On the solution of the purely nonlocal theory of elasticity as a limiting case of the two-phase theory

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Abstract

In the recent literature stance, purely nonlocal theory of elasticity is recognized to lead to ill-posed problems. Yet, we show that a meaningful energy bounded solution of the purely nonlocal theory may still be defined as the limit solution of the two-phase nonlocal theory. For this, we consider the problem of free vibrations of a flexural beam under the two-phase theory of nonlocal elasticity with an exponential kernel, in the presence of rotational inertia. After recasting the integro-differential governing equation and the boundary conditions into purely differential form, a singularly perturbed problem is met that is associated with a pair of end boundary layers. A multi-parametric asymptotic solution in terms of size-effect and local fraction is presented for the eigenfrequencies as well as for the eigenforms for a variety of boundary conditions. It is found that simply supported end conditions convey the weakest boundary layer and that, surprisingly, rotational inertia affects the eigenfrequencies only in the classical sense. Conversely, clamped and free conditions bring a strong boundary layer and eigenfrequencies are heavily affected by rotational inertia, even for the lowest mode, in a manner opposite to that brought by nonlocality. Remarkably, all asymptotic solutions admit a well defined and energy bounded limit as the local fraction vanishes and the purely nonlocal model is retrieved.

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Therefore, we may define this limiting case as the proper solution of the purely nonlocal model. Finally, numerical results support the accuracy of the proposed asymptotic approach.

Keywords:

Two-phase nonlocal elasticity, Nonlocal theory of elasticity, Asymptotic method, Free vibrations

1. Introduction

The classical linear theory of elasticity suffers from the well known defect of not encompassing an internal length scale, which feature gives rise to self-similar predictions. Yet, any real material possesses an internal microstructure and some characteristic length thereof. Consequently, classical elasticity may be assumed as a suitable model inasmuch as the physical phenomena of interest occur at a scale much greater than the internal characteristic length of the material. Failure to meet this condition is effectively demonstrated by, for instance, the singular stress field at the tip of a crack and by the non-dispersive nature of wave propagation. Extensions of classical elasticity have been proposed, in the form of generalized continuum mechanics (GCM), in an attempt to remediate these shortfalls. An excellent historical overview of GCM, together with extensive bibliographic details, may be found in [17]. Among GCM theories, we mention the theory of micro-polar elasticity [2, 3, 25], the couple-stress and strain-gradient elasticity theories [35, 23] and the nonlocal theory of elasticity [7]. In particular, following [7], "linear theory of nonlocal elasticity, which has been proposed independently by various authors [...], incorporates important features of lattice dynamics and yet it contains classical elasticity in the long wave length limit". Nonlocal elasticity is based on the idea that the stress state at a point is a convolution over the whole body of an attenuation function (sometimes named kernel or nonlocal modulus) with the strain field [34]. Although several attenuation functions may be considered, they need to comply with some important properties which warrant that (a) classical elasticity is re-

24 verted to in the limit of zero length scale and that (b) normalization is satisfied
 25 [6]. As an example, Helmholtz and bi-Helmoltz kernels have been widely used
 26 in 1-D problems, their name stemming from the differential operators they are
 27 Green's function of [8, 15]. Since nonlocal elasticity naturally leads to integro-
 28 differential equations whose solution is most often impractical, an equivalent
 29 differential nonlocal model (EDNM) was developed in [6]. In such form, non-
 30 local elasticity has been extensively applied to study elastodynamics of beams
 31 and shells as described in the recent review [4] and with special emphasis on
 32 the application to nanostructures [29]. Generally, EDNM leads to interesting
 33 mechanical effects, such as increased deflections and decreased buckling loads
 34 and natural frequencies (softening effect), when compared to classical elasticity.
 35 However, a number of pathological results have also emerged, which are often
 36 referred to as paradoxes [16, 10, 15]. For instance, for a cantilever beam under
 37 point loading, nonlocality brings no effect [24, 32, 1]. It should be remarked
 38 that many studies based on the EDNM employ boundary conditions in terms
 39 of macroscopic stresses, i.e. in classical form, and therefore they disregard the
 40 important effect of the boundary through nonlocality. Although this approach
 41 may be still adopted for long structures or in the case of localized deformations
 42 occurring away from the boundaries [20, 21], it is generally inaccurate.

43 Very recently, Romano et al. [30] claimed that Eringen's purely nonlocal
 44 model (PNLM) leads to ill-posed problems for the differential form of the model
 45 is consistent inasmuch as an extra pair of boundary conditions, termed consti-
 46 tutive, is satisfied. In [5], a two-phase nonlocal model (TPNL) was introduced
 47 which combines, according to the theory of mixtures, purely nonlocal elasticity
 48 with classical elasticity, by means of the volume fractions ξ_1 and $\xi_2 = 1 - \xi_1$.
 49 This model is immune from the inconsistencies of the PNLM and it has been
 50 adopted to solve the problem of static bending [33] and buckling [36] of Euler-
 51 Bernoulli (E-B) beams. Static axial deformation of a beam is considered in
 52 [26, 37], while semi-analytical solutions for the combined action of axial and
 53 flexural static loadings is given in [18]. Axial and flexural free vibrations of
 54 beams have also been considered in [19] and in [9]. In these works, either the

TPNM is solved numerically or it is reduced, by adopting the solution presented in [28], to an equivalent higher-order purely differential model with a pair of extra boundary conditions. Despite this reduction, the differential model is still difficult to analyse, especially in the neighbourhood of the PNLM, that is for ξ_1 small. In this respect, we believe that the asymptotic approach may be put to great advantage in predicting the mechanical behaviour of nanoscale structures for a vanishingly small ξ_1 [36, 19].

In this paper, we consider free vibrations of a flexural beam taking into account rotational inertia (Rayleigh beam), within the TPNM and having assumed the Helmholtz attenuation function. The integro-differential model is reduced to purely differential form with an extra pair of boundary conditions. Spotlight is set on developing asymptotic solutions valid for small microstructure and/or little local fraction. These solutions feature a pair of boundary layers located at the beam ends, whose strength depends on the constraining conditions. Numerical results support the accuracy of the expansions. Most remarkably, the asymptotic approach allows to investigate the behaviour of the solution in the neighbourhood of the PNLM, where the expansions are non-uniform. Nonetheless, they admit a perfectly meaningful, energy bounded limit, which may be taken as the solution of the PNLM. We point out that the existence of such limit has been observed numerically in [10] for free-free end conditions.

2. Problem formulation

2.1. Governing equations

For a flexural beam, vertical equilibrium gives

$$\rho S \frac{\partial^2 v}{\partial t^2} = \frac{\partial \hat{Q}}{\partial x} + q(x) \quad (1)$$

while rotational equilibrium lends

$$J \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\partial \hat{M}}{\partial x} + \hat{Q}. \quad (2)$$

Here, $v = v(x, t)$ is the vertical displacement, $\hat{Q}$ and $\hat{M}$ are the *dimensional* shearing force and the bending moment, respectively, ρ is the mass density,

81 $J = \rho I$ is the mass second moment of inertia per unit length of the beam, that
 82 is proportional to the second moment of area I , S is the cross-sectional area
 83 and $q(x)$ the vertical applied load. As well-known, it is $I = Sr_A^2$, where r_A is
 84 the radius of gyration. Assuming that the beam is homogeneous and that its
 85 cross-section is constant along the length, Eqs.(1,2) give

$$\frac{\partial^2 \hat{M}}{\partial x^2} - \rho S \frac{\partial^2 v}{\partial t^2} + J \frac{\partial^4 v}{\partial x^2 \partial t^2} + q = 0, \quad (3)$$

86 that governs transverse vibrations of flexural beams accounting for rotational
 87 inertia. In the mixed nonlocal theory (MNLT) of elasticity, we have [5, 7]

$$\hat{M} = -EI \left(\xi_1 \frac{\partial^2 v}{\partial x^2} + \xi_2 \int_0^L K(|x - \hat{x}|, \kappa) \frac{\partial^2 w}{\partial \hat{x}^2} d\hat{x} \right), \quad (4)$$

88 where EI is the beam flexural rigidity, L the beam length and $K(|x - \hat{x}|, \kappa)$
 89 is the kernel or attenuation function. The kernel is positive, symmetric, and it
 90 rapidly decays away from x ; the nonlocal parameter $\kappa = e_0 a$ depends on the
 91 scale coefficient e_0 as well as on the internal length scale a . ξ_1 and ξ_2 take up
 92 the role of volume fractions and they represent, respectively, the local and the
 93 nonlocal phase ratios, such that $\xi_1 + \xi_2 = 1$ and $\xi_1 \xi_2 \geq 0$. When $\xi_1 = 0$, Eq.(4)
 94 degenerates into the purely nonlocal model (PNLM), while, in contrast, the case
 95 $\xi_1 = 1$ corresponds to classical local elasticity.

96 In what follows, we consider the Helmholtz kernel

$$K(|x - \hat{x}|, \kappa) = \frac{1}{2\kappa} \exp\left(-\frac{|x - \hat{x}|}{\kappa}\right), \quad (5)$$

97 which is frequently used for 1D problems [30]. We note that for the Helmholtz
 98 kernel the following transformations are valid

$$\frac{d}{ds} \int_0^1 e^{-\frac{|s-\hat{s}|}{\varepsilon}} y(\hat{s}) d\hat{s} = \frac{1}{\varepsilon} \left[e^{\frac{s}{\varepsilon}} \int_s^1 e^{-\frac{\hat{s}}{\varepsilon}} y(\hat{s}) d\hat{s} - e^{-\frac{s}{\varepsilon}} \int_0^s e^{\frac{\hat{s}}{\varepsilon}} y(\hat{s}) d\hat{s} \right], \quad (6)$$

99 and

$$\frac{d^2}{ds^2} \int_0^1 e^{-\frac{|s-\hat{s}|}{\varepsilon}} y(\hat{s}) d\hat{s} = \frac{1}{\varepsilon^2} \int_0^1 e^{-\frac{|s-\hat{s}|}{\varepsilon}} y(\hat{s}) d\hat{s} - \frac{2}{\varepsilon} y(s). \quad (7)$$

In particular, Eq.(7) corresponds to [30, Eq.(6)] and it may be rewritten as

$$\int_0^1 \left[\varepsilon^2 \frac{d^2 K(|s - \hat{s}|, \varepsilon)}{d\hat{s}^2} - K(|s - \hat{s}|, \varepsilon) + \delta(|s - \hat{s}|) \right] y(\hat{s}) d\hat{s} = 0,$$

whereupon $K(|s - \hat{s}|, \varepsilon)$ is the Green's function of the singularly perturbed operator $\mathbf{H}_\varepsilon = 1 - \varepsilon^2 \frac{d^2}{d\hat{s}^2}$. It is trivial matter to prove impulsivity, i.e. $\lim_{\varepsilon \rightarrow 0} K(|s - \hat{s}|, \varepsilon) = \delta(s - \hat{s})$, where $\delta(s)$ is Dirac's delta function. Furthermore, we observe that Eq.(6), evaluated at the beam ends $s = 0, 1$ and for $\xi = 0$, lends the constitutive boundary conditions [30, Eq.(5)]

$$\frac{dM}{ds}(0) = \varepsilon^{-1} M(0), \quad \text{and} \quad \frac{dM}{ds}(1) = -\varepsilon^{-1} M(1),$$

100 where $M = L\hat{M}/EI$ is the dimensionless bending moment. Thus, the constitu-
101 tive boundary conditions are really the expression, on the domain boundary, of
102 a general feature of the solution that is related to the integral operator (4).

103 Introducing the dimensionless axial co-ordinate $s = x/L$, under the assump-
104 tion of time-harmonic motion (i is the imaginary unit)

$$v(s, t) = w(s) \exp(i\omega t),$$

105 and upon multiplying throughout by L^4/EI , Eq.(3) may be turned in dimen-
106 sionless form

$$\xi_1 \frac{d^4 w}{ds^4} + (\lambda^4 \theta - \varepsilon^{-2} \xi_2) \frac{d^2 w}{ds^2} + \frac{\xi_2}{2\varepsilon^3} \int_0^1 \exp\left(-\frac{|\hat{s} - s|}{\varepsilon}\right) \frac{d^2 w(\hat{s})}{d\hat{s}^2} d\hat{s} - \lambda^4 w = 0. \quad (8)$$

107 Here, use have been made of Eqs.(4,5) and we have let the dimensionless ratios

$$\theta = \frac{J}{\rho S L^2} = \left(\frac{r_A}{L}\right)^2, \quad \lambda^4 = \frac{\rho S L^4 \omega^2}{EI}, \quad (9)$$

together with the microstructure parameter

$$\varepsilon = \frac{\kappa}{L} \ll 1.$$

108 Clearly, θ plays the role of an aspect ratio squared and ε is a *scale effect*. As-
109 suming $w \in C^6[0, 1]$, twice differentiating Eq.(8), taking into account Eqs.(6,7)
110 and then subtracting, we get the governing equation in purely differential form

$$\varepsilon^2 \xi \frac{d^6 w}{ds^6} - (1 - \varepsilon^2 \theta \lambda^4) \frac{d^4 w}{ds^4} - \lambda^4 (\varepsilon^2 + \theta) \frac{d^2 w}{ds^2} + \lambda^4 w = 0, \quad (10)$$

111 where, hereinafter, we adopt the shorthand $\xi = \xi_1$. Eq.(10) is a singularly
112 perturbed ODE [14], with respect to the small parameter $\varepsilon\sqrt{\xi}$.

113 *2.2. Boundary conditions*

Eq.(10) is supplemented by suitable boundary conditions (BCs) at the ends. For clamped ends (C-C conditions), we have two pairs of kinematical conditions

$$w(0) = w'(0) = 0, \quad (11a)$$

$$w(1) = w'(1) = 0. \quad (11b)$$

For simply supported (S-S) ends

$$w(0) = 0, \quad M(0) = \xi w''(0) + M_0 = 0, \quad (12a)$$

$$w(1) = 0, \quad M(1) = \xi w''(1) + M_1 = 0, \quad (12b)$$

114 having let

$$M_0 = \frac{1-\xi}{2\varepsilon} \int_0^1 e^{-\frac{\hat{s}}{\varepsilon}} w''(\hat{s}) d\hat{s}, \quad M_1 = \frac{1-\xi}{2\varepsilon} e^{-\frac{1}{\varepsilon}} \int_0^1 e^{\frac{\hat{s}}{\varepsilon}} w''(\hat{s}) d\hat{s}. \quad (13)$$

For free-free (F-F) ends, one has

$$M(0) = 0, \quad Q(0) = \xi w'''(0) + \theta \lambda^4 w'(0) + \varepsilon^{-1} M_0 = 0, \quad (14a)$$

$$M(1) = 0, \quad Q(1) = \xi w'''(1) + \theta \lambda^4 w'(1) - \varepsilon^{-1} M_1 = 0. \quad (14b)$$

The nonlocal end bending moments (13) may be rewritten in differential form with the help of the original integro-differential equation (8):

$$M_0 = -\varepsilon^2 \xi w^{iv}(0) + [1 - \xi - \varepsilon^2 \theta \lambda^4] w''(0) + \varepsilon^2 \lambda^4 w(0), \quad (15a)$$

$$M_1 = -\varepsilon^2 \xi w^{iv}(1) + [1 - \xi - \varepsilon^2 \theta \lambda^4] w''(1) + \varepsilon^2 \lambda^4 w(1). \quad (15b)$$

Consequently, the BCs may be recast in differential form as

$$M(0) = w''(0) + \varepsilon^2 N_0 = 0, \quad (16a)$$

$$M(1) = w''(1) + \varepsilon^2 N_1 = 0, \quad (16b)$$

$$Q(0) = \xi w'''(0) + \theta \lambda^4 w'(0) + \varepsilon^{-1} M_0 = 0, \quad (16c)$$

$$Q(1) = \xi w'''(1) + \theta \lambda^4 w'(1) - \varepsilon^{-1} M_1 = 0, \quad (16d)$$

where, making use of the connections (6,7), we have

$$N_0 = \varepsilon^{-2}(\xi_2 w''(0) - M_0) = -\xi w^{iv}(0) - \theta \lambda^4 w''(0) + \lambda^4 w(0), \quad (17a)$$

$$N_1 = \varepsilon^{-2}(\xi_2 w''(1) - M_1) = -\xi w^{iv}(1) - \theta \lambda^4 w''(1) + \lambda^4 w(1). \quad (17b)$$

Besides, to rule out spurious solutions which may have appeared owing to double differentiation, we introduce a pair of additional BCs. Indeed, evaluating at the beam ends the differential with respect to s of the original governing equation (8), one arrives at

$$\begin{aligned} \varepsilon^3 \xi w^v(0) - \varepsilon^2 \xi w^{iv}(0) - (1 - \xi - \varepsilon^2 \theta \lambda^4) [\varepsilon w'''(0) - w''(0)] \\ - \varepsilon^3 \lambda^4 w'(0) + \varepsilon^2 \lambda^4 w(0) = 0, \end{aligned} \quad (18a)$$

$$\begin{aligned} \varepsilon^3 \xi w^v(1) + \varepsilon^2 \xi w^{iv}(1) - (1 - \xi - \varepsilon^2 \theta \lambda^4) [\varepsilon w'''(1) + w''(1)] \\ - \varepsilon^3 \lambda^4 w'(1) - \varepsilon^2 \lambda^4 w(1) = 0. \end{aligned} \quad (18b)$$

115 Dropping rotational inertia, the additional boundary conditions (18) coincide
 116 with the *constitutive boundary conditions* recently obtained by Fernández-Sáez
 117 and Zaera [9, Eqs.(59) and (60)], provided that we replace our ε and λ^4 with
 118 their h and λ_w , respectively. However, it should be remarked that in [9] the
 119 original integro-differential problem is reduced to the equivalent differential form
 120 extending to dynamics the original argument developed in [34] for statics. Such
 121 argument takes advantage of a result presented in [27], which really applies to
 122 inhomogeneous integral equations with a given right-hand side. In the case of
 123 dynamics, however, this right-hand side is a problem unknown, for it is really
 124 an acceleration term, and therefore the applicability of the reduction formula is
 125 questionable.

126 3. Exact solution of the boundary-value problems

The general solution of the ODE (10) is

$$w(s) = \sum_{j=0}^6 c_j \exp(b_j s),$$

127 where the constants b_j are the roots of the characteristic polynomial in ζ

$$\varepsilon^2 \xi \zeta^6 - (1 - \varepsilon^2 \theta \lambda^4) \zeta^4 - (\varepsilon^2 + \theta) \lambda^4 \zeta^2 + \lambda^4 = 0. \quad (19)$$

128 As detailed in [31, 22], this bi-cubic may be turned in canonical form by the
 129 substitution $Z = \zeta^2 - Z_0$, it being $Z_0 = (1 - \varepsilon^2 \theta \lambda^4)/(3\varepsilon^2 \xi)$, whence Eq.(19)
 130 becomes

$$Z^3 - pZ - q = 0,$$

where

$$p = (\xi \varepsilon^2)^{-1} \left[\frac{(\lambda^4 \theta \varepsilon^2 - 1)^2}{3\xi \varepsilon^2} + \lambda^4 (\theta + \varepsilon^2) \right] > 0,$$

$$q = -(\xi \varepsilon^2)^{-1} \left[\lambda^4 + \frac{\lambda^4 (\theta + \varepsilon^2) (\lambda^4 \theta \varepsilon^2 - 1)}{3\xi \varepsilon^2} + \frac{2(\lambda^4 \theta \varepsilon^2 - 1)^3}{27\xi^2 \varepsilon^4} \right].$$

This polynomial possesses three real roots provided that

$$\Delta = \frac{q^2}{4} - \frac{p^3}{27} < 0$$

131 and indeed, for $\varepsilon \sqrt{\xi} \ll 1$, we get, to leading order,

$$\Delta = -\lambda^4 \frac{4 + \theta^2 \lambda^4}{108(\xi \varepsilon^2)^4}.$$

132 Besides, we have, at leading order,

$$q = \frac{2}{27(\xi \varepsilon^2)^3}$$

and $q > 0$, whereupon out of the three real roots, two, say $Z_3 < Z_2 < 0$, are negative and one, say Z_1 , is positive. Upon reverting to the original variable ζ , we see that $\zeta_3^2 < 0 < \zeta_2^2 < \zeta_1^2$. Indeed, we get the expansions (the sign is immaterial)

$$\zeta_1 = \frac{1}{\varepsilon \sqrt{\xi}}, \quad \zeta_2 = \alpha, \quad \zeta_3 = \imath \beta,$$

with

$$\alpha = \lambda_0 \sqrt{-\frac{1}{2} \theta \lambda_0^2 + \sqrt{1 + \frac{\theta^2 \lambda_0^4}{4}}}, \quad (20a)$$

$$\beta = \lambda_0 \sqrt{\frac{1}{2} \theta \lambda_0^2 + \sqrt{1 + \frac{\theta^2 \lambda_0^4}{4}}}, \quad (20b)$$

133 whence $\zeta_{1,2}$ convey an exponential solution, while ζ_3 is related to an oscil-
 134 latory solution. It is worth noticing that ζ_1 blows up as $(\varepsilon\sqrt{\xi}) \rightarrow 0$, that is
 135 for a vanishingly small scale effect or in the purely nonlocal situation. Indeed,
 136 this very root accounts for the edge effect in this problem and it describes a
 137 boundary layer.

138 We observe that, in general, the frequency equation for the ODE (10), sub-
 139 ject to suitable boundary conditions, appears in transcendental form

$$F(\lambda; \xi, \varepsilon) = 0,$$

140 wherein λ is the sought-for eigenvalue. The numerical solution of this equation
 141 is not straightforward matter, especially for very small values of the local frac-
 142 tion ξ , see e.g. [9] and [34] where plots are given for $\xi > 0.1$ and $\xi > 0.05$,
 143 respectively. Indeed, when looking for the numerical roots of (19), we observe,
 144 after [31], that the transformation to canonical form lends a considerable numer-
 145 ical advantage over Cardano's formulas in that it provides purely real solutions.
 146 Conversely, Cardano's formulas are likely to introduce a very small spurious
 147 imaginary component, which is most likely the cause of the numerical difficulty
 148 encountered in the literature when dealing with small ξ . To estimate the eigen-
 149 value λ for any ξ and, in particular, in the limiting case of the PNLM (that
 150 occurs as $\xi \rightarrow 0$), we consider an asymptotic expansion in the small parameter
 151 ε .

152 4. Asymptotic solution of the boundary-value problems

153 Following a standard asymptotic argument [14, 19] and similarly to the ex-
 154 traction of the edge effect in shells [11, 12], we seek a solution of the eigenvalue
 155 problem through superposition of a solution, $w^{(m)}$, valid in the interior of the
 156 beam (the so-called outer solution), with a pair of boundary layers, $w_1^{(e)}$ and
 157 $w_2^{(e)}$, fading off away from the left and from the right beam end, respectively,

$$w(s, \varepsilon) = w^{(m)}(s) + \varepsilon^{\gamma_1} w_1^{(e)}(s, \varepsilon) + \varepsilon^{\gamma_2} w_2^{(e)}(s, \varepsilon), \quad (21)$$

158 where

$$\frac{\partial w^{(m)}}{\partial s} \sim w^{(m)}, \quad \frac{\partial w_i^{(e)}}{\partial s} \sim \varepsilon^{-\varsigma} w_i^{(e)} \quad \text{as } \varepsilon \rightarrow 0.$$

159 The parameter ς is named *the index of variation of the edge effect integrals*,
 160 while γ_1 and γ_2 are the *indices of intensity of the edge effect integrals* near
 161 the left and right ends, respectively. The positive values of γ_i depend on the
 162 boundary conditions and should be specified for each end.

163 4.1. Boundary layer

164 To derive an equation describing the beam behaviour in the vicinity of the
 165 ends (boundary layer), we zoom in by assuming $s = \varepsilon^\varsigma \sigma$ and $1 - s = \varepsilon^\varsigma \sigma$,
 166 respectively for the left and for the right end. For either case, one obtains the
 167 distinguished limit $\varsigma = 1$ and Eq. (10) is rewritten as

$$\xi \frac{d^6 w_i^{(e)}}{d\sigma^6} - (1 - \varepsilon^2 \theta \lambda^4) \frac{d^4 w_i^{(e)}}{d\sigma^4} - \varepsilon^2 \lambda^4 (\theta + \varepsilon^2) \frac{d^2 w_i^{(e)}}{d\sigma^2} + \varepsilon^4 \lambda^4 w_i^{(e)} = 0, \quad (22)$$

168 whose solution is sought in the form of an asymptotic series

$$w_i^{(e)} = w_{i0}^{(e)} + \varepsilon w_{i1}^{(e)} + \varepsilon^2 w_{i2}^{(e)} + \dots \quad (23)$$

169 Substitution of (23) into (22) lends a sequence of differential equations in the
 170 unknowns $w_{ij}^{(e)}(\sigma)$, $i = 1, 2$; $j = 0, 1, 2, \dots$. Here, we simply give the first two
 171 terms of the expansion in the original variable s

$$\begin{aligned} w_1^{(e)}(s, \varepsilon) &= a_{10} e^{-\frac{s}{\varepsilon\sqrt{\xi}}} + \varepsilon e^{-\frac{s}{\varepsilon\sqrt{\xi}}} \left[a_{11} + a_{10} \frac{\theta \lambda_0^4 (1-\xi)}{2\sqrt{\xi}} s \right] + O\left(\varepsilon^2 e^{-\frac{s}{\varepsilon\sqrt{\xi}}}\right), \\ w_2^{(e)}(s, \varepsilon) &= a_{20} e^{-\frac{1-s}{\varepsilon\sqrt{\xi}}} + \varepsilon e^{-\frac{1-s}{\varepsilon\sqrt{\xi}}} \left[a_{21} + a_{20} \frac{\theta \lambda_0^4 (1-\xi)}{2\sqrt{\xi}} (1-s) \right] + O\left(\varepsilon^2 e^{-\frac{1-s}{\varepsilon\sqrt{\xi}}}\right), \end{aligned} \quad (24)$$

172 where a_{ij} ($i = 1, 2$; $j = 0, 1, 2, \dots$) are constants that will be determined in the
 173 following from the boundary conditions.

174 4.2. The outer solution

175 The displacement $w^{(m)}$ as well as the eigenvalue λ are sought in the form of
 176 an asymptotic series

$$\begin{aligned} w^{(m)} &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots, \\ \lambda &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots \end{aligned} \quad (25)$$

177 The leading term in the series corresponds to the solution of the classical local
 178 problem and λ_0 is the classical eigenvalue. Substituting (25) into the governing
 179 Eq.(10) and equating coefficients of like powers of ε leads to the sequence of
 180 differential equations:

$$\sum_{j=0}^k \mathbf{L}_j w_{k-j} = 0, \quad k = 0, 1, 2, \dots, \quad (26)$$

where

$$\begin{aligned} \mathbf{L}_0 z &= \frac{d^4 z}{ds^4} + \theta \lambda_0^4 \frac{d^2 z}{ds^2} - \lambda_0^4 z, \quad \mathbf{L}_1 z = -4\lambda_0^3 \lambda_1 \mathbf{D}z, \quad \mathbf{D}z = z - \theta \frac{d^2 z}{ds^2}, \\ \mathbf{L}_2 z &= -\xi \frac{d^6 z}{ds^6} - \theta \lambda_0^4 \frac{d^4 z}{ds^4} + \lambda_0^4 \frac{d^2 z}{ds^2} - 2\lambda_0^2 (3\lambda_1^2 + 2\lambda_0 \lambda_2) \mathbf{D}z, \\ \mathbf{L}_3 z &= -4\theta \lambda_0^3 \lambda_1 \frac{d^4 z}{ds^4} + 4\lambda_0^3 \lambda_1 \frac{d^2 z}{ds^2} - 4\lambda_0 (\lambda_0^2 \lambda_3 + \lambda_1^3 + 2\lambda_0 \lambda_1 \lambda_2) \mathbf{D}z, \dots \end{aligned}$$

181 At leading order, one finds the homogeneous forth order ODE

$$\mathbf{L}_0 w_0 = 0, \quad (27)$$

182 whose general solution

$$w_0(s) = c_{01} \sin(\beta s) + c_{02} \cos(\beta s) + c_{03} e^{-\alpha s} + c_{04} e^{\alpha(s-1)}, \quad (28)$$

183 depends on the constants, c_{0i} , $i \in \{1, 2, 3, 4\}$, to be determined through the
 184 boundary conditions. However, the ODE (27) is subject to six boundary con-
 185 ditions and the problem is to determine which of these correspond to the outer
 186 solution and which pertain to the boundary layer [14]. The procedure of split-
 187 ting the boundary conditions also gives the indices of intensity of the boundary
 188 layer, γ_1, γ_2 , as well as the constants c_{0k}, a_{ij} . For this, one needs to insert the
 189 expansions (21,24,25) into the boundary conditions and equate coefficients of
 190 like powers of ε , while imposing the following requirements:

- 191 • in the leading approximation, every end condition should be homogeneous
 192 and coincide with those of the classical local theory;
- 193 • the k^{th} -order approximation generates two equations coupling the con-
 194 stants $a_{i(k-1)}$ with the previous order approximation $w_{k-1}(s)$ evaluated
 195 at the boundaries.

196 *4.3. Beam with simply supported ends*

197 Let both beam ends be simply supported (S-S conditions), as given by the
 198 boundary conditions (12) rewritten in differential form (16a,16b), together with
 199 the additional constraints (18). Substituting the expansions (21,24,25) into
 200 these conditions, we determine the strength of either boundary layer $\gamma_1 = \gamma_2 =$
 201 3.

At leading order, we arrive at the homogeneous classical boundary conditions

$$w_0(0) = w_0(1) = w_0''(0) = w_0''(1) = 0,$$

202 which give $c_{01} = C$, $c_{02} = c_{03} = c_{04} = 0$ and the classical eigenforms

$$w_0(s) = C \sin(\beta s), \quad \beta = \pi n, \quad n = 1, 2, \dots \quad (29)$$

203 In light of the definition (20b), we find the eigenfrequencies

$$\lambda_0 = \lambda_0^{(n)} \equiv \frac{\pi n}{[1 + \theta(\pi n)^2]^{1/4}}, \quad n = 1, 2, \dots, \quad (30)$$

204 and, by (9), the corresponding dimensional frequencies $\omega_0 = \sqrt{\frac{EI}{\rho S}} (\lambda_0/L)^2$.

205 Moving to first-order terms, we again obtain a set of homogeneous boundary
 206 conditions

$$w_1(0) = w_1(1) = w_1''(0) = w_1''(1) = 0, \quad (31)$$

as well as formulas for the leading amplitude in the boundary layer (24):

$$a_{10} = -\sqrt{\xi}(1 - \sqrt{\xi})w_0'''(0) = C\beta^3\sqrt{\xi}(1 - \sqrt{\xi}), \quad (32a)$$

$$a_{20} = \sqrt{\xi}(1 - \sqrt{\xi})w_0'''(1) = C(-1)^{n+1}\beta^3\sqrt{\xi}(1 - \sqrt{\xi}). \quad (32b)$$

207 Consideration of the inhomogeneous ODE (26) arising in this approximation,
 208 alongside the associated homogeneous boundary conditions (31), yields the com-
 209 patibility condition $\lambda_1 = 0$, whence

$$w_1 = C_1 \sin(\beta s),$$

210 where C_1 is an arbitrary constant. Without loss of generality, one can assume
 211 $w_1 \equiv 0$, for this amounts to taking $C = C_0 + \varepsilon C_1 + \dots$

212 In the second-order approximation, when taking into account the outcomes
 213 of the previous step, we have again a homogeneous set of boundary conditions

$$w_2(0) = w_2(1) = w_2''(0) = w_2''(1) = 0, \quad (33)$$

214 and $a_{11} = a_{21} = 0$. The associated differential equation for w_2 reads

$$\mathbf{L}_0 w_2 = -\mathbf{L}_2 w_0 \equiv \xi \frac{d^6 w_0}{ds^6} + \theta \lambda_0^4 \frac{d^4 w_0}{ds^4} - \lambda_0^3 (\lambda_0 + 4\theta \lambda_2) \frac{d^2 w_0}{ds^2} + 4\lambda_0^3 \lambda_2 w_0. \quad (34)$$

We thus arrive at the inhomogeneous BVP on "spectrum". Upon observing that the homogeneous boundary-value problem arising at leading order is self-conjugated and therefore possesses the solution $z(s) = w_0(s)$, we deduce the compatibility condition for the BVP (33,34)

$$\int_0^1 w_0(s) \mathbf{L}_2 w_0(s) ds = 0,$$

which readily gives the correction for the eigenvalue:

$$\lambda_2 = -\frac{\beta^2 [\lambda_0^4 (1 + \theta \beta^2) - \xi \beta^4]}{4\lambda_0^3 (1 + \theta \beta^2)}.$$

215 On taking into account this result, Eq. (34) turns homogeneous and, without
 216 loss of generality, we can assume $w_2 \equiv 0$.

217 Considering the third-order approximation, one obtains the inhomogeneous
 218 boundary conditions

$$\begin{aligned} w_3(0) &= -a_{10} = -C\beta^3 \sqrt{\xi}(1 - \sqrt{\xi}), \\ w_3(1) &= -a_{20} = C(-1)^n \beta^3 \sqrt{\xi}(1 - \sqrt{\xi}), \\ w_3''(0) &= \theta \lambda_0^4 a_{10} = C\theta \lambda_0^4 \beta^3 \sqrt{\xi}(1 - \sqrt{\xi}), \\ w_3''(1) &= \theta \lambda_0^4 a_{20} = (-1)^{n+1} C\theta \lambda_0^4 \beta^3 \sqrt{\xi}(1 - \sqrt{\xi}) \end{aligned} \quad (35)$$

219 for the inhomogeneous ODE

$$\mathbf{L}_0 w_3 = -\mathbf{L}_3 w_0 \equiv 4\lambda_0^3 \lambda_3 \mathbf{D} w_0. \quad (36)$$

The compatibility condition for the boundary-value problem (35,36) works out

$$\begin{aligned} & -w_3''(1)w_0'(1) + w_3''(0)w_0'(0) - w_3(1)w_0'''(1) + w_3(0)w_0'''(0) \\ & + \theta\lambda_0^4[w_3(0)w_0'(0) - w_3(1)w_0'(1)] + 4\lambda_0^3\lambda_3 \int_0^1 (w_0 - \theta w_0'')w_0 ds = 0, \end{aligned}$$

220 whence we get the next correction term for the eigenvalue

$$\lambda_3 = \frac{\beta^6 \sqrt{\xi}(1 - \sqrt{\xi})}{\lambda_0^3(1 + \theta\beta^2)}. \quad (37)$$

The eigenform correction w_3 , satisfying the boundary conditions (35), is given by the sum of a particular solution w_{3p} of Eq.(36), with the homogenous solution w_{3o} . The former reads

$$w_{3p}(s) = C_{3p} s \cos(\beta s),$$

where

$$C_{3p} = 2C\lambda_0^3\lambda_3 \frac{1 + \theta\beta^2}{\beta(\alpha^2 + \beta^2)} = 2C \frac{\beta^5}{\alpha^2 + \beta^2} \sqrt{\xi}(1 - \sqrt{\xi}).$$

Consequently, making use of (37), we get

$$\begin{aligned} w_3(s) = & C\beta^3 \sqrt{\xi}(1 - \sqrt{\xi}) \{c_{32} \cos(\beta s) + c_{33} \exp(-\alpha s) \\ & + c_{34} \exp[\alpha(s - 1)] - 2c_{32}s \cos(\beta s)\}, \end{aligned}$$

with the constants

$$\begin{aligned} c_{32} &= -\beta^2/(\alpha^2 + \beta^2), \\ c_{33} &= \frac{1}{2}\alpha^2 e^\alpha (1 - \coth \alpha) [e^\alpha + (-1)^n]/(\alpha^2 + \beta^2), \\ c_{34} &= -\frac{1}{2}\alpha^2 e^\alpha (1 - \coth \alpha) [(-1)^n e^\alpha + 1]/(\alpha^2 + \beta^2). \end{aligned}$$

Breaking at this step the asymptotic procedure for seeking the eigenvalues λ_k and the associated eigenfunctions w_k , we obtain the asymptotic expansion

$$\lambda = \lambda_0 \left[1 - \frac{1}{4}\varepsilon^2 \beta^2 (1 - \xi) + \varepsilon^3 \beta^2 \sqrt{\xi}(1 - \sqrt{\xi}) + O(\varepsilon^4) \right],$$

Figure 1: 1st (left) and 2nd (right) eigenfrequencies ω for a S-S beam (solid, black), with $\varepsilon = 0.01, 0.05$ and 0.075 , superposed onto the 1-term (dashed, red) and the 2-term (dotted, blue) asymptotic approximation, normalized with respect to the classical local frequency ω_0 , Eq.(39)

where β and λ_0 are determined by (29) and (30), respectively. Up to an undetermined factor, the associated eigenmode reads

$$\begin{aligned}
 w(s) = & \sin(\pi ns) + \varepsilon^3 (\pi n)^3 \sqrt{\xi} (1 - \sqrt{\xi}) \left\{ c_{32} \cos(\pi ns) + c_{33} \exp(-\alpha s) \right. \\
 & + c_{34} \exp[\alpha(s-1)] - 2c_{32}s \cos(\pi ns) \\
 & \left. + \exp\left(-\frac{s}{\varepsilon\sqrt{\xi}}\right) + (-1)^{n+1} \exp\left(\frac{s-1}{\varepsilon\sqrt{\xi}}\right) \right\} + O(\varepsilon^4).
 \end{aligned} \tag{38}$$

It is of interest to compare the dimensional natural frequency, ω , determined with the TPNM, with its classical counterpart, ω_0 , evaluated within the framework of local elasticity, i.e. for $\xi = 1$. When taking into account Eq.(9), we arrive at the relation

$$\frac{\omega}{\omega_0} = (\lambda/\lambda_0)^2 = 1 - \frac{1}{2}\varepsilon^2(\pi n)^2(1-\xi) + 2\varepsilon^3(\pi n)^2\sqrt{\xi}(1-\sqrt{\xi}) + O(\varepsilon^4). \tag{39}$$

Remarkably, this expression is independent of θ and this unexpected feature is indeed confirmed by the numerical solution of the TPNM, see Fig.5. Fig.1 plots the approximation (39) in the range $0 < \xi < 1$ againsts the numerical solution of the TPNM (given for $\xi > 0.01$) for the scale parameter $\varepsilon = 0.01, 0.05$ and 0.075 . It appears that the 1-term asymptotic approximation is remarkably effective for small values of ε . The numerical solution of the TPNM given in Fig.1 compares favourably with the corresponding solution depicted in Fig.4 of [10] that, however, pertains to the range $\xi_1 > 0.1$, presumably owing to the

235 numerical difficulties that may arise in the neighbourhood of the PNLM.

236 As a special case of Eq.(39), one obtains the eigenfrequency ratio correspond-
 237 ing to the PNLM (i.e. for $\xi = 0$)

$$\frac{\omega}{\omega_0} = 1 - \frac{1}{2}\varepsilon^2(\pi n)^2 + O(\varepsilon^4). \quad (40)$$

238 4.4. Beam with clamped ends

Consideration of a beam with clamped ends requires enforcing (11) and (18) on Eqs.(21,24,25). We thus get the strength of the boundary layer $\gamma_1 = \gamma_2 = 2$. In the leading approximation, one has the classical boundary conditions

$$w_0(0) = w_0(1) = w'_0(0) = w'_0(1) = 0,$$

239 that give the constants

$$\begin{aligned} c_{01} &= 2\alpha(\cosh \alpha - \cos \beta) \\ c_{02} &= 2\alpha \sin \beta - 2\beta \sinh \alpha, \\ c_{03} &= \beta(e^\alpha - \cos \beta) - \alpha \sin \beta, \\ c_{04} &= -e^\alpha \alpha \sin \beta + \beta(e^\alpha \cos \beta - 1), \end{aligned} \quad (41)$$

240 as well as the frequency equation

$$\frac{1}{2}\theta\lambda_0^2 \sin \beta \sinh \alpha + \cos \beta \cosh \alpha - 1 = 0. \quad (42)$$

In particular, if $\theta = 0$, one arrives at the classical frequency equation, $\cosh \lambda_0 \cos \lambda_0 = 1$, valid for a Bernoulli-Euler beam that disregards the rotational inertia of the cross-section, the corresponding eigenmode being

$$w_0(s) = C \left[U(\lambda_0 s) - \frac{U(\lambda_0)}{V(\lambda_0)} V(\lambda_0 s) \right],$$

where $S(x), T(x), U(x), V(x)$ are the well-known Krylov-Duncan functions [13, §14.4.3]

$$\begin{aligned} S(x) &= \frac{1}{2}(\cosh x + \cos x), & T(x) &= \frac{1}{2}(\sinh x + \sin x), \\ U(x) &= \frac{1}{2}(\cosh x - \cos x), & V(x) &= \frac{1}{2}(\sinh x - \sin x). \end{aligned}$$

Besides, we get

$$a_{10} = \sqrt{\xi} \left(1 - \sqrt{\xi}\right) w_0''(0), \quad (43a)$$

$$a_{20} = \sqrt{\xi} \left(1 - \sqrt{\xi}\right) w_0''(1). \quad (43b)$$

241 In the first-order approximation, one has the inhomogeneous ODE (26)

$$\mathbf{L}_0 w_1 = 4\lambda_0^3 \lambda_1 \mathbf{D} w_0, \quad (44)$$

242 and the procedure of splitting the boundary conditions gives

$$\begin{aligned} w_1(0) = w_1(1) = 0, \\ w_1'(0) = (1 - \sqrt{\xi}) w_0''(0), \quad w_1'(1) = -(1 - \sqrt{\xi}) w_0''(1). \end{aligned} \quad (45)$$

The compatibility conditions for the BVP (44,45) reads

$$\begin{aligned} w_1'(1)w_0''(1) - w_1'(0)w_0''(0) - w_1(1)w_0'''(1) + w_1(0)w_0'''(0) \\ - 4\lambda_0^3 \lambda_1 \int_0^1 \mathbf{D} w_0(s) w_0(s) ds = 0, \end{aligned}$$

243 whence, accounting for Eqs.(45), one obtains the correction

$$\lambda_1 = -\lambda_0 \frac{(1 - \sqrt{\xi}) \{ [w_0''(0)]^2 + [w_0''(1)]^2 \}}{4 \int_0^1 [w_0''(s)]^2 ds}, \quad (46)$$

244 where part-integration has been used at the denominator. Now, we can write
245 the problem solution

$$w_1(s) = c_{11} \sin(\beta s) + c_{12} \cos(\beta s) + c_{13} e^{-\alpha s} + c_{14} e^{\alpha(s-1)} + w_{1p}(s), \quad (47)$$

246 where

$$\begin{aligned} w_{1p}(s) = 2 \frac{\lambda_0^3 \lambda_1}{\alpha^2 + \beta^2} s \left\{ \frac{1 + \theta \beta^2}{\beta} [-c_{01} \cos(\beta s) + c_{02} \sin(\beta s)] \right. \\ \left. + \frac{1 - \theta \alpha^2}{\alpha} [c_{03} e^{-\alpha s} - c_{04} e^{\alpha(s-1)}] \right\} \end{aligned} \quad (48)$$

is the particular solution of Eq.(44) with the coefficients c_{0j} being given by Eqs.(41). In the special case of no rotational inertia, $\theta = 0$, Eq.(46) may be reduced to the very simple expression

$$\lambda_1 = -2\lambda_0(1 - \sqrt{\xi}),$$

Figure 2: 1st (left) and 2nd (right) eigenfrequencies ω for a C-C beam (solid, black) in the absence of rotatory inertia, $\theta = 0$, and with $\varepsilon = 0.01, 0.05$ and 0.075 , superposed onto the 1-term (dotted, blue) asymptotic approximation, normalized with respect to the classical local frequency ω_0 , Eq.(50)

and Eq.(48) gives

$$w_{1p}(s) = \frac{\lambda_1}{\lambda_0} s w'_0(s) = -2C(1 - \sqrt{\xi})\lambda_0 s \left[T(\lambda_0 s) - \frac{U(\lambda_0)}{V(\lambda_0)} U(\lambda_0 s) \right].$$

Similarly, Eq.(47) becomes

$$w_1(s) = C(1 - \sqrt{\xi})\lambda_0 \left[T(\lambda_0 s) - \frac{T(\lambda_0)}{V(\lambda_0)} V(\lambda_0 s) \right] + w_{1p}(s).$$

247 Breaking the asymptotic procedure at this step, we can write down the
 248 approximate formula for the nonlocal-to-local frequency ratio

$$\frac{\omega}{\omega_0} = 1 - \frac{1}{2}\varepsilon \left(1 - \sqrt{\xi} \right) \frac{[w''_0(0)]^2 + [w''_0(1)]^2}{\int_0^1 [w''_0(s)]^2 ds} + O(\varepsilon^2), \quad (49)$$

249 that, in the absence of rotary inertia, reduces to

$$\frac{\omega}{\omega_0} = 1 - 4\varepsilon(1 - \sqrt{\xi}) + O(\varepsilon^2). \quad (50)$$

250 Fig.2 plots the approximated ratio (50) onto the numerical solution of the TPNM
 251 and shows that the 1-term correction provides excellent agreement for the fun-
 252 damental mode. It is also clear from Eq.(50) that, as in the S-S situation, a
 253 perfectly reasonable limit is retrieved for the PNLN, i.e. for $\xi \rightarrow 0$.

254 The asymptotic expansion for the eigenmode reads

$$w = w_0 + \varepsilon w_1 + O(\varepsilon^2), \quad (51)$$

where w_0 and w_1 belong to the outer solution and they are given by (28), with coefficients (41), and by (47), respectively. We observe that the boundary layer terms are $O(\varepsilon^2)$ and therefore they do not appear explicitly in (51). To incorporate them consistently, one needs to consider the successive approximation term, $\varepsilon^2 w_2$, for the outer solution.

4.5. Beam with clamped and simply supported ends

To fix ideas, let the left beam end be clamped and the right simply supported. The correspondent boundary conditions are given by (11a), (12b) and the pair of additional conditions (18). In this case, we arrive at $\gamma_1 = 2$ and $\gamma_2 = 3$ for the left and for the right boundary layer, respectively.

At leading order, one has the classical boundary conditions

$$w_0(0) = w'_0(0) = w_0(1) = w''_0(1) = 0,$$

whence we get the constants in the general solution (28)

$$c_{01} = -2\lambda_0^2 (\alpha^2 \beta^{-2} \cosh \alpha + \cos \beta), \quad (52a)$$

$$c_{02} = 2 (\lambda_0^2 \sin \beta + \alpha^2 \sinh \alpha), \quad (52b)$$

$$c_{03} = -\lambda_0^2 \sin \beta - \beta^2 \cos \beta - e^\alpha \alpha^2, \quad (52c)$$

$$c_{04} = e^\alpha (\beta^2 \cos \beta - \lambda_0^2 \sin \beta) + \alpha^2, \quad (52d)$$

together with Eq.(43a). The eigenvalues $\lambda_0 = \lambda_0^{(n)}$ are found from the transcendental equation

$$\alpha \cosh \alpha \sin \beta - \beta \cos \beta \sinh \alpha = 0,$$

that, when $\theta = 0$, boils down to

$$T(\lambda_0)U(\lambda_0) = S(\lambda_0)V(\lambda_0).$$

The last equation amounts to the well known classical equation $\tanh \lambda_0 = \tan \lambda_0$, while the correspondent eigenmodes are given by

$$w_0(s) = C \left[U(\lambda_0 s) - \frac{S(\lambda_0)}{T(\lambda_0)} V(\lambda_0 s) \right]. \quad (53)$$

268 The first-order approximation yields

$$w_1(0) = 0, \quad w_1'(0) = (1 - \sqrt{\xi}) w_0''(0), \quad w_1(1) = w_1'(1) = 0, \quad (54)$$

269 and a_{10} and a_{20} are defined by Eqs.(43a,32b)

$$\begin{aligned} a_{10} &= C \lambda_0^2 \sqrt{\xi} (1 - \sqrt{\xi}), \\ a_{20} &= C \lambda_0^3 \sqrt{\xi} (1 - \sqrt{\xi}) \left[V(\lambda_0) - \frac{S^2(\lambda_0)}{T(\lambda_0)} \right]. \end{aligned} \quad (55)$$

270 The inhomogeneous equation (44), subject to the boundary conditions (54),
271 possesses a solution provided that compatibility is satisfied, whereby we get the
272 first eigenfrequency correction

$$\lambda_1 = -\lambda_0 \frac{(1 - \sqrt{\xi}) [w_0''(0)]^2}{4 \int_0^1 [w_0''(s)]^2 ds}. \quad (56)$$

273 The solution of the BVP (44,54) has the form (47) as for the C-C case, yet with
274 different coefficients. Indeed, in the special case $\theta = 0$, Eq.(56) simplifies to

$$\lambda_1 = -\lambda_0(1 - \sqrt{\xi}),$$

and the particular solution becomes

$$w_{1p}(s) = \frac{\lambda_1}{\lambda_0} s w_0'(s) = C \lambda_1 s \left[T(\lambda_0 s) - \frac{S(\lambda_0)}{T(\lambda_0)} U(\lambda_0 s) \right],$$

whence

$$\begin{aligned} w_1(s) &= C \lambda_0 (1 - \sqrt{\xi}) \left[T(\lambda_0 s) - \frac{S(\lambda_0) U(\lambda_0)}{T(\lambda_0) V(\lambda_0)} V(\lambda_0 s) \right] + w_{1p}(s) \\ &= C \lambda_0 (1 - \sqrt{\xi}) \left[(1 - s) T(\lambda_0 s) + \frac{S(\lambda_0)}{T(\lambda_0)} \left(s U(\lambda_0 s) - \frac{U(\lambda_0)}{V(\lambda_0)} V(\lambda_0 s) \right) \right]. \end{aligned} \quad (57)$$

275 Finally, we arrive at the following asymptotic expansion for the frequency
276 ratio

$$\frac{\omega}{\omega_0} = 1 - \frac{1}{2} \varepsilon (1 - \sqrt{\xi}) \frac{[w_0''(0)]^2}{\int_0^1 [w_0''(s)]^2 ds} + O(\varepsilon^2) \quad (58)$$

277 that, in the case $\theta = 0$, reduces to

$$\frac{\omega}{\omega_0} = 1 - 2\varepsilon (1 - \sqrt{\xi}) + O(\varepsilon^2). \quad (59)$$

Figure 3: 1st (left) and 2nd (right) eigenfrequencies ω for a C-S beam (solid, black) in the absence of rotatory inertia, $\theta = 0$, and with $\varepsilon = 0.01, 0.05$ and 0.075 , superposed onto the 1-term (dotted, blue) asymptotic approximation, normalized with respect to the classical local frequency ω_0 , Eq.(59)

Eq.(59) is plotted in Fig.3 alongside the numerical solution of the TPNM. Although the accuracy of the expansion is restricted to small values of ε , we still appreciate a limit as the TPNM tends to the PNLN.

4.6. Cantilever Beam

For a cantilever beam we have, at leading order,

$$w_0(0) = w'_0(0) = w''_0(1) = w'''_0(1) + \theta\lambda_0^4 w'_0(1) = 0,$$

and the constants in the general solution (28) are given by Eqs.(52), i.e. they are the same as in the C-S case. The secular equation now reads

$$(1 + \frac{1}{2}\theta^2\lambda_0^4) \cosh \alpha \cos \beta - \frac{1}{2}\theta\lambda_0^2 \sinh \alpha \sin \beta + 1 = 0,$$

that, in the special case of vanishing rotational inertia, reduces to

$$S^2(\lambda_0) - T(\lambda_0)V(\lambda_0) = 0.$$

This formula coincides with the classical result $\cosh \lambda_0 \cos \lambda_0 + 1 = 0$ and the corresponding eigenforms are still given by Eq.(53).

In the first-order approximation, one arrives at the following boundary conditions

$$\begin{aligned} w_1(0) &= 0, & w'_1(0) &= (1 - \sqrt{\xi}) w''_0(0), \\ w''_1(1) &= 0, & w'''_1(1) + \lambda_0^4 \theta w'_1(1) &= -4\lambda_0^3 \lambda_1 \theta w'_0(1). \end{aligned} \tag{60}$$

Figure 4: 1st (left) and 2nd (right) eigenfrequencies ω for a cantilever beam (solid, black) in the absence of rotatory inertia, $\theta = 0$, and with $\varepsilon = 0.01, 0.05$ and 0.075 , superposed onto the 1-term (dotted, blue) asymptotic approximation, normalized with respect to the classical local model frequency ω_0 , according to Eq.(59)

together with the right boundary layer amplitude

$$a_{20} = \sqrt{\xi} \left(1 - \sqrt{\xi} \right) [w_1''(1) + w_0'''(1)],$$

the left being given by Eq.(43a). The compatibility condition for the inhomogeneous BVP (44, 60) is still given by Eq.(56) and, as a consequence, the ratio ω/ω_0 and the corresponding eigenmode correction are once again retrieved. Fig.4 compares the normalized eigenfrequency ω/ω_0 as numerically evaluated for the TPNM with the 1-term expansion (59) and shows good accuracy. Besides, the numerical solution curve matches the corresponding result given in Fig.5 of [10].

5. Purely nonlocal model

From the previous analysis, it clearly appears that the situation $\xi \rightarrow 0$ lends a perfectly admissible eigenfrequency which, therefore, can be assumed as the proper solution to the PNLN. We now consider what happens to the eigenmodes and for this we need to investigate the behavior of the boundary layer term $B_\xi(s) = \sqrt{\xi} \exp[-s/(\varepsilon\sqrt{\xi})]$, $0 \leq s \leq 1$, as $\xi \rightarrow 0$. Clearly, this is a transcendentally small term for $s > 0$ and $B_\xi(s) \rightarrow 0$ uniformly. Non uniformity arises when we consider $s = 0$ for then a boundary layer appears that may be studied taking the rescaled variable $s^* = s/(\varepsilon\sqrt{\xi})$, see [14]. This boundary layer

is vanishingly small as $\xi \rightarrow 0$ but not so are its derivatives with respect to s

$$B'_\xi(s) \rightarrow \begin{cases} 0, & s > 0, \\ -\varepsilon^{-1}, & s = 0, \end{cases} \quad \text{and} \quad B''_\xi(s) \rightarrow \begin{cases} 0, & s \neq 0, \\ +\infty, & s = 0, \end{cases}.$$

295 This result is the analogue of the steep boundary layer described in [37] under
 296 static axial deformation. We may now ask whether this unboundedness in the
 297 second derivative leads to an unbounded bending energy. To answer this we
 298 first observe that $\forall \eta > 0, \int_0^\eta B''_\xi(s) ds \rightarrow \varepsilon^{-1}$ uniformly and therefore $B''_\xi(s)$ is
 299 proportional to Dirac's delta function. Indeed, when considering the contribu-
 300 tion M_ξ of the boundary layer B_ξ to the bending moment M through Eq.(4),
 301 we find

$$M_\xi(0) \rightarrow (2\varepsilon^2)^{-1},$$

302 at leading order. If we use this result in, say, the eigenmodes (38) for a S-S beam,
 303 we easily see that the boundary condition $M(0) = 0$ is satisfied at leading order,
 304 for the boundary layer cancels out the contribution of the outer solution. At the
 305 same time, the constitutive BCs are asymptotically satisfied for a vanishingly
 306 small ξ due to the asymptotic procedure applied above. We then conclude that,
 307 in the limit as $\xi \rightarrow 0$, the boundary layer warrants the fulfilment of all boundary
 308 conditions and it brings a finite contribution to the bending energy. From the
 309 standpoint of displacements, we get

$$w(s) \rightarrow w^{(m)} + \varepsilon^{\gamma_1-1} a_{10} R(-s) + \varepsilon^{\gamma_2-1} a_{20} R(s-1),$$

310 where $R(s)$ is the ramp function. For a S-S beam, we have $\gamma_1 = \gamma_2 = 3$ and

$$a_{10} = (-1)^{n+1} a_{20} = C\beta^3.$$

311 Whence, a finite jump in the rotation and a concentrated couple at the beam
 312 ends is produced. This is perhaps not so surprising, for solutions in the sense
 313 of distributions are to be expected when an integral form of the constitutive
 314 equation is adopted. Consequently, from a mathematical standpoint, an energy
 315 bounded solution of the PNLM may be consistently defined as the limit of the
 316 TPNM, although it is meaningful in the sense of distributions and we may want
 317 to reject it on physical grounds.

Figure 5: Eigenfrequency ω for modes 1, 2 and 4 for a S-S beam, normalized over the classical frequency ω_0 , for $\theta = 0, 1/100$ and $1/10$, as a function of the local model fraction ξ . As it occurs for the asymptotic expansion (39), the frequency ratio is unaffected by rotational inertia and curves overlap

Figure 6: Eigenfrequency ratio ω/ω_0 for modes 1 (left panel) and 4 (right) for a C-C beam for $\theta = 0$ (solid, black), $\theta = 1/100$ (dashed, blue) and $1/10$ (dotted, red), as a function of the local model fraction ξ

6. Influence of rotational inertia

We now consider the effect of including rotational inertia when considering the solution of the TPNM. Fig. 5 plots the frequency ratio ω/ω_0 for mode numbers $n = 1, 4$ and 8 for a S-S beam and $\theta = 0, 1/100$ and $1/10$. It appears that, for the S-S end conditions, rotational inertia is irrelevant for the purpose of determining the frequency ratio (yet it still affects ω_0). Fig. 6 plots the frequency ratio ω/ω_0 for mode numbers $n = 1$ and 4 for $\theta = 0, 1/100$ and $1/10$ in a C-C beam. This time, rotational inertia plays an important role in the direction of contrasting the softening effect induced by the nonlocal fraction. Indeed, this hardening effect is already well manifest in the fundamental mode and, as ex-

Figure 7: Eigenfrequency ratio ω/ω_0 for modes 1 (left panel) and 4 (right) for a C-S beam for $\theta = 0$ (solid, black), $\theta = 1/100$ (dashed, blue) and $1/10$ (dotted, red), as a function of the local model fraction ξ

Figure 8: Eigenfrequency ratio ω/ω_0 for modes 1 (left panel) and 4 (right) for a C-F beam for $\theta = 0$ (solid, black), $\theta = 1/100$ (dashed, blue) and $1/10$ (dotted, red), as a function of the local model fraction ξ

pected, it becomes stronger for higher modes. Besides, encompassing rotational inertia of the cross-section has a significant bearing on higher modes, regardless of the actual value of θ . The same qualitative picture appears in Fig.7 and in Fig.8, respectively for C-S and C-F beams. It appears that the softening effect is stronger moving from S-S to C-C, C-F and then to C-S.

7. Conclusions

The purely nonlocal theory of elasticity has recently attracted considerable attention for the controversial results it conveys. Indeed, this model is believed to lead to ill-posed problems, owing to the appearance of a pair of constitutive boundary conditions which are generally at odd with the natural boundary conditions. In this paper, we approach the problem from a different perspective and carry out an asymptotic analysis of the free vibrations of flexural beams endowed with rotational inertia, within the two-phase theory of nonlocal elasticity. We show that the nonlocal term contributes with a boundary layer whose strength greatly varies for different end conditions. In the case of simply supported beams, the boundary layer is the weakest and we provide a two-term correction for the classical solution. Remarkably, this situation is affected by the presence of rotational inertia only in the classical sense. Conversely, clamped-clamped, clamped-supported and clamped-free (i.e. cantilever) conditions bring a much stronger boundary layer, a for these we provide a single correction term. Numerical results confirm the accuracy of the asymptotic approach and show that rotational inertia is very relevant in contrasting the softening effect connected to the nonlocal phase. Most interestingly, for any end condition, the asymptotic solution still exists and its energy remains bounded in the limit of the purely nonlocal theory, that is for a vanishingly small local phase. This is in contrast to what is anticipated in the literature, see, for instance, [30]. We are therefore in the position of attaching a meaning to the purely nonlocal theory, as the limit of the two-phase theory. In so doing, we encounter a solution that is defined in the sense of distributions (for the curvature) and, although maybe questionable

357 from a physical standpoint, it is mathematically sound.

358 Acknowledgments

359 GM gratefully acknowledges a Visiting Professor Position granted by the
360 University of Modena and Reggio Emilia in the AY 2018/2019.

361 AN welcomes support from FAR2019, Piano di sviluppo dipartimentale
362 DIEF, DR nr. 498/2019.

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