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Splitting Methods for a Class of Horizontal Linear Complementarity Problems

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Abstract

In this paper, we propose two splitting methods for solving horizontal linear complementarity problems characterized by matrices with positive diagonal elements. The proposed procedures are based on the Jacobi and on the Gauss–Seidel iterations and differ from existing techniques in that they act directly and simultaneously on both matrices of the problem. We prove the convergence of the methods under some assumptions on the diagonal dominance of the matrices of the problem. Several numerical experiments, including large-scale problems of practical interest, demonstrate the capabilities of the proposed methods in various situations.

Keywords Horizontal linear complementarity problem · Matrix splitting · Projected methods

Mathematics Subject Classification 65K05 · 65H10 · 90C33

1 Introduction

Horizontal linear complementarity problems (HLCPs) are a well-known generalization of linear complementarity problems (LCPs) [1] and have applications in many different fields, including structural mechanics, mechanical and electrical engineering and transportation science. Several solution techniques have, then, been devised and new ones are proposed to this day. In particular, some popular approaches are based on Interior Point (IP) methods (see, e.g., [2]) or rely on reducing the HLCP to an LCP

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[3,4]. Homotopy approaches [5] and, more recently, neural networks [6] have been studied as well.

Some drawbacks to these procedures, however, exist. Reduction techniques, for instance, require permutations or matrix inversions and can be computationally onerous. The other aforementioned methods, on the other hand, avoid reductions but generally involve complex iterations. It would nonetheless be desirable to tackle more directly the HLCP, especially for applications to large, practical problems.

We here do so by introducing two projected splitting methods that act directly on the matrices of the complementarity problem. Conceptually, the idea is similar to projected splitting methods for LCPs, like [7–9]. However, HLCPs are characterized by the presence of two matrices, which we hereafter denote as A and B. Then, we need to split them both, which we do simultaneously. Thus, we set up Jacobi or Gauss–Seidel iterations which involve elements of both A and B, while complementarity is enforced by two projections.

For these projections to be meaningful, the diagonals of A and B must be positive. Thus, we consider such a class of HLCPs and introduce our projected Jacobi and projected Gauss–Seidel methods in Sects. 2 and 3, respectively. We there also prove the convergence of the procedures when A and B are strictly diagonally dominant by columns. In Sect. 4, we then provide some insights into cases where diagonal dominance does not hold in a strict sense. In particular, we do so in the framework of irreducibly diagonally dominant matrices [10, p. 23]. In this context, we also provide some results on the uniqueness of the solution of the HLCP in these cases. In Sects. 5 and 6, we then present and solve several numerical examples, ranging from random test problems to HLCPs of practical interest. We thus validate the procedures, demonstrate their efficiency and also perform a comparison with an IP method recently introduced for solving HLCPs in hydrodynamic lubrication [11]. Finally, the conclusions of this work are summarized in Sect. 7.

2 The Projected Jacobi Iteration for HLCPs

Given $A, B \in \mathbb{R}^{n\times n}$ and a source term $\boldsymbol{c} \in \mathbb{R}^n$, the horizontal complementarity problem HLCP($A, B, \boldsymbol{c}$) consists in finding a pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ satisfying

$$A\boldsymbol{x} - B\boldsymbol{y} = \boldsymbol{c}, \qquad \boldsymbol{x} \geq \boldsymbol{0}, \ \ \boldsymbol{y} \geq \boldsymbol{0}, \qquad \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0. \tag{1}$$

Throughout the paper, we assume that A and B have positive diagonal. To solve this problem, let us apply a Jacobi splitting to both A and B. Thus, given two arbitrary vectors $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ and $\boldsymbol{y}^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, \ldots$, let $\{\boldsymbol{x}^{(k+1)}, \boldsymbol{y}^{(k+1)}\}$ be the solution of the complementarity problem

$$\begin{cases} a_{11}x_1 + \sum_{j=2}^{n} a_{1j}x_j^{(k)} - b_{11}y_1 - \sum_{j=2}^{n} b_{1j}y_j^{(k)} &= c_1 \\ a_{22}x_2 + \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j^{(k)} - b_{22}y_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} b_{2j}y_j^{(k)} &= c_2 \\ a_{33}x_3 + \sum_{\substack{j=1 \\ j\neq 3}}^{n} a_{3j}x_j^{(k)} - b_{33}y_3 - \sum_{\substack{j=1 \\ j\neq 3}}^{n} b_{3j}y_j^{(k)} &= c_3 \\ &\vdots \\ a_{nn}x_n + \sum_{j=1}^{n-1} a_{nj}x_j^{(k)} - b_{nn}y_n - \sum_{j=1}^{n-1} b_{nj}y_j^{(k)} &= c_n \end{cases}$$
$$\boldsymbol{x} \geq \boldsymbol{0}; \qquad \boldsymbol{y} \geq \boldsymbol{0}; \qquad \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0,$$

where x_i and y_i denote the i-th component of the unknown vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and $x_i^{(k)}$ and $y_i^{(k)}$ indicate the i-th component of $\boldsymbol{x}^{(k)}$ and $\boldsymbol{y}^{(k)}$, respectively.
More compactly, collecting terms in $\boldsymbol{x}^{(k)}$ and in $\boldsymbol{y}^{(k)}$ on the right-hand side of the system, for $i = 1, \ldots, n$ we obtain $\{x_i^{(k+1)}, y_i^{(k+1)}\}$ as the solution of the horizontal complementarity problem

$$\begin{aligned} &a_{ii}x_i - b_{ii}y_i = w_i^{(k)} \\ &x_i \geq 0; \qquad y_i \geq 0; \qquad x_i y_i = 0, \end{aligned} \tag{2}$$

with $a_{ii}, b_{ii} > 0$ by the positivity of the diagonals of A and B and with

$$w_i^{(k)} = c_i - \sum_{j\neq i} a_{ij}x_j^{(k)} + \sum_{j\neq i} b_{ij}y_j^{(k)}, \tag{3}$$

where, for compactness, we have set $\sum_{j\neq i} \square = \sum_{\substack{j=1 \\ j\neq i}}^{n} \square$.

Thus, for $i = 1, \ldots, n$, the solution $\{x_i^{(k+1)}, y_i^{(k+1)}\}$ of (2) has the simple form

$$x_i^{(k+1)} = \max\left\{0, \frac{w_i^{(k)}}{a_{ii}}\right\}; \qquad y_i^{(k+1)} = \max\left\{0, -\frac{w_i^{(k)}}{b_{ii}}\right\}. \tag{4}$$

We refer to this method as *projected Jacobi method* for HLCPs.

In the following, it is useful to denote

$$w_{i+}^{(k)} := \max\left\{0, w_i^{(k)}\right\}; \qquad w_{i-}^{(k)} := \max\left\{0, -w_i^{(k)}\right\} \tag{5}$$

and to introduce a few results involving sums and differences of terms $w_{i+}^{(k)}$ and $w_{i-}^{(k)}$. In particular, it is easy to notice that $w_{i+}^{(k)} + w_{i-}^{(k)} = |w_i^{(k)}|$. Moreover, for any two arbitrary real numbers c, d, we have

$$|c^+ - d^+| + |c^- - d^-| = |c - d|. \tag{6}$$

Indeed, if $c > d \geq 0$ or if $0 > c \geq d$, one of the two terms in absolute value in the left-hand side vanishes and the equality is readily proved. Similarly, if $c \geq 0 > d$, the left-hand side becomes $|c| + |d|$, with $|c| = c$ and $|d| = -d$. Then, $c - d = |c - d|$ and the equality holds true. All other possible cases easily follow.

Using these results, we hereafter analyze the convergence of the projected Jacobi method. We first assume A, B strictly diagonally dominant, case where (assuming also the positivity of the diagonals) the solution of the HLCP$(A, B, \boldsymbol{c})$ is unique [12].

Theorem 2.1 *Let $A, B \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant by columns with positive diagonal elements. Then, the sequence $\{\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}\}$ generated by (3)–(4) converges to the unique solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of the HLCP$(A, B, \boldsymbol{c})$ for all initial vectors $\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}$.*

Proof For $i = 1, \dots, n$, define

$$w_i^* = c_i - \sum_{j\neq i} a_{ij} x_j^* + \sum_{j\neq i} b_{ij} y_j^* \tag{7}$$

and consider that we can thus write[1] $x_i^* = \max\{0, w_i^*/a_{ii}\}$, $y_i^* = \max\{0, -w_i^*/b_{ii}\}$. Then, noticing that, by (5), we can express the iterates in (4) as

$$x_i^{(k+1)} = \frac{w_{i+}^{(k)}}{a_{ii}}; \qquad y_i^{(k+1)} = \frac{w_{i-}^{(k)}}{b_{ii}}, \tag{8}$$

and considering the definitions of $w_i^{(k)}$ in (3) and of w_i^* in (7), we can express the residual at the i-th index and at k-th iteration as

$$\begin{aligned}\left|w_i^{(k)} - w_i^*\right| &= \left|-\sum_{j\neq i} a_{ij}\left(x_j^{(k)} - x_j^*\right) + \sum_{j\neq i} b_{ij}\left(y_j^{(k)} - y_j^*\right)\right| \\ &= \left|-\sum_{j\neq i} \frac{a_{ij}}{a_{jj}}\left(w_{j+}^{(k-1)} - w_{j+}^*\right) + \sum_{j\neq i} \frac{b_{ij}}{b_{jj}}\left(w_{j-}^{(k-1)} - w_{j-}^*\right)\right| \\ &\leq \sum_{j\neq i}\left|\frac{a_{ij}}{a_{jj}}\right|\left|w_{j+}^{(k-1)} - w_{j+}^*\right| + \sum_{j\neq i}\left|\frac{b_{ij}}{b_{jj}}\right|\left|w_{j-}^{(k-1)} - w_{j-}^*\right|.\end{aligned} \tag{9}$$

In the trivial case of A and B diagonal, the residual is evidently zero in a single iteration. Otherwise, let us define

$$\rho := \max_{j\neq i}\left\{\left|\frac{a_{ij}}{a_{jj}}\right|, \left|\frac{b_{ij}}{b_{jj}}\right|\right\}.$$

[1] It is easy to prove that $\max\{0, w_i^*/a_{ii}\}$ and $\max\{0, -w_i^*/b_{ii}\}$ are the solution x_i^*, y_i^* of the HLCP$(A, B, \boldsymbol{c})$, $i = 1, \dots, n$. Indeed, replacing in the i-th row of $A\boldsymbol{x} - B\boldsymbol{y} = \boldsymbol{c}$, we have $a_{ii}x_i^* - b_{ii}y_i^* = \max\{0, w_i^*\} - \max\{0, -w_i^*\} = w_i^*$. By the positivity of a_{ii} and b_{ii}, we then have the nonnegativity of x_i and y_i and that x_i is positive when $y_i = 0$ (and vice versa).

Applying (6), we obtain

$$
\begin{aligned}
\left|w_i^{(k)} - w_i^*\right| &\leq \rho \sum_{j \neq i}\left(\left|w_{j^+}^{(k-1)} - w_{j^+}^*\right| + \sum_{j \neq i}\left|w_{j^-}^{(k-1)} - w_{j^-}^*\right|\right) \\
&= \rho \sum_{j \neq i}\left|w_j^{(k-1)} - w_j^*\right| \leq \rho \sum_{j=1}^{n}\left|w_j^{(k-1)} - w_j^*\right|.
\end{aligned}
$$

Let us then evaluate this new expression. Proceeding as in (9), we can write

$$
\begin{aligned}
\sum_{j=1}^{n}\left|w_j^{(k-1)} - w_j^*\right| &\leq \sum_{j=1}^{n}\left(\sum_{l \neq j}\left|\frac{a_{jl}}{a_{ll}}\right|\left|w_{l^+}^{(k-2)} - w_{l^+}^*\right| + \sum_{l \neq j}\left|\frac{b_{jl}}{b_{ll}}\right|\left|w_{l^-}^{(k-2)} - w_{l^-}^*\right|\right) \\
&= \sum_{l=1}^{n}\left|w_{l^+}^{(k-2)} - w_{l^+}^*\right| \sum_{j \neq l}\left|\frac{a_{jl}}{a_{ll}}\right| + \sum_{l=1}^{n}\left|w_{l^-}^{(k-2)} - w_{l^-}^*\right| \sum_{j \neq l}\left|\frac{b_{jl}}{b_{ll}}\right|,
\end{aligned}
\tag{10}
$$

where, in the last passage, we have reversed the order of the summations [13, p. 36] and have brought terms not dependent on the index j out of the inner sum.

If we now define

$$
\mu := \max_{l=1,n}\left\{\sum_{j \neq l}\left|\frac{a_{jl}}{a_{ll}}\right|, \sum_{j \neq l}\left|\frac{b_{jl}}{b_{ll}}\right|\right\},
\tag{11}
$$

we can make a further evaluation and write

$$
\sum_{j=1}^{n}\left|w_j^{(k-1)} - w_j^*\right| \leq \mu \sum_{l=1}^{n}\left|w_l^{(k-2)} - w_l^*\right|,
$$

where $\mu \in]0, 1[$ by the strict column diagonal dominance of A and B.

Then, proceeding iteratively, we find

$$
\begin{aligned}
\left|w_i^{(k)} - w_i^*\right| &\leq \rho \sum_{j=1}^{n}\left|w_j^{(k-1)} - w_j^*\right| \leq \rho\mu \sum_{j=1}^{n}\left|w_j^{(k-2)} - w_j^*\right| \\
&\leq \rho\mu^2 \sum_{j=1}^{n}\left|w_j^{(k-3)} - w_j^*\right| \leq \ldots \leq \rho\mu^{k-1} \sum_{j=1}^{n}\left|w_j^{(0)} - w_j^*\right|
\end{aligned}
\tag{12}
$$

for any generic i-th index. Thus, $\lim_{k\to\infty}\left|w_i^{(k)} - w_i^*\right| = 0$ for $i = 1, \ldots, n$. This implies that $w_i^{(k)} \to w_i^*$ for $k \to \infty$ for all $i = 1, \ldots, n$ and then $\lim_{k\to\infty} x_i^{(k)} = \max\{0, w_i^*/a_{ii}\} = x_i^*$ and $\lim_{k\to\infty} y_i^{(k)} = \max\{0, -w_i^*/b_{ii}\} = y_i^*$. □

3 The Projected Gauss–Seidel Iteration for HLCPs

If we apply a Gauss–Seidel splitting to A and B instead of the Jacobi splitting employed in the previous section, we can analogously formulate a projected Gauss–Seidel method. In this regard, let us again consider (1) with A, B real matrices with positive diagonal elements. Then, given two arbitrary vectors $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ and $\boldsymbol{y}^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, \dots$ and for $i = 1, \dots, n$, let $\{x_i^{(k+1)}, y_i^{(k+1)}\}$ be the solution of the horizontal linear complementarity problem

$$\begin{aligned} a_{ii}x_i - b_{ii}y_i &= \tilde{w}_i^{(k)} \\ x_i \geq 0; \qquad y_i \geq 0; &\qquad x_i y_i = 0, \end{aligned} \tag{13}$$

with $a_{ii}, b_{ii} > 0$ by the positivity of the diagonals of A and B and with

$$\tilde{w}_i^{(k)} = c_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + \sum_{j=1}^{i-1} b_{ij}y_j^{(k+1)} + \sum_{j=i+1}^{n} b_{ij}y_j^{(k)}. \tag{14}$$

Thus, for $i = 1, \dots, n$, the solution of (13) has the simple form

$$x_i^{(k+1)} = \max\left\{0, \frac{\tilde{w}_i^{(k)}}{a_{ii}}\right\}; \qquad y_i^{(k+1)} = \max\left\{0, -\frac{\tilde{w}_i^{(k)}}{b_{ii}}\right\}. \tag{15}$$

We refer to this method as *projected Gauss–Seidel method* for HLCPs. Let us now prove that the iterates computed by (15) converge to the solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$.

Theorem 3.1 *Let $A, B \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant by columns with positive diagonal elements and let $\boldsymbol{c} \in \mathbb{R}^n$. Then, the sequence $\{\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}\}$ generated by (14)–(15) converges to the unique solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of the HLCP($A, B, \boldsymbol{c}$) for all initial vectors $\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}$.*

Proof Similarly to what done for the projected Jacobi method, we define

$$\tilde{w}_{i^+}^{(k)} := \max\left\{0, \tilde{w}_i^{(k)}\right\}; \qquad \tilde{w}_{i^-}^{(k)} := \max\left\{0, -\tilde{w}_i^{(k)}\right\}. \tag{16}$$

Then, by (15), we can write the solution $\{x_i^{(k+1)}, y_i^{(k+1)}\}$ of (13) as

$$x_i^{(k+1)} = \frac{\tilde{w}_{i^+}^{(k)}}{a_{ii}}; \qquad y_i^{(k+1)} = \frac{\tilde{w}_{i^-}^{(k)}}{b_{ii}}. \tag{17}$$

By (14) and (17) and proceeding similarly as in Theorem 2.1, we find that, at the generic k-th iteration, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|\tilde{w}_i^{(k)}-w_i^*\right| &= \sum_{i=1}^{n}\left|-\sum_{j=1}^{i-1}a_{ij}\left(x_j^{(k+1)}-x_j^*\right)-\sum_{j=i+1}^{n}a_{ij}\left(x_j^{(k)}-x_j^*\right)\right.\\ &\quad\left.+\sum_{j=1}^{i-1}b_{ij}\left(y_j^{(k+1)}-y_j^*\right)+\sum_{j=i+1}^{n}b_{ij}\left(y_j^{(k)}-y_j^*\right)\right|\\ &\leq \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^*\right|\sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-1)}-w_{j^+}^*\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|\\ &\quad+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^*\right|\sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-1)}-w_{j^-}^*\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|.\end{aligned} \tag{18}$$

On the other hand, by (6), we also have

$$\begin{aligned}\sum_{h=1}^{n}\left|\tilde{w}_h^{(k)}-w_h^*\right| &= \sum_{h=1}^{n}\left(\left|\tilde{w}_{h^+}^{(k)}-w_{h^+}^*\right|+\left|\tilde{w}_{h^-}^{(k)}-w_{h^-}^*\right|\right)\\ &> \sum_{h=1}^{n}\left|\tilde{w}_{h^+}^{(k)}-w_{h^+}^*\right|\sum_{l\neq h}\left|\frac{a_{lh}}{a_{hh}}\right|+\sum_{h=1}^{n}\left|\tilde{w}_{h^-}^{(k)}-w_{h^-}^*\right|\sum_{l\neq h}\left|\frac{b_{lh}}{b_{hh}}\right|,\end{aligned} \tag{19}$$

since $\sum_{l\neq h}|a_{lh}/a_{hh}|$ and $\sum_{l\neq h}|b_{lh}/b_{hh}|$ are strictly smaller than one by strict column diagonal dominance of A and B. Combining (18) and (19), we then obtain

$$\begin{aligned}&\sum_{h=1}^{n}\left|\tilde{w}_{h^+}^{(k)}-w_{h^+}^*\right|\sum_{l\neq h}\left|\frac{a_{lh}}{a_{hh}}\right|+\sum_{h=1}^{n}\left|\tilde{w}_{h^-}^{(k)}-w_{h^-}^*\right|\sum_{l\neq h}\left|\frac{b_{lh}}{b_{hh}}\right| < \sum_{h=1}^{n}\left|\tilde{w}_h^{(k)}-w_h^*\right|\\ &\leq \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^*\right|\sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-1)}-w_{j^+}^*\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|\\ &\quad+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^*\right|\sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-1)}-w_{j^-}^*\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|.\end{aligned}$$

Hence, subtracting $\sum_{j=1}^{n}|\tilde{w}_{j^+}^{(k)}-w_{j^+}^*|\sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}|\tilde{w}_{j^-}^{(k)}-w_{j^-}^*|\sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right|$ from all members of the previous chain of inequalities, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| \\
&\quad < \sum_{j=1}^{n}\left|\tilde{w}_{j}^{(k)} - w_{j}^{*}\right| - \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^{*}\right|\sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right| - \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^{*}\right|\sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right| \\
&\quad \le \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-1)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-1)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|.
\end{aligned}
\tag{20}
$$

Consider the middle expression. By (6), we can write any of its terms as

$$
\left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^{*}\right|\left(1 - \sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|\right) + \left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^{*}\right|\left(1 - \sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right|\right), \tag{21}
$$

where

$$
1 - \sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right| = \sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \xi_{1_j}; \quad 1 - \sum_{i=j+1}^{n}\left|\frac{b_{ij}}{b_{jj}}\right| = \sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| + \xi_{2_j}, \tag{22}
$$

with $\xi_{1_j} := 1 - \sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| - \sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right| > 0$ by strict column diagonal dominance of A and, analogously, $\xi_{2_j} > 0$ by strict column diagonal dominance of B, for $j = 1, \ldots, n$. Moreover, considering the first and the last member of (20) and proceeding iteratively, we also have

$$
\begin{aligned}
&\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| \\
&\quad < \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-1)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-1)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| \\
&\quad < \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-2)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-2)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| < \ldots \\
&\quad < \sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(0)} - w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(0)} - w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|.
\end{aligned}
\tag{23}
$$

Thus, the sequence $\left\{\sum_{j=1}^{n}|\tilde{w}_{j^+}^{(k)} - w_{j^+}^{*}|\sum_{i=1}^{j-1}|\frac{a_{ij}}{a_{jj}}| + \sum_{j=1}^{n}|\tilde{w}_{j^-}^{(k)} - w_{j^-}^{*}|\sum_{i=1}^{j-1}|\frac{b_{ij}}{b_{jj}}|\right\}$ for $k = 0, 1, \ldots$ is bounded, monotonic decreasing and, hence, convergent. Then,

$$\lim_{k\to\infty}\left(\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|\right)$$

$$=\lim_{k\to\infty}\left(\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^{*}\right|\xi_{1_j}\right.$$

$$\left.+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^{*}\right|\xi_{2_j}\right)$$

$$=\lim_{k\to\infty}\left(\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k-1)}-w_{j^+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k-1)}-w_{j^-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|\right),$$

where we have exploited (20) with the middle member written by using (21) and (22). Thus,

$$\sum_{j=1}^{n}\left|\tilde{w}_{j^+}^{(k)}-w_{j^+}^{*}\right|\xi_{1_j}+\sum_{j=1}^{n}\left|\tilde{w}_{j^-}^{(k)}-w_{j^-}^{*}\right|\xi_{2_j}\to 0 \text{ for } k\to\infty, \tag{24}$$

which, in turn, implies that $\tilde{w}_j^{(k)} \to w_j^*$ for $k \to \infty$ for all $j = 1,\dots,n$, since $\xi_{1_j}, \xi_{2_j} > 0$ for any $j = 1\dots,n$. Therefore, the method converges to the solution of the HLCP(A, B, $\boldsymbol{c}$). ◻

4 Convergence for *A*, *B* Not Strictly Diagonally Dominant

We now provide some remarks on the uniqueness of solution and on the convergence of the presented procedures when A and B are not strictly diagonally dominant by columns. The proofs here provided exploit the concepts of column representativeness and of column $\mathcal{W}$-property [12].

Theorem 4.1 *Let A, $B \in \mathbb{R}^{n\times n}$ be column diagonally dominant matrices with positive diagonal elements. Then,*

(i) *if A, B and all their column representative matrices are irreducibly diagonally dominant by columns, the solution of the HLCP(A,B,$\boldsymbol{c}$) is unique;*
(ii) *if $a_{jj} > \sum_{i\neq j}|a_{ij}|$ and $b_{jj} > \sum_{i\neq j}|b_{ij}|$ for at least one same index j[2], with $\sum_{i=k+1}^{j} a_{ik}, \sum_{i=k+1}^{j} b_{ik} \neq 0$ for $k = 1,\dots,j-1$ and $\sum_{i=j}^{k-1} a_{ik}, \sum_{i=j}^{k-1} b_{ik} \neq 0$ for $k = j+1,\dots,n$, the iterates generated by (3)–(4) and by (14)–(15) converge to the solution of the HLCP(A, B, $\boldsymbol{c}$).*

Proof Regarding uniqueness, the hypotheses of the first point of the theorem imply that all eigenvalues of all column representative matrices of the set $\mathcal{M} := \{A, B\}$ have positive real part [14, §6.2.27]. Then, all column representative matrices of $\mathcal{M}$ have positive determinants. Hence, $\mathcal{M}$ has the column $\mathcal{W}$-property and the solution of the HLCP(A, B, $\boldsymbol{c}$) is unique [12, Theorem 2].

[2] It is easy to notice that this is necessarily true if the hypotheses at the first point of the theorem hold.

Regarding convergence of the methods, let us start from the projected Gauss–Seidel method. Assume, for simplicity and with no loss of generality, that only the first column of A and B is strictly diagonally dominant. Proceeding as in Theorem 3.1, we find the same result as in (24), where, however,

$$\xi_{1_1}, \xi_{2_1} > 0; \quad \xi_{1_j} = \xi_{2_j} = 0 \ \text{ for } j = 2, \ldots, n.$$

We thus only find that $\tilde{w}_1^{(k)} \to w_1^*$ for $k \to \infty$.

Next, consider $\sum_{i=2}^n |\tilde{w}_i^{(k)} - w_i^*|$. By proceeding as in Theorem 3.1, we find

$$\begin{aligned}\sum_{i=2}^n \left|\tilde{w}_i^{(k)} - w_i^*\right| \le \sum_{j=1}^n \left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^*\right| \sum_{i=j+1}^n \left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^+}^{(k-1)} - w_{j^+}^*\right| \sum_{i=2}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| \\ + \sum_{j=1}^n \left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^*\right| \sum_{i=j+1}^n \left|\frac{b_{ij}}{b_{jj}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^-}^{(k-1)} - w_{j^-}^*\right| \sum_{i=2}^{j-1} \left|\frac{b_{ij}}{b_{jj}}\right|.\end{aligned}$$

Moreover, by the column diagonal dominance of A and B, we can also write

$$\sum_{h=2}^n \left|\tilde{w}_h^{(k)} - w_h^*\right| > \sum_{h=2}^n \left|\tilde{w}_{h^+}^{(k)} - w_{h^+}^*\right| \sum_{l \ne h, l \ge 2} \left|\frac{a_{lh}}{a_{hh}}\right| + \sum_{h=2}^n \left|\tilde{w}_{h^-}^{(k)} - w_{h^-}^*\right| \sum_{l \ne h, l \ge 2} \left|\frac{b_{lh}}{b_{hh}}\right|.$$

This inequality still holds in strict sense, since $\sum_{i=1}^{j-1} |a_{ij}| \ne 0$ and $\sum_{i=1}^{j-1} |b_{ij}| \ne 0$ for $j = 2, \ldots n$ implies $\sum_{i \ne 2, i \ge 2} \left|\frac{a_{i2}}{a_{22}}\right| < 1$ and $\sum_{i \ne 2, i \ge 2} \left|\frac{b_{i2}}{b_{22}}\right| < 1$. Thus, combining these inequalities and subtracting the same term $\sum_{j=2}^n |\tilde{w}_{j^+}^{(k)} - w_{j^+}^*| \sum_{i=j+1}^n \left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^n |\tilde{w}_{j^-}^{(k)} - w_{j^-}^*| \sum_{i=j+1}^n \left|\frac{b_{ij}}{b_{jj}}\right|$ from all of them, we obtain

$$\begin{aligned}&\sum_{j=2}^n \left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^*\right| \sum_{i=2}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^*\right| \sum_{i=2}^{j-1} \left|\frac{b_{ij}}{b_{jj}}\right| \\ &< \sum_{j=2}^n \left|\tilde{w}_{j^+}^{(k)} - w_{j^+}^*\right| \sum_{i=1}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^-}^{(k)} - w_{j^-}^*\right| \sum_{i=1}^{j-1} \left|\frac{b_{ij}}{b_{jj}}\right| \\ &\le \left|\tilde{w}_{1^+}^{(k)} - w_{1^+}^*\right| \sum_{i=2}^n \left|\frac{a_{i1}}{a_{11}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^+}^{(k-1)} - w_{j^+}^*\right| \sum_{i=2}^{j-1} \left|\frac{a_{ij}}{a_{jj}}\right| \\ &\quad + \left|\tilde{w}_{1^-}^{(k)} - w_{1^-}^*\right| \sum_{i=2}^n \left|\frac{b_{i1}}{b_{11}}\right| + \sum_{j=2}^n \left|\tilde{w}_{j^-}^{(k-1)} - w_{j^-}^*\right| \sum_{i=2}^{j-1} \left|\frac{b_{ij}}{b_{jj}}\right|,\end{aligned} \tag{25}$$

where the middle expression has been obtained analogously as in (21)–(22), with $\xi_{1_j} = \xi_{2_j} = 0$ for $j = 2, \ldots, n$, as A and B are (not strictly) diagonally dominant in the columns $j = 2, \ldots, n$. For compactness, let us denote by α_k the first member of (25). Then, the inequality between the first and the last member can be written compactly as $\alpha_k < \beta_k + \alpha_{k-1}$, where β_k contains the terms of iteration k of the last member of (25). Moreover, α_k is bounded, since all of its terms are contained in the k–th term of (23), which is bounded. Then, since we also have that β_k tends to zero for $k \to \infty$, the sequence of the $\{\alpha_k\}$ converges and, by (25), we can write

$$\begin{aligned}
&\lim_{k\to\infty}\left(\sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k)} - w_{j+}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k)} - w_{j-}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|\right)\\
&\quad= \lim_{k\to\infty}\left(\sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k)} - w_{j+}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k)} - w_{j-}^{*}\right|\sum_{i=1}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|\right)\\
&\quad= \lim_{k\to\infty}\left(\sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k-1)} - w_{j+}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k-1)} - w_{j-}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right|\right).
\end{aligned}$$

Thus, since the middle member of the previous expression can be rewritten as

$$\begin{aligned}
&\lim_{k\to\infty}\left(\sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k)} - w_{j+}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k)} - w_{j+}^{*}\right|\left|\frac{a_{1j}}{a_{jj}}\right|\right.\\
&\quad\left.+\sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k)} - w_{j-}^{*}\right|\sum_{i=2}^{j-1}\left|\frac{b_{ij}}{b_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k)} - w_{j-}^{*}\right|\left|\frac{b_{1j}}{b_{jj}}\right|\right),
\end{aligned}$$

we finally find

$$\sum_{j=2}^{n}\left|\tilde{w}_{j+}^{(k)} - w_{j+}^{*}\right|\left|\frac{a_{1j}}{a_{jj}}\right| + \sum_{j=2}^{n}\left|\tilde{w}_{j-}^{(k)} - w_{j-}^{*}\right|\left|\frac{b_{1j}}{b_{jj}}\right| \to 0 \text{ for } k \to \infty.$$

However, $\left|\frac{a_{12}}{a_{22}}\right| > 0$ and $\left|\frac{b_{12}}{b_{22}}\right| > 0$ by hypothesis. Thus, $\tilde{w}_2^{(k)} \to w_2^{*}$ for $k \to \infty$ must hold true. By induction, the same applies also to successive components of the residual.

Finally, the convergence of the projected Jacobi method can be proved analogously. Indeed, proceeding as done for Gauss–Seidel, by (10) and (19), we find

$$\begin{aligned}
&\sum_{j=1}^{n}\left|w_{j+}^{(k)} - w_{j+}^{*}\right|\sum_{i\neq j}\left|\frac{a_{ij}}{a_{jj}}\right| + \sum_{j=1}^{n}\left|w_{j-}^{(k)} - w_{j-}^{*}\right|\sum_{i\neq j}\left|\frac{b_{ij}}{b_{jj}}\right|\\
&\quad< \sum_{j=1}^{n}\left|w_{j+}^{(k)} - w_{j+}^{*}\right|\left(\sum_{i\neq j}\left|\frac{a_{ij}}{a_{jj}}\right| + \xi_{1_j}\right) + \sum_{j=1}^{n}\left|w_{j-}^{(k)} - w_{j-}^{*}\right|\left(\sum_{i\neq j}\left|\frac{b_{ij}}{b_{jj}}\right| + \xi_{2_j}\right)
\end{aligned}$$

$$\leq \sum_{j=1}^{n} \left| w_{j^+}^{(k-1)} - w_{j^+}^* \right| \sum_{i \neq j} \left| \frac{a_{ij}}{a_{jj}} \right| + \sum_{j=1}^{n} \left| w_{j^-}^{(k-1)} - w_{j^-}^* \right| \sum_{i \neq j} \left| \frac{b_{ij}}{b_{jj}} \right|.$$

where, if A and B are, for instance, strictly diagonally dominant only at the first column,

$$\xi_{1_1}, \xi_{2_1} > 0; \quad \xi_{1_j} = \xi_{2_j} = 0 \quad \text{for } j = 2, \ldots, n.$$

Proceeding as we did for the projected Gauss–Seidel method, we thus find $|w_1^{(k)} - w_1^*| \to 0$ for $k \to \infty$ and, by induction, we finally have $w_j^{(k)} \to w_j^*$ for $k \to \infty$ also for the following $j = 2, \ldots, n$. □

5 Numerical Experiments

In this section, we present the numerical experiments that we use to show the effectiveness of the proposed methods.

First, we solve a set of problems where A and B are randomly generated. Full, random matrices are generated by the *rand* function in MATLAB 2015b and scaled so that every component belongs to the interval $[-10, 10]$. Then, diagonal elements are set equal to the sum of the absolute values of the respective column plus, potentially, a random number in $]0, 1[$. In particular, we consider both matrices where strict diagonal dominance holds for all columns and full, column diagonally dominant matrices where strict diagonal dominance holds only at the first column. The HLCPs defined by these matrices admit a unique solution for any $\boldsymbol{c}$. Thus, also $\boldsymbol{c}$ is generated at random, with values between -10 and 10. We also consider problems where random matrices have a particular structure, like triangular strictly diagonally dominant matrices or matrices where all off-diagonal elements have the same sign. In all cases, initial iterates are chosen as random sequences of zeros and ones satisfying complementarity. Since all these problems are meant as a first validation of the procedure, we do not impose a stopping condition, but we simply evaluate the residual after a given number k^* of iterations by replacing $\boldsymbol{x}$ and $\boldsymbol{y}$ in $A\boldsymbol{x} - B\boldsymbol{y} = \boldsymbol{c}$ by the computed iterates $\boldsymbol{x}^{(k^*)}$ and $\boldsymbol{y}^{(k^*)}$ (which satisfy also the complementarity by the projection).

Then, we consider HLCPs arising in mechanical engineering to model cavitation (which is the formation of gaseous bubbles) in hydrodynamic lubrication. The formulation of the problem in complementarity form can be found in [15], while we follow [11] for the HLCP formulation arising from finite difference discretizations of the differential problem. The complementarity variables represent here pressure and density and are thus denoted by $\boldsymbol{p}$ and $\boldsymbol{r}$ (instead of $\boldsymbol{x}$ and $\boldsymbol{y}$). Moreover, they depend on the space variables, which we denote by ζ in 1D problems and by (ζ, η) in 2D problems.

We solve the four test problems presented in [11] and identify them by $P1$, $P2$, $P3$ and $P4$, respectively. These problems, whose solution is known, encompass various cases which can occur. In particular, $P1$, $P2$ and $P3$ regard 1D cases, while $P4$ refers to a 2D case. In all these problems, A and B are M-matrices. Moreover, in 1D, A is tridiagonal and B is lower bidiagonal. In 2D, A is block tridiagonal and B is block

Table 1 Mean and maximum residuals after 20 iterations for 100 HLCP(A, B, $\boldsymbol{c}$) with A, B generated at random as in Sect. 5

Problem	n	Mean	Max	Mean	Max
SDD	20	2.59e−9	5.54e−8	9.16e−15	1.40e−13
	100	1.52e−14	2.02e−14	1.49e−14	1.98e−14
	1000	1.43e−13	1.58e−13	1.43e−13	1.58e−13
DD	20	2.23e−9	2.53e−8	9.25e−15	1.84e−13
	100	1.53e−14	2.13e−14	1.52e−14	2.06e−14
	1000	1.44e−13	1.63e−13	1.44e−13	1.63e−13

Projected Jacobi: left. Projected Gauss–Seidel: right

diagonal. Numerical experiments are here performed in Fortran and compared with the results in [11], where the test problems were solved by an IP method. In order to perform a consistent comparison, we enforce a stopping criterion as similar as possible to that of the IP method in [11]. We then stop the procedures as the norms of the residual and of the difference between two successive iterates become smaller than a given tolerance tol. Due to the low computational complexity of the iterations, the stopping criterion is checked every 1000 iterations.

All experiments have been performed in Unix environment on a laptop equipped with a dual core 2.7 GHz Intel Core i5 processor (*Broadwell* series).

6 Results and Analysis

We first validate the procedures by applying the projected Jacobi and Gauss–Seidel methods to solving series of 100 HLCPs(A, B, $\boldsymbol{c}$) with random A, B and $\boldsymbol{c}$, generated as described in Sect. 5. For each problem, we compute the l^2-norm of the residual after 20 iterations of the methods. Finally, we evaluate whether the algorithms converged by analyzing the arithmetic mean and the maximum of all these residuals, thus ensuring that the procedures converged for every single problem. We repeat the process for problems of different dimensions n. In Table 1, by SDD we refer to problems with strictly column diagonally dominant matrices, while DD refers to A, B random, full and column diagonally dominant matrices with positive diagonal entries and with strict diagonal dominance holding only at the first column. Moreover, in all tables, we use the e-notation to denote the powers of 10.

In all cases, we compute the correct solution of the considered HLCPs. Indeed, the mean and the maximum of the l^2-norms of the residuals of the considered problems are always small. Moreover, both methods converge quickly for this kind of problems. Indeed, both the mean and the maximum of residuals decrease rapidly to values in the order of 10^{-13}–10^{-14}. More significant differences between the Jacobi and the Gauss–Seidel procedures can only be seen when n is small, but the residuals are nonetheless in the order of at most 10^{-8}–10^{-9} also in this case.

After this first validation by full, random matrices, we then pass to the analysis of structured matrices, which are not only more interesting but also potentially more

Table 2 Mean and maximum residuals after 20, 000 iterations for 100 HLCP(A, B, $\boldsymbol{c}$) with A, B generated at random as in Sect. 5 and with off-diagonal elements of uniform sign

n	Mean	Max	Mean	Max
100	4.68e−8	7.79e−7	7.11e−13	4.42e−12

Projected Jacobi: left. Projected Gauss–Seidel: right

Table 3 Analysis of row and column diagonal dominance in random, triangular matrices

SDD	Mean	Max	Mean	Max
by columns	1.29e−8	3.45e−7	3.18e−11	1.02e−9
by rows	1.19e+41	1.19e+43	1.73e+87	1.74e+89

Projected Jacobi: left. Projected Gauss–Seidel: right

challenging. Let us then consider random matrices that present a structure which, arguably, could hinder convergence.

In this regard, let us consider, for instance, a series of HLCPs(A, B, $\boldsymbol{c}$) where A and B are column diagonally dominant matrices (strictly at the first column) with positive diagonal elements, A has negative off-diagonal elements and B has positive off-diagonal elements. In this case, several evaluations performed in the convergence theorems apply with the equality sign. The results for these problems for $n = 100$ after 20,000 iterations are reported in Table 2.

Again, both algorithms converge in all cases. As expected, convergence is however slower. Notwithstanding this, the proposed procedures are not computationally onerous and, as we will see later, they are competitive also when structured matrices are used. Finally, we also notice a significant difference between the projected Jacobi and the Gauss–Seidel iterations: indeed, the convergence of the Gauss–Seidel method appears much faster for these problems.

Lastly, we provide also experimentally a few insights into the convergence of the proposed methods. On one hand, experiments conducted with completely random, full matrices generally converge also when A and B are row diagonally dominant. However, row-diagonal dominance does not ensure, in general, the convergence of the algorithm. To see this experimentally, let us consider A lower triangular matrix with positive diagonal elements and negative off-diagonal elements and B upper triangular matrix with positive elements. By the structure of A and B, column diagonal dominance implies that several rows are not diagonally dominated. Analogously, A and B row diagonally dominant have, in general, several columns which are not diagonally dominated. We can then more accurately study the effect of diagonal dominance on convergence. Let us then consider the cases of A and B strictly column or row diagonally dominant and apply the methods to solving 100 problems of this kind with $n = 100$. The results after 20 iterations are reported in Table 3.

As expected and consistently with the convergence analysis earlier performed, column diagonal dominance ensures the convergence of both projected Jacobi and Gauss–Seidel iterations. Instead, row-diagonal dominance is, in general, not sufficient for the methods to converge. This is consistent also with the role of the columns of

A and B in horizontal complementarity, highlighted by the role of the column $\mathcal{W}$-property for the uniqueness of solution of HLCPs.

Let us now pass to the analysis of horizontal complementarity problems arising in practical applications. In particular, let us consider the HLCPs arising in hydrodynamic lubrication and described in Sect. 5. Both the projected Jacobi and the Gauss–Seidel iterations have then been implemented so to exploit sparsity, in order to increase the efficiency of the procedures. Thus, computational times remain reasonable also when thousands of iterations are run in large problems.

Starting from 1D problems, the results for $P1$ and $P3$ with $tol = 10^{-8}$ are shown in Table 4, where we report the number of iterations it, the computational time t and the l^2-norm of the residual $\boldsymbol{res}$ and of the difference between two successive iterates at convergence, $\boldsymbol{\Delta p}$ and $\boldsymbol{\Delta r}$. The computed solutions for $n = 100$ are reported in Fig. 1. The results for $P2$ are analogous to those for $P1$ and hence not reported.

Both the Jacobi and the Gauss–Seidel iterations converge in all cases and the efficiency of the algorithm is apparent, especially when n is quite small. Nonetheless, this is enough to compute accurate solutions of 1D problems: indeed, the curves in Fig. 1, obtained with $n = 100$ in $t \simeq 10^{-2}$ s, are in perfect agreement with results in the literature (see [11]), further validating the proposed approaches. Increasing n, more iterations are triggered but computational times remain, nonetheless, reasonable. In this context, it is also interesting to notice that the Jacobi iteration converges in roughly twice as many iterations as Gauss–Seidel's.

Finally, we analyze 2D problems with tol $= 10^{-4}$. In this context, we analyze the behavior of the proposed procedure in larger scale problems. Table 5 reports the results obtained applying the projected Gauss–Seidel iteration to solving problem $P4$. Analogous results can be obtained using the projected Jacobi method, with a difference in efficiency comparable to 1D cases. For instance, with $n = 100$, projected Jacobi requires 10,000 iterations and $t = 2.86$ to compute a solution with the same accuracy of the one computed by projected Gauss–Seidel in Table 5. Figure 2 represents the complementarity solutions computed by the Gauss–Seidel method for $n = 500$. We remark that, dealing with 2D problems, A and B are block matrices of order n^2. We therefore consider cases where A and B are up to $250{,}000 \times 250{,}000$.

Again, both Jacobi and Gauss–Seidel methods converge for all considered n. Computational times are remarkable as well. For instance, in [11], $P4$ was solved by an IP method with a tolerance of 10^{-4} on the norm of the residual and of the increment. Using an efficient implementation (where inner linear systems were solved by efficient PETSc routines [16]), solving the problem with $n = 100$ required more than 600 s. Here, on the other hand, we achieved the same accuracy in less than 2 s using the same discretization.

Considering larger dimensions, which were not treated in previous works, the number of iterations and computational times increase, but the procedure always converge. Moreover, also computational times are not problematic, especially considering that all computations have been performed in series.

Table 4 Results for 1D HLCPs arising in hydrodynamic lubrication

n	it	t	$\Vert\mathbf{res}\Vert$	$\Vert\boldsymbol{\Delta p}\Vert$	$\Vert\boldsymbol{\Delta r}\Vert$	it	t	$\Vert\mathbf{res}\Vert$	$\Vert\boldsymbol{\Delta p}\Vert$	$\Vert\boldsymbol{\Delta r}\Vert$
Problem P1										
100	1.10e4	$3e-2$	$5.4e-13$	$3.7e-9$	$8.1e-11$	6.00e3	$1e-2$	$1.2e-13$	$1.6e-9$	$3.3e-11$
250	5.80e4	0.19	$3.5e-12$	$9.4e-9$	$2.1e-10$	3.10e4	0.15	$1.3e-12$	$7.1e-9$	$1.5e-10$
500	2.13e5	1.17	$7.1e-12$	$9.7e-9$	$2.1e-10$	1.12e5	0.92	$3.6e-12$	$9.8e-9$	$2.1e-10$
1000	7.79e5	8.31	$1.4e-11$	$9.9e-9$	$2.1e-10$	4.12e5	6.35	$7.2e-12$	$9.8e-9$	$2.1e-10$
Problem P3										
100	1.60e4	$3e-2$	$6.4e-13$	$4.6e-9$	$9.8e-10$	8.00e3	$2e-2$	$5.3e-13$	$7.5e-9$	$1.9e-11$
250	8.30e4	0.27	$2.9e-12$	$8.5e-9$	$3.1e-9$	4.30e4	0.19	$1.5e-12$	$8.8e-9$	$3.4e-9$
500	3.05e5	1.81	$6.6e-12$	$9.8e-9$	$5.5e-9$	1.60e5	1.30	$3.1e-12$	$9.1e-9$	$5.0e-9$
1000	1.12e6	12.3	$1.3e-11$	$9.9e-9$	$7.6e-9$	5.90e5	9.39	$6.6e-12$	$9.8e-9$	$7.6e-9$

Projected Jacobi: left. Projected Gauss–Seidel: right

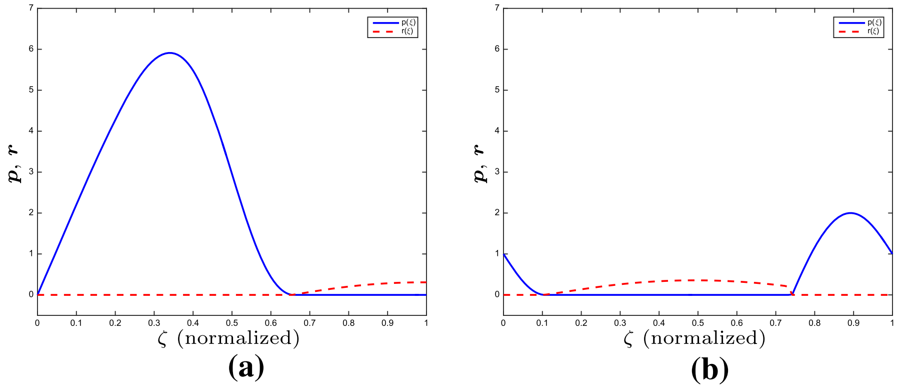

Fig. 1 Plots of solutions of $P1$ and $P3$ computed by projected Gauss–Seidel with $n = 100$. **a** Results for problem $P1$, **b** results for problem $P3$

Table 5 Results for $P4$ solved by the projected Gauss–Seidel method with various n

n	it	t	$\|\mathbf{res}\|$	$\|\boldsymbol{\Delta p}\|$	$\|\boldsymbol{\Delta r}\|$
100	6000	1.46	8.60e−9	2.71e−5	1.34e−5
200	19,000	40.00	2.35e−8	7.39e−5	5.00e−5
500	98,000	1668	2.98e−8	9.38e−5	9.94e−5

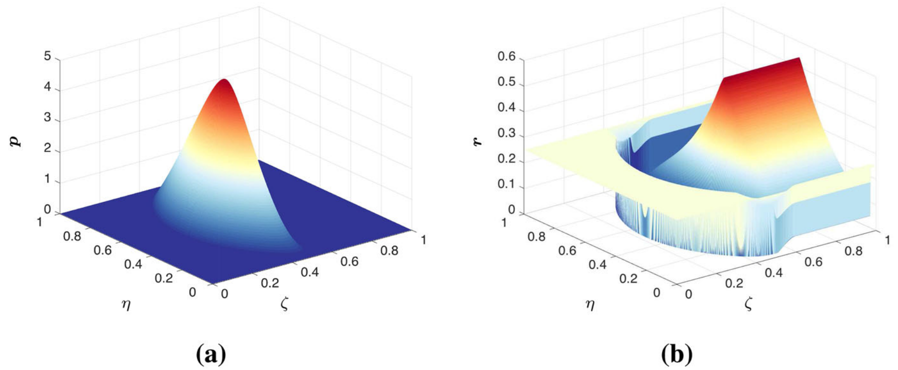

Fig. 2 Plots of the solution of $P4$ with $n = 500$. **a** plot of $\boldsymbol{p}$ for Problem 4. **b** plot of $\boldsymbol{r}$ for Problem 4

7 Conclusions

We have presented two splitting procedures for solving HLCPs characterized by matrices with positive diagonal elements. Contrarily to existing techniques, the proposed methods act directly on splittings of the matrices of the problem. Hence, the iteration is fast and simple to implement. We have proved the convergence of the proposed methods under some conditions over the diagonal dominance of A and of B. Then, we have performed several numerical experiments, demonstrating the capabilities of the procedures in both random test problems and in HLCPs of practical interest. For these latter problems, in several cases, computational times have been smaller than those in the literature. This highlights the efficiency of the procedure and is interesting also in view of future developments of this study to include other splitting techniques.

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