say that  $A$  is a  $G$ -group. If  $G$  does not stabilise any nontrivial proper subgroup of  $A$ , then  $A$  is called an irreducible  $G$ -group. Two  $G$ -groups  $A$  and  $B$  are said to be  $G$ -isomorphic, or  $A \cong_G B$ , if there exists a group isomorphism  $\phi : A \rightarrow B$  such that  $\phi(g(a)) = g(\phi(a))$  for all  $a \in A, g \in G$ . Following [11], we say that two  $G$ -groups  $A$  and  $B$  are  $G$ -equivalent and we put  $A \equiv_G B$ , if there are isomorphisms  $\phi : A \rightarrow B$  and  $\Phi : A \rtimes G \rightarrow B \rtimes G$  such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & A \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \phi & & \downarrow \Phi & & \parallel & & \\ 1 & \longrightarrow & B & \longrightarrow & B \rtimes G & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Note that two  $G$ -isomorphic  $G$ -groups are  $G$ -equivalent. In the abelian case, the converse is true: if  $A_1$  and  $A_2$  are abelian and  $G$ -equivalent, then  $A_1$  and  $A_2$  are also  $G$ -isomorphic. It is known (see for example [11, Proposition 1.4]) that two chief factors  $A_1$  and  $A_2$  of  $G$  are  $G$ -equivalent if and only if either they are  $G$ -isomorphic, or there exists a maximal subgroup  $M$  of  $G$  such that  $G/\text{Core}_G(M)$  has two minimal normal subgroups,  $N_1$  and  $N_2$ ,  $G$ -isomorphic to  $A_1$  and  $A_2$  respectively. Let  $A = X/Y$  be a chief factor of  $G$ . We say that  $A = X/Y$  is a Frattini chief factor if  $X/Y$  is contained in the Frattini subgroup of  $G/Y$ ; this is equivalent to saying that  $A$  is abelian and there is no complement to  $A$  in  $G$ . The number of non-Frattini chief factors  $G$ -equivalent to  $A$  in any chief series of  $G$  does not depend on the series, and so this number is well-defined: we will denote it by  $\delta_A(G)$ .

The following numerical results will be useful.

**Lemma 8.** [7, 9.15 p. 54] *Let  $n > 0$ , then*

$$\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \cdot e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \cdot e^{\frac{1}{12n}}.$$

**Corollary 9.** *If  $t < n$ , then*

$$\frac{n!}{(n-t)!} \geq \frac{9}{10} \frac{n^t}{e^t}.$$

*Proof.*

$$\begin{aligned} \frac{n!}{(n-t)!} &\geq \frac{n^{n+\frac{1}{2}}}{e^n} \frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}} \cdot \frac{e^{\frac{1}{12n+1}}}{e^{\frac{1}{12(n-t)}}} \geq \frac{n^{n+\frac{1}{2}}}{e^n} \frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}} \cdot \frac{1}{e^{\frac{1}{12}}} \geq \\ &\geq \frac{n^{n+\frac{1}{2}}}{e^n} \frac{e^{n-t}}{(n-t)^{n-t+\frac{1}{2}}} \cdot \frac{9}{10} \geq \frac{9}{10} \frac{n^{n+\frac{1}{2}}}{(n-t)^{n-t+\frac{1}{2}}} \cdot \frac{1}{e^t} \geq \\ &\geq \frac{9}{10} \frac{n^{n+\frac{1}{2}}}{n^{n-t+\frac{1}{2}}} \cdot \frac{1}{e^t} = \frac{9}{10} \frac{n^t}{e^t}. \quad \square \end{aligned}$$

**Proposition 10.** *Let  $G$  be a finite group and let  $B$  be a non-abelian chief factor of  $G$ . Denote by  $t = \delta_G(B)$  the number of factors  $G$ -equivalent to  $B$  in a given chief series of  $G$ . More precisely let  $X_1/Y_1, X_2/Y_2, \dots, X_t/Y_t$ , with  $Y_t \leq X_t \leq \dots \leq Y_1 \leq X_1$ , be the factors  $G$ -equivalent to  $B$  in a given chief series of  $G$ . For  $1 \leq i \leq t$ , let  $\alpha_i = \alpha(G/Y_i, X_i/Y_i, d)$ . We have*

$$\prod_{1 \leq i \leq t} \alpha_i \geq \frac{9}{10} \left( \frac{53|B|}{90e} \right)^t.$$

*Proof.* Let  $L = G/C_G(B)$  be the monolithic primitive group associated to  $B$  and assume  $L = \langle l_1, \dots, l_d \rangle$ . Moreover define  $\Gamma := C_{\text{Aut}(B)}(L/B)$ ,  $\gamma = |\Gamma|$ ,  $\Phi := \Phi_B(l_1, \dots, l_d)$ . By [5, Proposition 16], for  $1 \leq i \leq t$ , we have

$$\alpha_i = \frac{|\Phi|}{|B|^{d-1}} - \frac{(i-1)\gamma}{|B|^{d-1}}.$$

Let  $\rho = |\Phi|/\gamma$  (notice that  $\rho$  is an integer) and let  $\tau = |B|^{d-1}/\gamma$ . It follows from [6, Theorem 1.1] that  $\rho/\tau \geq \frac{53}{90}|B|$ . In view of Corollary 9 we have

$$\prod_{1 \leq i \leq t} \alpha_i = \frac{\rho(\rho-1) \cdots (\rho-(t-1))}{\tau^t} \geq \frac{9}{10 \cdot e^t} \left( \frac{\rho}{\tau} \right)^t \geq \frac{9}{10} \left( \frac{53|B|}{90e} \right)^t. \quad \square$$

Next we deal with the proof of Theorem 2.

*Proof of Theorem 2.* Let  $X_t \leq X_{t-1} \leq \cdots \leq X_1 = G$  be a chief series of  $G$  and for  $1 \leq i \leq t-1$ , let  $\alpha_i = \alpha(G/X_{i+1}, X_i/X_{i+1}, d)$ . Since  $d(G) = d$ , it must be  $\delta_G(C_2) \leq d$  and this implies in particular that there exists at most a unique index  $j^*$  such that  $X_{j^*}/X_{j^*+1}$  has order 2, is complemented in  $G/X_{j^*+1}$  and the quotient  $G/X_{j^*}$  admits  $C_2^{d-1}$  as an epimorphic image. If  $|X_i/X_{i+1}| = 2$  and  $i \neq j^*$ , then, by Lemma 7,  $\alpha_i \geq 3/2 \geq \sqrt{2} = \sqrt{|X_i/X_{i+1}|}$ . If  $X_i/X_{i+1}$  is abelian and  $|X_i/X_{i+1}| = p_i^{n_i} > 2$ , then, again by Lemma 7,  $\alpha_i \geq p_i^{n_i-1}(p_i - 1) \geq \sqrt{p_i^{n_i}} = \sqrt{|X_i/X_{i+1}|}$ . Now assume that  $B$  is a non-abelian chief factor of  $G$  and let

$$I_B = \{1 \leq k \leq t-1 \mid X_k/X_{k+1} \cong_G B\}.$$

By Proposition 10, noticing that  $\delta_B(G) = |I_B|$  and  $|B| \geq 6\sqrt{|B|}$  since  $|B| \geq 60$ , we have

$$\begin{aligned} \prod_{k \in I_B} \alpha_k &\geq \frac{9}{10} \left( \frac{53|B|}{90e} \right)^{\delta_B(G)} \geq \left( \frac{53|B|}{100e} \right)^{\delta_B(G)} \geq \\ &\geq \left( \frac{|B|}{6} \right)^{\delta_B(G)} \geq \left( \sqrt{|B|} \right)^{\delta_B(G)} = \prod_{k \in I_B} \sqrt{|X_k/X_{k+1}|}. \end{aligned}$$

The result follows since  $\alpha(G, d) = \prod_{1 \leq i \leq t-1} \alpha_i$  and  $|G| = \prod_{1 \leq i \leq t-1} |X_i/X_{i+1}|$ .  $\square$

We close this section with the proof of Corollary 3.

*Proof of Corollary 3.* By Theorem 2,

$$\rho(G)^d \geq \phi_G(d) = \alpha(G, d)|G|^{d-1} \geq \frac{|G|^{\frac{1}{2}}|G|^{d-1}}{2} = \frac{|G|^{d-\frac{1}{2}}}{2}. \quad \square$$

### 3. PROOF OF THEOREM 1

Before proving Theorem 1, we recall some general results in graph theory concerning lower bounds for the genus, the thickness and the crossing number of a simple graph  $\Delta$ .

**Proposition 11.** [10, 7.2.4 - F35] *If  $\Delta$  is a simple graph with  $e$  edges and  $v$  vertices, then*

$$\gamma(\Delta) \geq 1 - \frac{v}{2} + \frac{e}{6} \geq \frac{v}{6} \left( \frac{e}{v} - 3 \right).$$

**Proposition 12.** [3, 10.3.6 (a)]. *If  $\Delta$  is a simple graph with  $e$  edges and  $v \geq 3$  vertices, then*

$$\theta(\Delta) \geq \frac{e}{3v-6}.$$

**Proposition 13.** [2, Theorem 6] *If  $\Delta$  is a simple graph with  $e$  edges and  $v$  vertices, then*

$$\text{cr}(\Delta) \geq \frac{e^3}{29v^2} - \frac{35}{29}v.$$

Assume that  $G$  is a finite group and let  $a$  and  $b$  be positive integers. Let  $d = a + b \geq d(G)$ . If  $a \neq b$  then  $\Gamma_{a,b}(G)$  is a bipartite graphs with two parts, one corresponding to the elements of  $G^a$  and the other to the elements of  $G^b$ . In particular  $\Gamma_{a,b}(G)$  has  $|G|^a + |G|^b$  vertices and there exists a bijective correspondence between the set of the generating  $d$ -uples of  $G$  and the set of the edges

of  $\Gamma_{a,b}(G)$ : indeed if  $\langle g_1, \dots, g_d \rangle = G$ , then  $(g_1, \dots, g_a)$  and  $(g_{a+1}, \dots, g_d)$  are adjacent vertices of the graph. Hence the number of edges of  $\Gamma_{a,b}(G)$  is  $\phi_G(d)$ . The situation is different if  $a = b$ . In that case  $\Gamma_{a,a}(G)$  has  $|G|^a$  vertices,  $\phi_G(a)$  loops and other  $(\phi_G(d) - \phi_G(a))/2$  edges connecting two different vertices (in other words if  $e$  is the the number of edges, excluding the loops, and  $l$  is the number of loops, then  $2e + l = \phi_G(d)$ ); indeed the two elements  $(g_1, \dots, g_a, g_{a+1}, \dots, g_d)$  and  $(g_{a+1}, \dots, g_d, g_1, \dots, g_a)$  give rise to the same edge in  $\Gamma_{a,a}(G)$ . Summarizing, let  $\nu$  and  $\eta$  be, respectively, the number of vertices and edges of  $\Gamma_{a,b}(G)$ , excluding the loops. We have

$$|G|^b \leq \nu \leq |G|^a + |G|^b \leq 2|G|^{d-1}.$$

Moreover  $\eta = \phi_G(a+b)$  if  $a \neq b$ ,  $\eta = (\phi_G(2a) - \phi_G(a))/2$  if  $a = b$ . If  $\phi_G(a) \neq 0$ , then  $\phi_G(2a) \geq \phi_G(a)|G|^a$ , so  $\phi_G(a) \leq \phi_G(2a)/|G|^a$ . So if  $|G| \geq 2$ , then  $\eta \geq \phi_G(d)/4$ . By applying Theorem 2 and Propositions 11,12 and 13 respectively it follows that if  $G \neq 1$ , then we have the following inequalities.

$$\begin{aligned} \gamma(\Gamma_{a,b}(G)) &\geq \frac{\nu}{6} \left( \frac{\eta}{\nu} - 3 \right) \geq \frac{|G|^b}{6} \left( \frac{\phi_G(d)}{8|G|^{d-1}} - 3 \right) \geq \frac{|G|^b}{6} \left( \frac{\sqrt{|G|}}{16} - 3 \right). \\ \theta(\Gamma_{a,b}(G)) &\geq \frac{\eta}{3\nu} \geq \frac{\phi_G(d)}{24|G|^{d-1}} \geq \frac{\sqrt{|G|}}{48}. \\ \text{cr}(\Gamma_{a,b}(G)) &\geq \frac{\eta^3}{29 \cdot \nu^2} - \frac{35}{29} \cdot \nu \geq \frac{(\phi_G(d))^3}{29 \cdot 4^3 \cdot 4 \cdot (|G|^{d-1})^2} - \frac{70 \cdot |G|^{d-1}}{29} \\ &\geq \frac{\phi_G(d)|G|}{29 \cdot 4^5} - \frac{70 \cdot |G|^{d-1}}{29} \geq \frac{|G|^{d+\frac{1}{2}}}{29 \cdot 2^{11}} - \frac{70 \cdot |G|^{d-1}}{29}. \end{aligned}$$

This concludes the proof of Theorem 1.

#### 4. PROOF OF THEOREM 4

The main goal of this section is to prove Theorem 4. We start with two preliminary results.

**Proposition 14.** [4, Lemma 9.23]. *A simple bipartite planar graph on  $v$  vertices, whose every connected component contains at least three vertices, can have not more than  $2v - 4$  edges.*

**Lemma 15.** *Let  $G$  be a finite group and let  $b \geq d(G)$ . Consider the set  $W = \{(x_1, \dots, x_b) \in G^b \mid \langle x_1, \dots, x_b \rangle = G\}$ . If  $G$  is not cyclic, then  $|W| \geq 3$ .*

*Proof.* Assume  $d = d(G)$  and  $G = \langle g_1, \dots, g_d \rangle$ . Then  $(g_1, g_2, g_3, \dots, g_d, 1, \dots, 1)$ ,  $(g_1 g_2, g_2, g_3, \dots, g_d, 1, \dots, 1)$  and  $(g_1, g_1 g_2, g_3, \dots, g_d, 1, \dots, 1)$  are three different elements of  $W$ .  $\square$

We are now ready to embark on the proof of Theorem 4.

*Proof of Theorem 4.* Let  $a$  and  $b$  be positive integers with  $a + b \geq d(G)$ . We want to discuss when  $\Gamma_{a,b}(G)$  is planar. We assume  $a + b \geq d(G)$  and  $a \leq b$ . If  $a = 0$ , then  $\Gamma_{a,b}(G)$  is a star, so it is planar. We may exclude from our discussion the case  $a = b = 1$ , since the result in this case follows from the main result in [12] (notice that the cyclic group  $C_5$  appears in the statement of [12, Theorem 1.1] but not in the statement of Theorem 4: this is because in [12] the identity element is not included in the vertex-set of  $\Gamma_{1,1}(G)$ ).

First assume that  $G = \langle g \rangle$  is cyclic.

- If  $a \geq 3$ , take

$$\alpha_1 = (1, 1, g, 1, \dots, 1), \alpha_2 = (1, g, g, 1, \dots, 1), \alpha_3 = (1, g, 1, 1, \dots, 1) \in G^a,$$

$$\beta_1 = (g, 1, g, 1, \dots, 1), \beta_2 = (g, g, g, 1, \dots, 1), \beta_3 = (g, g, 1, 1, \dots, 1) \in G^b.$$

- If  $a = 2$  and  $|G| \neq 2$ , take

$$\alpha_1 = (1, g), \alpha_2 = (g, 1), \alpha_3 = (g, g) \in G^2,$$

$$\beta_1 = (1, g^2, 1, \dots, 1), \beta_2 = (g^2, 1, \dots, 1), \beta_3 = (g^2, g^2, 1, \dots, 1) \in G^b.$$

- If  $a = 2$  and  $|G| = 2$  and  $b \geq 3$ , take

$$\alpha_1 = (1, g), \alpha_2 = (g, 1), \alpha_3 = (g, g) \in G^2,$$

$$\beta_1 = (1, g, g, 1, \dots, 1), \beta_2 = (g, 1, g, 1, \dots, 1), \beta_3 = (g, g, g, 1, \dots, 1) \in G^b.$$

In all these cases, since  $\alpha_i$  and  $\beta_j$  are adjacent for every  $1 \leq i, j \leq 3$ ,  $\Gamma_{a,b}(G)$  contains  $K_{3,3}$ , so it is not planar. If  $a = b = 2$  and  $|G| = 2$ , then  $\Gamma_{2,2}(G) \cong K_4$  is planar. If  $a = 1$  and  $|G| > 2$ , then we may consider the subgraph of  $\Gamma_{1,b}(G)$  induced by the following vertices:  $(1), (g), (g^2), (g, x, \dots, x) \in G^b$  for  $x \in G$ . This subgraph is bipartite with  $3 + |G|$  vertices and  $3|G|$  edges. Since  $3|G| > 2(3 + |G|) - 4$ , it follows from Proposition 14, that this graph is not planar. On the other hand, if  $a = 1$  and  $|G| = 2$ , then it can be easily seen that the graph  $\Gamma_{1,b}(G)$  is planar.

Now assume that  $G$  is not cyclic. Let  $d = d(G)$  and  $G = \langle g_1, \dots, g_d \rangle$ .

First assume that  $a \geq 2$ . If  $a + b = d$ , then set

$$\alpha_1 = (g_1, g_2, g_3, \dots, g_a) \in G^a,$$

$$\alpha_2 = (g_1, g_1 g_2, g_3, \dots, g_a) \in G^a,$$

$$\alpha_3 = (g_1 g_2, g_2, g_3, \dots, g_a) \in G^a,$$

$$\beta_1 = (g_{a+1}, g_{a+2}, g_{a+3}, \dots, g_b) \in G^b,$$

$$\beta_2 = (g_{a+1} g_{a+2}, g_{a+2}, g_{a+3}, \dots, g_b) \in G^b,$$

$$\beta_3 = (g_{a+1}, g_{a+1} g_{a+2}, g_{a+3}, \dots, g_b) \in G^b.$$

If  $a + b > d$ , choose three different elements  $x, y, z$  of  $G$  and set

$$\alpha_1 = (g_1, g_2, g_3, \dots, g_a) \in G^a,$$

$$\alpha_2 = (g_1, g_1 g_2, g_3, \dots, g_a) \in G^a,$$

$$\alpha_3 = (g_1 g_2, g_2, g_3, \dots, g_a) \in G^a,$$

$$\beta_1 = (g_{a+1}, g_{a+2}, g_{a+3}, \dots, g_b, x, \dots, x) \in G^b,$$

$$\beta_2 = (g_{a+1}, g_{a+2}, g_{a+3}, \dots, g_b, y, \dots, y) \in G^b,$$

$$\beta_3 = (g_{a+1}, g_{a+2}, g_{a+3}, \dots, g_b, z, \dots, z) \in G^b.$$

In both cases, since  $\alpha_i$  and  $\beta_j$  are adjacent for every  $1 \leq i, j \leq 3$ ,  $\Gamma_{a,b}(G)$  contains  $K_{3,3}$ , so it is not planar.

Assume  $a = 1$  and  $a + b > d$ . Let  $W = \{(x_1, \dots, x_b) \in G^b \mid \langle x_1, \dots, x_b \rangle = G\}$  and let  $x, y, z$  be three different elements of  $G$ . We may consider the subgraph of  $\Gamma_{1,b}(G)$  induced by following vertices:  $(x), (y), (z), w \in W$ . This subgraph is bipartite with  $3 + |W|$  vertices and  $3|W|$  edges. Since, by Lemma 15,  $|W| \geq 3$ , it

follows  $3|W| > 2(3 + |W|) - 4$ , and consequently, by Proposition 14, this graph is not planar.

Finally assume  $a = 1$  and  $a + b = d$ . Let  $H = \langle g_2, \dots, g_b \rangle$ . If  $H$  is cyclic, then  $d(G) \leq 2$ , in contradiction with  $d(G) = 1 + b$  and  $b > 1$ . Let  $x, y, z$  be three different elements of  $H$  and  $W = \{(x_1, \dots, x_b) \in G^b \mid \langle x_1, \dots, x_b \rangle = H\}$ . We may consider the subgraph of  $\Gamma_{1,b}(G)$  induced by following vertices:  $(g_1x), (g_1y), (g_1z), w \in W$ . It is bipartite with  $3 + |W|$  vertices and  $3|W|$  edges. Since  $H$  is not cyclic, we have  $|W| \geq 3$  by Lemma 15. It follows  $3|W| > 2(3 + |W|) - 4$ , and consequently, by Proposition 14, this graph is not planar.  $\square$

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