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3 **Analytical estimates of the pull-in voltage for carbon**
4 **nanotubes considering tip-charge concentration**
5 **and intermolecular forces**

6 **Giovanni Bianchi · Enrico Radi**

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9 **Abstract** Two-side accurate analytical estimates of
10 the pull-in parameters of a carbon nanotube switch
11 clamped at one end under electrostatic actuation are
12 provided by considering the proper expressions of the
13 electrostatic force and van der Waals interactions for a
14 carbon nanotube, as well as the contribution of the
15 **AQ1** charge concentration at the free end. According to the
16 Euler–Bernoulli beam theory, the problem is governed
17 by a fourth-order nonlinear boundary value problem.
18 Two-side estimates on the centreline deflection are
19 derived. Then, very accurate lower and upper bounds
20 to the pull-in voltage and deflection are obtained as
21 **AQ2** function of the geometrical and material parameters.
22 The analytical predictions are found to agree remark-
23 ably well with the numerical results provided by the
24 shooting method, thus validating the proposed
25 approach. Finally, a simple closed-form relation is
26 proposed for the minimum feasible gap and maximum
27 realizable length for a freestanding CNT cantilever.

Keywords Carbon nanotube · Pull-in voltage · 28
NEMS · Nanocantilever · van der Waals interactions · 29
Charge concentration 30

1 Introduction 31

Carbon nanotubes (CNTs) display a number of smart 32
electronic and mechanical properties that are currently 33
exploited in a wide variety of industrial applications, 34
such as sensors, nanoactuators, memory devices, 35
switches, high frequency nanoresonators and nan- 36
otweezers [1–3]. Due to their tiny size, CNTs display 37
ultra-low mass and very high resonance frequency. 38
Moreover, they undergo purely elastic behaviour, they 39
are able to carry huge electrical currents and to sustain 40
high current densities. These attractive properties, in 41
conjunction with the significant progress recently 42
made in the fabrication of carbon nanostructures, 43
allow CNTs to become essential components in the 44
production of enhanced nano-electro-mechanical sys- 45
tems (NEMS) [1]. As a consequence, a considerable 46
amount of research interest has been dedicated to the 47
accurate modelling of the structural and electric 48
behavior of CNTs in the last few years. 49

A typical CNT switch consists in a moveable 50
nanowire suspended over a fixed conductive ground 51
plane, usually made of graphite. By applying DC 52
voltage difference between the components, the CNT 53

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54 deflects toward the ground electrode until at the pull-in
 55 voltage it sticks on the ground plane, thus shortening
 56 the electric circuit. The atomic interactions at the
 57 nano-scale separations, modelled by the van der Waals
 58 force, substantially affects the pull-in instability of
 59 NEMS. Both the electrostatic and van der Waals
 60 forces depend on the CNT deflection non-linearly.
 61 This occurrence makes every attempt to describe their
 62 response in closed form a very difficult task. Specif-
 63 ically, no exact solution can be found for the non-
 64 linear ordinary differential equation (ODE) governing
 65 the CNT deflection under electrostatic actuation. As a
 66 consequence, a variety of numerical and approximated
 67 approaches has been proposed in the technical liter-
 68 ature, ranging from the reduction to 1D lumped
 69 models, based on the assumption of appropriate shape
 70 functions for the CNT deflection, to the use of
 71 powerful numerical techniques to generate reduced-
 72 order models, such as the Differential Quadrature
 73 Method, the Galerkin Discretization Method or the
 74 Finite Element Method [4–11]. However, these
 75 approximated methods may provide significant error
 76 percentages as the CNT deflection increases and gets
 77 closer to the pull-in limit. Moreover, they predict
 78 arbitrary estimates of the effective pull-in parameters,
 79 whereas an effective approach should provide accu-
 80 rate lower and upper bounds that can be exploited for
 81 ensuring the safe operation of the device. Alterna-
 82 tively, molecular dynamics approaches have been
 83 adopted to study CNTs pull-in behavior [12]. How-
 84 ever, these methods are very time-consuming and can
 85 not be easily employed for large structures.

86 As remarked by Ke et al. [13, 14], electric charges
 87 tend to concentrate at the ends of a linear conductor
 88 and thus for proper modeling of the pull-in instability
 89 phenomenon the effect of the concentrated load due to
 90 charge concentration at the end of a CNT cantilever is
 91 expected to provide a significant contribution on the
 92 deflection of CNT and consequently on the pull-in
 93 instability. Therefore, it must be necessarily consid-
 94 ered for the accurate evaluation of the pull-in voltage.
 95 In particular, Ke et al. [13] showed that the pull-in
 96 voltage decreases by about 14% due to the effect of the
 97 tip-charge concentration. They also provided an
 98 approximate relation for the pull-in voltage that
 99 account for the effects of tip-charge concentration
 100 and finite kinematics. They found that the finite
 101 kinematic effect is negligible for a CNT-based
 102 cantilever switch, but the effect of charge

concentration is quite significant. Ke [15] also pre-
 103 sented a detailed review of the recent advances in the
 104 electro-mechanical modeling and characterization of
 105 CNT cantilevers and their applications. 106

The development of analytical models that can
 107 predict the pull-in response of the device becomes
 108 extremely relevant for identifying the most efficient
 109 geometries and materials required for meeting the
 110 requests of ultralow power consumption, strength and
 111 durability. Despite the amount of numerical and
 112 approximated investigations, analytical models and
 113 closed form expressions for assessing the occurring of
 114 CNT pull-in instability still appears to be limited. An
 115 accurate determination of the stable actuating range
 116 and the pull-in instability threshold is a crucial issues
 117 for the design of reliable and optimized CNT-based
 118 NEMS. In two previous works, Radi et al. [16, 17]
 119 provided an analytical methodology for assessing
 120 accurate lower and upper bounds to the pull-in
 121 parameters of an electro-statically actuated micro- or
 122 nano-cantilever, by taking the contributions of flexible
 123 support and compressive axial load into consideration.
 124 Both contributions are found to reduce the pull-in
 125 voltage and to increase the critical gap spacing for a
 126 freestanding nano-cantilever, namely in the absence of
 127 electrical actuation. The investigations [16, 17] have
 128 focused on the pull-in instability in micro- and
 129 nanobeams with rectangular cross-section only. More-
 130 over, the contribution of the charge concentrated at the
 131 nanocantilever tip has been neglected in these works. 132

In the present work, attention is paid to investigate
 133 the pull-in phenomenon in CNT with circular cross-
 134 section rolled up by graphene sheets, by considering
 135 the proper expressions of the electrostatic force as well
 136 as the significant effect induced by the tip-charge
 137 concentration [13, 14, 18, 19]. The van der Waals
 138 force acting on the CNT has been derived in [4]
 139 starting from the Lennard–Jones potential (see also
 140 [8, 20–22]). The finite kinematic effect has been
 141 neglected here, Ke et al. [13, 14] found indeed that for
 142 a clamped CNT it becomes significant only for very
 143 slender CNTs and large gap spacing. Indeed, the pull-
 144 in instability generally occurs as the CNT tip deflec-
 145 tions attains about $1/3 \div 1/2$ of the gap spacing,
 146 which is much smaller than the CNT length. Within
 147 this range, the CNT can be reasonably supposed to
 148 experience small deformations and small displace-
 149 ment. Therefore, reference is made here to the classic
 150 Euler–Bernoulli (EB) beam theory, which is valid for
 151

152 most of the CNT applications as switches and
 153 actuators [23]. The main advantage of the present
 154 approach with respect to other ones proposed in
 155 literature consists in providing accurate analytical
 156 bounds from above and below for the pull-in voltage
 157 and pull-in deflection, thus avoiding the numerical
 158 integration of the nonlinear fourth-order ODE derived
 159 from the EB beam theory. Moreover, the present work
 160 extends previous investigations on nanobeams with
 161 rectangular cross section [16, 17], which are not
 162 specifically addressed to CNTs and do not take the
 163 contribution of the concentrated-tip charge into
 164 account.

165 By introducing few non-dimensional parameters,
 166 the nonlinear ODE for the CNT centreline deflection
 167 and the corresponding boundary conditions are pre-
 168 sented in Sect. 2. Moreover, an equivalent integral
 169 equation formulation is derived therein. The nonlinear
 170 response is due to the electrostatic force and van der
 171 Waals interactions, which depends on the beam
 172 deflection nonlinearly, whereas the CNT is modelled
 173 by using a linear elastic EB beam. The solution of the
 174 boundary value problem is then proved to be positive,
 175 increasing and convex. Upper and lower estimates for
 176 the CNT deflection are obtained in Sect. 3. Accurate
 177 two-side analytical bounds to the pull-in parameters
 178 are derived in Sect. 4 by exploiting the estimates
 179 obtained in Sect. 3. The accuracy of the proposed
 180 bounds are then validated in Sect. 5 by comparing the
 181 analytical estimates and the numerical results pro-
 182 vided by the shooting method. A remarkable agree-
 183 ment is observed therein. On the basis of the obtained
 184 results, an approximated closed-form expression is
 185 finally proposed for permissible gap spacing and CNT

length under the influence of intermolecular
 attractions.

The approach here proposed refers to a single-
 walled CNT. However, it can be easily generalized to
 multi-walled CNTs, e.g. by considering the expres-
 sions of the electrostatic and van der Waals forces
 provided in [24], as well as to other kinds of
 interactions, such as capillary and electrochemical
 forces [25, 26].

2 Mathematical modeling

A schematic view of a CNT-based cantilever switch is
 shown in Fig. 1. A movable single-walled or multi-
 walled CNT is placed above a fixed ground plane and
 subject to van der Waals interactions and attractive
 electrostatic force due to applied voltage. The nano-
 tube length and the cross section mean radius are
 denoted with L and R , respectively. The gap spacing
 between the nanotube and the ground plane is denoted
 by H . The deflection $v(z)$ of the CNT centreline is
 described by the following non-linear fourth-order
 ODE written in terms of the nondimensional variables
 $u = v/H$ and $x = z/L$ for $0 \leq x \leq 1$ and $0 \leq u \leq 1$

$$u^{IV}(x) = f(u(x)), \quad \text{for } x \in [0, 1], \quad (1)$$

where the prime denotes differentiation with respect to
 the function argument. The CNT actuation is modelled
 by considering both contributions of electrostatic
 force and van der Waals interactions, namely

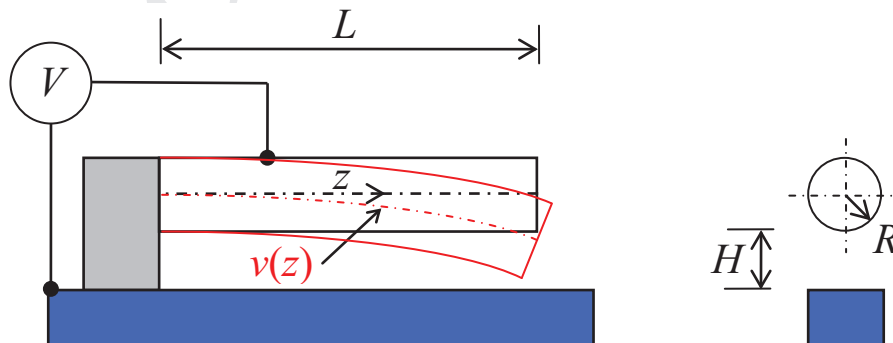


Fig. 1 A CNT based cantilever switch under electrostatic loading

$$f(u) = \beta f_e(u) + \gamma F_c(u), \tag{2}$$

214 where the normalized electrostatic and van der Waals
215 forces for the cylindrical geometry are given by
216 [4, 8, 13, 14, 18–22, 27, 28]

$$f_e(u) = \frac{1}{\sqrt{(1-u)(1-u+2/k)[\cosh^{-1}(1+k-ku)]^2}},$$

$$f_c(u) = \frac{8k^4(1-u)^4 + 32k^3(1-u)^3 + 72k^2(1-u)^2 + 80k(1-u) + 35}{k^{10}[(1-u)(1-u+2/k)]^{9/2}}, \tag{3}$$

218 where $k = H/R$ is a geometric ratio and the non-
219 dimensional parameters β and γ are proportional to the
220 magnitude of the electrostatic force and van der Waals
221 interactions, respectively, namely

$$\beta = \frac{\pi \epsilon_0 V^2 L^4}{H^2 EI}, \quad \gamma = \frac{C_6 \sigma^2 \pi^2 L^4}{2R^5 EI} \tag{4}$$

223 where $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ is the per-
224 mittivity of vacuum, V is the electric voltage applied to
225 the electrodes, $C_6 = 15.2 \text{ Ev} \text{ \AA}^6$ is a constant charac-
226 terizing the interaction between carbon–carbon atoms,
227 $\sigma = 38 \text{ nm}^{-2}$ is the graphene surface density, $I \approx \pi t$

$$f'_e(u) = \frac{1}{(1-u)(1-u+2/k)[\cosh^{-1}(1+k-ku)]^2} \left[\frac{2}{\cosh^{-1}(1+k-ku)} + \frac{1-u+1/k}{\sqrt{(1-u)(1-u+2/k)}} \right], \tag{8}$$

$$f'_c(u) = 5 \frac{8k^5(1-u)^5 + 40k^4(1-u)^4 + 120k^3(1-u)^3 + 200k^2(1-u)^2 + 175k(1-u) + 63}{k^{11}[(1-u)(1-u+2/k)]^{11/2}}.$$

228 R^3 is the moment of inertia of the CNT cross-section,
229 where t is the CNT wall thickness, and E is the
230 Young’s modulus of the graphene. A number of
231 studies based on experimental tests and atomistic
232 simulations found that the Young’s modulus of the
233 graphene varies from 0.5 to 5.5 TPa and the single
234 wall thickness ranges between 0.7 and 3.4 Å, see the
235 summary of results given in [29]. The mean values
236 suggested in [29] are $t = 1.34 \text{ \AA}$ and $E = 2.52 \text{ TPa}$.

237 The van der Waals force per unit length (3)₂ has
238 been derived in [4] by taking the derivative with
239 respect to the deflection of the van der Waals energy
240 determined by double volume integral of the Lennard–
241 Jones potential.

242 The boundary conditions for the cantilever EB
243 beam then require vanishing of displacement and

rotation at the clamped end ($x = 0$), vanishing of the 244
bending moment and assigned shearing force at the 245
free end ($x = 1$), namely 246

$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \tag{5}$$

$$u'''(1) = -\beta q(\delta),$$

where $\delta = u(1)$ is the tip displacement and 248

$$q(\delta) = \frac{0.85\rho(1+k)^{2/3}}{\sqrt{(1-\delta)(1-\delta+2/k)[\cosh^{-1}(1+k-k\delta)]^2}}, \tag{6}$$

is the normalized shearing force due to the electro- 250
static attraction of the charge concentrated at the CNT 251
tip [13, 14, 27], being $\rho = R/L$ the inverse of the CNT 252
slenderness. 253

By taking the derivative of Eq. (2) with respect to u , 254
one obtains 255

$$f'(u) = \beta f'_e(u) + \gamma F'_c(u), \tag{7}$$

where 257
258

Note that the functions $f(u)$ and $f'(u)$ defined in (2) and 259
(7) are positive and monotonically increasing for 260
 $0 \leq u \leq 1$ and $k > 0$, namely 261

$$f(u) \geq f(0) \geq 0, \quad f'(u) \geq f'(0) \geq 0, \tag{9}$$

where 263

$$f(0) = \frac{\beta}{\sqrt{(1+2/k)[\cosh^{-1}(1+k)]^2}} + \gamma \frac{8k^4 + 32k^3 + 72k^2 + 80k + 35}{k^{10}(1+2/k)^{9/2}}, \tag{10}$$

265

$$f'(0) = \beta \frac{2\sqrt{k(2+k)} + (1+k) \cosh^{-1}(1+k)}{(2+k)\sqrt{(1+2/k)[\cosh^{-1}(1+k)]^3}} + 5\gamma \frac{8k^5 + 40k^4 + 120k^3 + 200k^2 + 175k + 63}{k^{11}(1+2/k)^{11/2}}. \tag{11}$$

267 2.1 Nonlinear integral equation for $u(x)$

268 In this section, the governing ODE (1) is integrated
 269 four times by using the boundary conditions (5), in
 270 order to obtain preliminary estimates about the
 271 solution $u(x)$ and its derivatives up to the third order.
 272 Moreover, a nonlinear integral equation for the
 273 deflection u is obtained, which will be used later for
 274 achieving accurate bounds for the pull-in parameters.
 275 A first integration of the governing ODE (1) between x
 276 and 1 by using the boundary condition (5)₄ yields

$$u'''(x) = - \int_x^1 f(u(t))dt - \beta q(\delta). \tag{12}$$

278 Integration of Eq. (12) between x and 1, by using the
 279 boundary condition (5)₃ and integration by parts, then
 280 yields

$$u''(x) = \int_x^1 (t-x)f(u(t))dt + (1-x)\beta q(\delta). \tag{13}$$

282 Integration of Eq. (13) between 0 and x , by using the
 283 boundary condition (5)₂ and integration by parts, then
 284 yields

$$u'(x) = \frac{1}{2} \left\{ x \int_x^1 (2t-x)f(u(t))dt + \int_0^x t^2 f(u(t))dt + \beta q(\delta)(2-x)x \right\}. \tag{14}$$

286 Finally, integration of Eq. (14) between 0 and x by
 287 using the boundary condition (5)₁ and integration by
 288 parts gives the following nonlinear integral equation
 289 for $u(x)$

$$u(x) = \frac{1}{6} \left\{ x^2 \int_x^1 (3t-x)f(u(t))dt + \int_0^x (3x-t)t^2 f(u(t))dt + \beta q(\delta)(3-x)x^2 \right\}. \tag{15}$$

The normalized deflection of the cantilever tip, $\delta = u(1)$, then must satisfy the following condition derived from Eq. (15) for $x = 1$

$$\delta = \frac{1}{6} \int_0^1 (3-t)t^2 f(u(t))dt + \frac{1}{3} \beta q(\delta). \tag{16}$$

Considering that $f(u) \geq 0$ and $q(\delta) \geq 0$, from Eqs. (1), (12)–(15) the following conditions then hold true for $x \in [0, 1]$:

$$\begin{aligned} u(x) &\geq \frac{x^2}{6} (3-x)\beta q(\delta) \geq 0, \\ u'(x) &\geq \left(x - \frac{x^2}{2}\right) \beta q(\delta) \geq 0, \\ u''(x) &\geq (1-x)\beta q(\delta) \geq 0, \\ u'''(x) &\leq -\beta q(\delta). \end{aligned} \tag{17}$$

Therefore, the function $u(x)$ is positive, increasing and convex for $x \in (0, 1)$.

3 Two-side estimates for the deflection 301

In order to define upper and lower bounds to the pull-in parameters, two-side estimates are first derived for the deflection $u(x)$. 302 303 304

3.1 Upper bounds to the deflection $u(x)$ 305

Let $u(x)$ denotes the solution to the BVP (1) and (5), then it can be proved that $u(x) \leq u_U(x)$ for $x \in [0, 1]$, where 306 307 308

$$u_U(x) = \delta b_1(x) + \beta q(\delta) [b_2(x) + f'(0)b_3(x)], \tag{18}$$

and 310

Author Proof

$$b_1(x) = \frac{1}{3}x^2(x^2 - 4x + 6) \geq 0, \tag{19}$$

$$b_2(x) = \frac{1}{18}x^2(1-x)(2x-3) \geq 0,$$

$$b_3(x) = \frac{1}{5040}x^2(1-x)(x^4 - 6x^3 - 6x^2 + 38x - 33) \leq 0. \tag{20}$$

Indeed, let us define the function

$$h(x) = \delta b_1(x) + \beta q(\delta) [b_2(x) + f'(0)b_3(x)] - u(x), \tag{21}$$

then the derivatives of $h(x)$ up to the fourth order become

$$\begin{aligned} h'(x) &= 4\delta \left(\frac{x^3}{3} - x^2 + x \right) - \frac{\beta}{18} q(\delta) \left[\frac{f'(0)}{280} (7x^6 - 42x^5 + 176x^3 - 213x^2 + 66x) + 8x^3 - 15x^2 + 6x \right] - u'(x), \\ h''(x) &= 4\delta(1-x)^2 + \frac{\beta}{3} q(\delta) (1-x) \left[\frac{f'(0)}{280} (7x^4 - 28x^3 - 28x^2 + 60x - 11) + 4x - 1 \right] - u''(x), \\ h'''(x) &= -8\delta(1-x) - \frac{\beta}{3} q(\delta) \left[\frac{f'(0)}{280} (35x^4 - 140x^3 + 176x - 71) + 8x - 5 \right] - u'''(x), \\ h^{IV}(x) &= 8\delta - \frac{\beta}{3} q(\delta) \left[\frac{f'(0)}{70} (35x^3 - 105x^2 + 44) + 8 \right] - u^{IV}(x). \end{aligned} \tag{22}$$

Moreover, by taking the derivative of $h^{IV}(x)$, using Eq. (1), one has

$$h^V(x) = \beta q(\delta) f'(0) \left(x - \frac{1}{2}x^3 \right) - f'(u)u'(x) \leq 0, \tag{23}$$

where the last inequality follows from (9)₂ and (17)₂, thus implying that the function $h'''(x)$ is concave. Then, the following conditions are met by function $h(x)$ and its derivatives

$$\begin{aligned} h(0) = 0, \quad h(1) = 0, \quad h'(0) = 0, \quad h'(1) = 0, \\ h'''(1) = 0, \quad h^V(x) \leq 0. \end{aligned} \tag{24}$$

Therefore, the function $h(x)$ satisfies all the requirements for the application of Lemma A reported in the “Appendix”, and thus $h(x) \geq 0$ for $x \in [0, 1]$, so that, by using the definition (21), the upper bound (18) for the CNT deflection holds true. \square

The term $\delta b_1(x)$ appearing in (18) coincides with the quartic polynomial used for approximating nanobeam deflection in [30]. Moreover, from conditions (18), by using (9)₂ and (20) it follows that

$$u(x) \leq \delta b_1(x) + \beta q(\delta) b_2(x), \quad \text{for } x \in [0, 1], \tag{25}$$

Obviously, the upper bound (25) is less accurate than (18), but it depends linearly on the parameter β .

Therefore, two slightly different procedures for deriving lower bound to the pull-in parameters will be developed in Sect. 4.1 starting from the bounds (18) and (25), respectively.

3.2 Lower bounds to the deflection $u(x)$

Let $u(x)$ denote the solution to the BVP (1) and (5), then the lower bound $u(x) \geq u_L(x)$ holds true for $x \in [0, 1]$, where

$$u_L(x) = \delta a_1(x) + f(0)a_2(x) \tag{26}$$

and

$$\begin{aligned} a_1(x) &= \frac{1}{2}(3x^2 - x^3) \geq 0, \\ a_2(x) &= \frac{1}{48}(3x^2 - 5x^3 + 2x^4) \geq 0. \end{aligned} \tag{27}$$

Let us indeed define the following function

$$g(x) = u(x) - \frac{\delta}{2}(3x^2 - x^3) - \frac{f(0)}{48}(3x^2 - 5x^3 + 2x^4), \quad (28)$$

then the derivatives of $g(x)$ write

$$\begin{aligned} g'(x) &= u'(x) - \frac{3}{2}\delta(2x - x^2) - \frac{f(0)}{48}(6x - 15x^2 + 8x^3), \\ g''(x) &= u''(x) - 3\delta(1 - x) - \frac{f(0)}{8}(1 - 5x + 4x^2), \\ g'''(x) &= u'''(x) + 3\delta + (5 - 8x), \\ g^{IV}(x) &= u^{IV}(x) - f(0) \geq 0, \end{aligned} \quad (29)$$

where the latter inequality follows from Eqs. (1) and (9)₁. Therefore, the function $g(x)$ satisfies the following boundary conditions

$$g(0) = 0, \quad g(1) = 0, \quad g'(0) = 0, \quad g''(1) = 0. \quad (30)$$

Therefore, the function $g(x)$ satisfies all the requirements for the application of Lemma B proved in "Appendix". It follows that $g(x) \geq 0$ for $x \in [0, 1]$, so that, by using the definition (28), the lower bound (26) for the CNT deflection holds true. \square

4 Bounds to the pull-in parameters

By introducing the estimates (18), (25) and (26) on the deflection $u(x)$ in relation (16), the following lower and upper bounds to the pull-in parameters β_{PI} and δ_{PI} can be derived analytically.

4.1 Accurate lower bounds to the pull-in parameters

By using (9)₂ and the upper bound to the CNT deflection (18) one has $f(u) \leq f(u_U)$, then from (16) it follows

$$\delta \leq F(\delta, \beta) + \frac{\beta}{3}q(\delta), \quad (31)$$

where the function

$$F(\delta, \beta) = \frac{1}{6} \int_0^1 t^2(3-t)f(u_U(t))dt \quad (32)$$

can be calculated numerically.

Condition (31) defines a lower bound to the relation between the electrostatic loading parameter β and the normalized pull-in deflection δ . The maximum value of the parameter β and the corresponding tip deflection δ obtained from relation (31) by using the stationary condition

$$\frac{\partial \beta}{\partial \delta} = 0, \quad (33)$$

then define the lower bounds of the pull-in parameters β_L and δ_L , such that $\beta_{PI} \geq \beta_L$ and $\delta_{PI} \geq \delta_L$, which are given by the following two conditions

$$\begin{aligned} F(\delta_L, \beta_L) + \frac{\beta_L}{3}q(\delta_L) &= \delta_L, \\ \Phi(\delta_L, \beta_L) + \frac{\beta_L}{3}q'(\delta_L) &= 1, \end{aligned} \quad (34)$$

where the function

$$\begin{aligned} \Phi(\delta, \beta) &= \frac{1}{6} \int_0^1 t^2(3-t)\{b_1(t) + \beta q'(\delta)[b_2(t) \\ &\quad + f'(0)b_3(t)]\}f'(u_U(t))dt \end{aligned} \quad (35)$$

can be calculated numerically and is given by the derivative with respect to δ of the function $F(\delta, \beta)$ defined in (32), performed by considering the maximization condition (33) and the definition (18) of u_U .

4.1.1 Lower bounds to the pull-in parameters

By using the estimate (25) and the monotony conditions $f_e'(u) \geq 0$ and $f_c'(u) \geq 0$, from (16) it follows

$$\delta \leq \beta f_e(\delta) + \gamma F_c(\delta) + \frac{\beta}{3}q(\delta), \quad (36)$$

where the functions

$$F_e(\delta) = \frac{1}{6} \int_0^1 t^2(3-t)f_e(\delta b_1(t) - \beta q(\delta)b_2(t))dt, \quad (37)$$

$$F_c(\delta) = \frac{1}{6} \int_0^1 t^2(3-t)f_c(\delta b_1(t) - \beta q(\delta)b_2(t))dt, \quad (38)$$

402 can be calculated numerically.

403 Condition (36) defines a lower bound to the relation
 404 between the electrostatic loading parameter β and the
 405 normalized pull-in deflection δ . In this case, inequality
 406 (36) can be easily solved for the parameter β . The
 407 maximum value of the parameter β and the corre-
 408 sponding tip deflection δ obtained from relation (36)
 409 by using the stationary condition (33) then provides
 410 the lower bounds of the pull-in parameters β_L and δ_L .
 411 Namely, the latter values are given by the conditions

$$\begin{aligned} \beta_L F_e(\delta_L) + \gamma F_c(\delta_L) + \frac{\beta_L}{3} q(\delta_L) &= \delta_L, \\ \beta_L F'_e(\delta_L) + \gamma F'_c(\delta_L) + \frac{\beta_L}{3} q'(\delta_L) &= 1, \end{aligned} \quad (39)$$

413 where the apex denotes the derivative with respect to
 414 the function argument within the brackets, namely

$$\begin{aligned} F'_e(\delta) &= \frac{1}{6} \int_0^1 t^2 (3-t) [b_1(t) - \beta q'(\delta) b_2(t)] f'_e(\delta b_1(t) \\ &\quad - \beta q(\delta) b_2(t)) dt, \end{aligned} \quad (40)$$

$$\begin{aligned} F'_c(\delta) &= \frac{1}{6} \int_0^1 t^2 (3-t) [b_1(t) - \beta q'(\delta) b_2(t)] f'_c(\delta b_1(t) \\ &\quad - \beta q(\delta) b_2(t)) dt. \end{aligned} \quad (41)$$

418 The latter functions can be calculated numerically and
 419 are given by the derivative with respect to δ of the
 420 functions defined in (37) and (38), performed by
 421 considering the maximization condition (33).

422 4.2 Upper bounds to the pull-in parameters

423 By using (9)₂ and the lower bound to the CNT
 424 deflection (26) it follows that $f(u) \geq f(u_L)$, then from
 425 (16) one has

$$\delta \geq G(\delta, \beta) + \frac{\beta}{3} q(\delta), \quad (42)$$

427 where the function

$$G(\delta, \beta) = \frac{1}{6} \int_0^1 t^2 (3-t) f(u_L(x)) dt \quad (43)$$

can be calculated numerically.

Inequality (42) implicitly defines an upper bound to
 the relation between the parameters β and δ . The
 maximum value of the parameters β and the corre-
 sponding tip deflection δ obtained from this relation by
 using the stationary condition (33) then provides the
 upper bounds of the pull-in parameters β_U and δ_U ,
 such that $\beta_{PI} \leq \beta_U$ and $\delta_{PI} \leq \delta_U$. Therefore, the upper
 bounds follow from the two conditions

$$\begin{aligned} G(\delta_U, \beta_U) + \frac{\beta_U}{3} q(\delta_U) &= \delta_U, \\ \Gamma(\delta_U, \beta_U) + \frac{\beta_U}{3} q'(\delta_U) &= 1, \end{aligned} \quad (44)$$

where the function

$$\Gamma(\delta, \beta) = \frac{1}{6} \int_0^1 t^2 (3-t) a_1(t) f'(u_L(x)) dt, \quad (45)$$

can be calculated numerically and is given by the
 derivative with respect to δ of the function $G(\delta, \beta)$
 defined in (43), performed by considering the maxi-
 mization condition (33) and the definition (26) of u_L .

4.3 Ke et al. estimates to the pull-in voltage

The following approximated relation for the pull-in
 voltage of a CNT whose radius R is much smaller than
 the gap spacing H between the CNT and ground plane,
 namely for $k \gg 1$ has been proposed in [14]

$$V_{PI} = 0.85 \sqrt{\frac{1 + K^{FK}}{1 + K^{TIP}}} \frac{H}{L^2} \ln\left(2 \frac{H}{R}\right) \sqrt{\frac{EI}{\epsilon_0}}, \quad (46)$$

where the parameters

$$\begin{aligned} K^{FK} &= \frac{8H^2}{9L^2} = \frac{8}{9} k^2 \rho^2, \\ K^{TIP} &= 2.55 \frac{R^{1/3} (H+R)^{2/3}}{L} = 2.55 \rho (k+1)^{2/3}, \end{aligned} \quad (47)$$

take into account for the effects of finite kinematics
 and concentrated-tip charge, respectively. Considering
 the definition (4)₁ of the normalized pull-in voltage,
 from (46) and (47) it follows

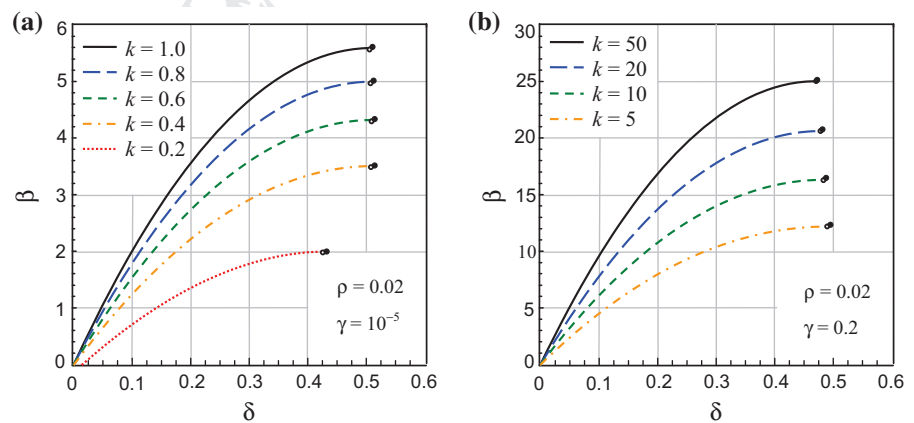
Table 1 Lower and upper bounds of the pull-in parameters for a CNT switch with $k = 1.0$, for various values of the van der Waals parameter γ and geometric ratio ρ

$k = 1$	$\rho = 0.01$				$\rho = 0.05$				$\rho = 0.1$			
	δ_L	β_L	δ_U	β_U	δ_L	β_L	δ_U	β_U	δ_L	β_L	δ_U	β_U
0	0.5119	5.8346	0.5193	5.9082	0.4884	4.8695	0.4923	4.9062	0.4700	4.0432	0.4722	4.0619
0.1	0.4017	4.2413	0.4088	4.3134	0.3942	3.6010	0.3996	3.6477	0.3874	3.0298	0.3917	3.0601
0.2	0.3477	3.0363	0.3550	3.1098	0.3437	2.5965	0.3500	2.6497	0.3401	2.1983	0.3455	2.2368
0.3	0.3091	1.9975	0.3167	2.0724	0.3070	1.7162	0.3139	1.7744	0.3050	1.4591	0.3113	1.5043
0.4	0.2785	1.0613	0.2862	1.1374	0.2775	0.9150	0.2848	0.9774	0.2766	0.7805	0.2836	0.8313
0.5	0.2527	0.1975	0.2606	0.2744	0.2526	0.1708	0.2603	0.2365	0.2524	0.1460	0.2600	0.2016

Table 2 Lower and upper bounds of the pull-in parameters for a CNT switch with $k = 10$, for various values of the van der Waals parameter γ and geometric ratio ρ

$k = 10$	$\rho = 0.01$				$\rho = 0.05$				$\rho = 0.1$			
	δ_L	β_L	δ_U	β_U	δ_L	β_L	δ_U	β_U	δ_L	β_L	δ_U	β_U
0	0.4978	18.515	0.5029	18.687	0.4565	11.939	0.4578	11.975	0.4363	8.2986	0.4368	8.3090
2×10^4	0.4176	14.924	0.4235	15.114	0.3985	9.8578	0.4015	9.9198	0.3876	6.9270	0.3895	6.9538
4×10^4	0.3725	12.060	0.3789	12.263	0.3607	8.0684	0.3648	8.1521	0.3537	5.7083	0.3567	5.7512
6×10^4	0.3394	9.5547	0.3461	9.7693	0.3316	6.4495	0.3365	6.5520	0.3269	4.5860	0.3309	4.6443
8×10^4	0.3127	7.2818	0.3197	7.5054	0.3075	4.9490	0.3132	5.0683	0.3043	3.5334	0.3092	3.6063
10×10^4	0.2901	5.1768	0.2973	5.4081	0.2868	3.5381	0.2930	3.6725	0.2847	2.5347	0.2904	2.6215

Fig. 2 Relations between electrostatic loading parameter β and tip deflection δ obtained from the shooting method, for various geometric ratios k and two different values of γ . Lower and upper estimates of the pull-in parameters are denoted by small circles and small points, respectively



$$\beta_{ke} = \frac{\pi \epsilon_0 V_{PI}^2 L^4}{EI H^2} = \pi 0.85^2 \frac{1 + K^{FK}}{1 + K^{TP}} \ln^2 \left(2 \frac{H}{R} \right) = \pi 0.85^2 \frac{(1 + 8k^2 \rho^2 / 9) \ln^2(2k)}{1 + 2.55 \rho (k + 1)^{2/3}} \quad (48)$$

5 Results

Lower and upper estimates for the normalized pull-in voltage β_L and β_U and the corresponding estimates of the normalized pull-in deflection δ_U and δ_L have been reported in Tables 1 and 2. In these tables, two different values of the geometric ratio k are considered

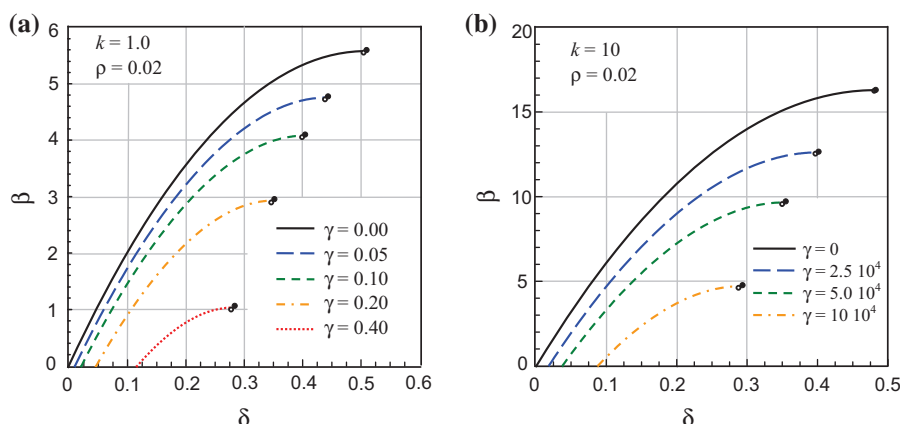


Fig. 3 Relations between electrostatic loading parameter β and tip deflection δ obtained from the shooting method, for the geometric ratios $k = 1$ (a) and $k = 10$ (b) and various values of

the van der Waals parameter γ . Lower and upper estimates of the pull-in parameters are denoted by small circles and small points, respectively

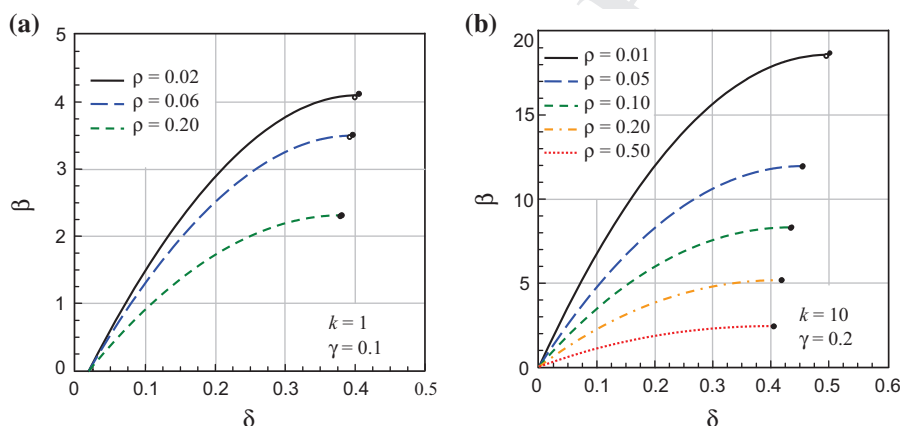


Fig. 4 Relations between electrostatic loading parameter β and tip deflection δ obtained from the shooting method, for $k = 1$ and $\gamma = 0.1$ (a) and $k = 10$ and $\gamma = 0.2$ (b), for various

geometric ratios ρ . Lower and upper estimates of the pull-in parameters are denoted by small circles and small points, respectively

465 and results are listed for three specific values of the
 466 ratio $\rho = R/L$, which denotes the inverse of the CNT
 467 slenderness, and for some specific set of the normal-
 468 ized van der Waals parameter γ defined in Eq. (4)₂.

469 In order to validate the analytical estimates provided
 470 here, the solution to the nonlinear BVP defined by
 471 Eqs. (1) and (5) has been calculated by using the
 472 numerical integration scheme available in *Mathemat-*
 473 *ica*[®], which is based on the shooting method. Fig-
 474 ures 2, 3, and 4 show the relationships between the
 475 electrostatic loading parameter β and tip deflection
 476 of the CNT, $\delta = u(1)$, obtained by using the func-
 477 tion DSolve of *Mathematica*[®], varying the geom-
 478 etric and material parameters of the CNT switch. In these

479 figures, the lower and upper analytical estimates of the
 480 pull-in parameters β_{PI} and δ_{PI} calculated by using the
 481 accurate method described in Sects. 4.1 and 4.2 are
 482 marked with small circles and points, respectively. In
 483 particular, the curves in Fig. 2 display the variation of
 484 normalized CNT tip deflection δ with the electrostatic
 485 loading parameter β obtained from the shooting
 486 method, for various values of the geometric ratio k .
 487 A slender CNT ($\rho = 0.02$) subject to weak inter-
 488 molecular surface forces ($\gamma = 10^{-5} \div 0.2$) is consid-
 489 ered therein. These results confirm that the lower and
 490 upper analytical bounds for β and δ are very close each
 491 other (for all the values of the parameter k considered
 492 here), thus ensuring extremely accurate estimates of

493 the exact pull-in parameters β_{PI} and δ_{PI} , which
 494 correspond to the maximum of the curves β versus δ
 495 obtained by the numerical integration procedure. As
 496 expected, the pull-in voltage β_{PI} is found to increase
 497 with the gap spacing H between the electrodes, which
 498 is proportional to the parameter k . The pull-in tip
 499 displacement δ_{PI} displays a non monotonic behavior
 500 as k is increased. Indeed, it grows for small values of
 501 k and then it decreases as k becomes larger. The
 502 contribution of the charge concentrated at the CNT
 503 free end has been neglected in most investigations,
 504 which thus overestimate the pull-in voltage. Actually,
 505 the pull-in voltage is significantly reduced when the
 506 contribution of the concentrated load acting at the free
 507 end is taken into account.

508 Figure 3 is similar to Fig. 2 except that it focuses on
 509 the effects of the van der Waals attractions on the pull-
 510 in parameters. The same geometric ratio $\rho = 0.02$
 511 considered in Fig. 2 has been assumed here. As the
 512 beam deflection increases and the normalized gap
 513 spacing $1-u$ decreases, the van der Waals interaction
 514 becomes stronger than the electrostatic force. Their
 515 magnitude indeed varies with the gap spacing accord-
 516 ing to the different laws introduced in (3). If the
 517 magnitude γ of the van der Waals interaction
 518 increases, then it becomes effective at larger gap
 519 spacing and, thus, both the pull-in voltage and the pull-
 520 in tip deflection are found to decrease, as it can be
 521 observed in Fig. 3a, b. These plots also confirm that
 522 the analytical lower and upper bounds for β and δ are
 523 very close each other and, thus, also to the exact pull-
 524 in parameters β_{PI} and δ_{PI} , which should lay in
 525 between.

The effects of the geometric ratio ρ on the pull-in
 parameters can be observed in Fig. 4 for two sets of
 values of γ and k . As ρ decreases, namely for slender
 CNT, the normalized pull-in voltage β_{PI} increases
 together with the corresponding normalized tip deflec-
 tion δ_{PI} . Note the effects of the CNT slenderness ratio
 ρ are more evident for large gap spacing, namely for
 $k \gg 1$ (Fig. 4b).

The variations of the van der Waals parameters γ
 with the tip displacement δ for a freestanding CNT
 cantilever ($\beta = 0$) obtained by numerical integration
 are plotted in Fig. 5 for various values of the
 geometric ratio k . If the parameter γ exceeds its
 critical value γ_{PI} , which is given by the maximum of
 the γ - δ curve obtained by numerical integration, then
 pull-in instability occurs even if no electric voltage is
 applied to the electrodes. It can be observed that the
 estimated values of γ_{PI} and the corresponding pull-in
 deflection δ_{PI} agree very well with the results of the
 numerical procedure. These plots also show that the
 critical values of van der Waals parameter is increased
 by increasing the geometric ratio k . No significant
 influence of k has been observed on the normalized
 pull-in tip deflection δ_{PI} , which turns out to be about
 constant and equal to 0.25, independently of k . Lower
 and upper estimates of critical van der Waals param-
 eter γ_{PI} and tip deflection δ_{PI} for a freestanding CNT
 can be found in Table 3 for some values of the
 geometric ratio k . There, it can be noted that a stronger
 van der Waals force is required to induce the pull-in
 instability as the normalized gap spacing k increases,
 whereas the normalized pull-in tip deflection δ is
 almost independent of k . Note that the geometric ratio
 ρ has no effect on the pull-in value γ_{PI} of the van der

Fig. 5 Relations between van der Waals parameter γ and tip deflection δ obtained from the shooting method, for a freestanding nanotube ($\beta = 0$) and for small (a) and large (b) values of the geometric ratio k . Lower and upper estimates of the pull-in parameters are denoted by small circles and small points, respectively

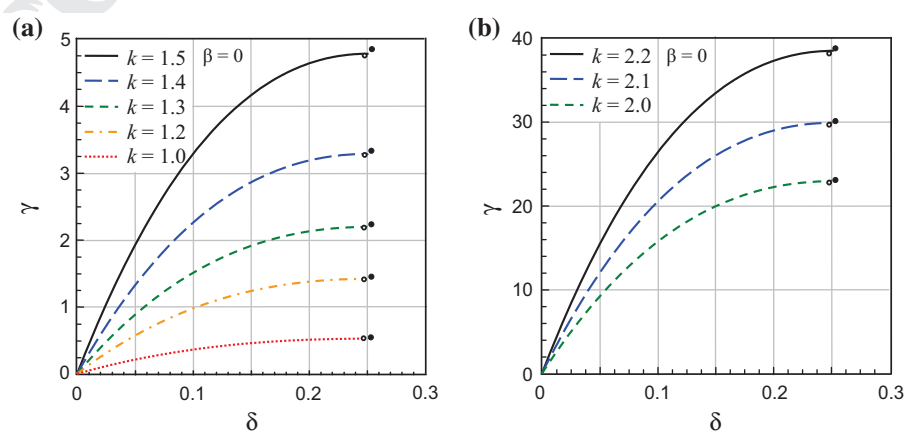


Table 3 Lower and upper bounds for the parameters γ and δ causing the pull-in instability in the absence of electrostatic actuation ($\beta = 0$) and approximated value γ_{PI}^* provided by Eq. (49), for various values of the geometric ratio k

k	γ_{PI}^*	γ_L	δ_L	γ_U	δ_U
0.2	7.16×10^{-5}	7.72×10^{-5}	0.2456	7.86×10^{-5}	0.2513
0.4	3.24×10^{-3}	3.47×10^{-3}	0.2459	3.53×10^{-3}	0.2517
0.6	3.01×10^{-2}	3.21×10^{-2}	0.2463	3.26×10^{-2}	0.2521
0.8	1.47×10^{-1}	1.55×10^{-1}	0.2467	1.58×10^{-1}	0.2525
1	5.00×10^{-1}	5.24×10^{-1}	0.2471	5.33×10^{-1}	0.2529
2	2.26×10^1	2.28×10^1	0.2485	2.32×10^1	0.2543
3	2.10×10^2	2.06×10^2	0.2485	2.09×10^2	0.2543
4	1.02×10^3	9.83×10^2	0.2477	1.00×10^3	0.2535
5	3.49×10^3	3.32×10^3	0.2465	3.38×10^3	0.2523
10	1.58×10^5	1.55×10^5	0.2406	1.58×10^5	0.2463
20	7.15×10^6	7.99×10^6	0.2345	8.14×10^6	0.2401
50	1.10×10^9	1.67×10^9	0.2294	1.70×10^9	0.2348

Fig. 6 Variation of the van der Waals parameter γ_{PI}^* with the geometric ratio k . The upper and lower values of γ causing the pull-in instability for a freestanding CNT cantilever are marked by full and empty circles, respectively

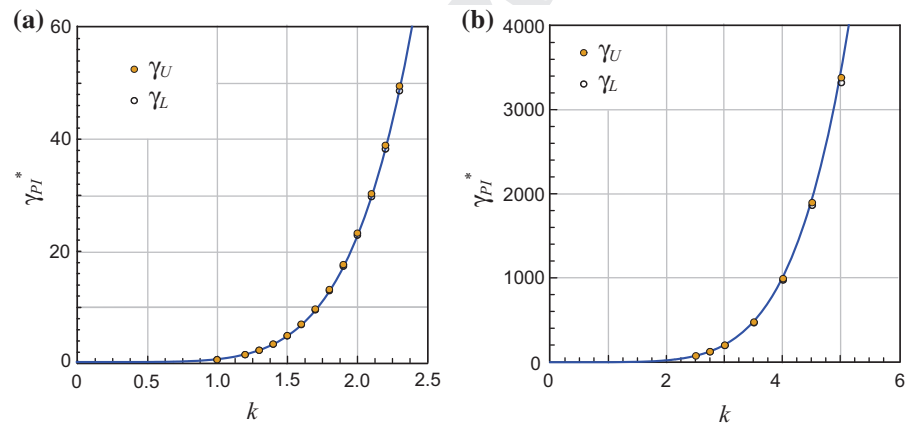
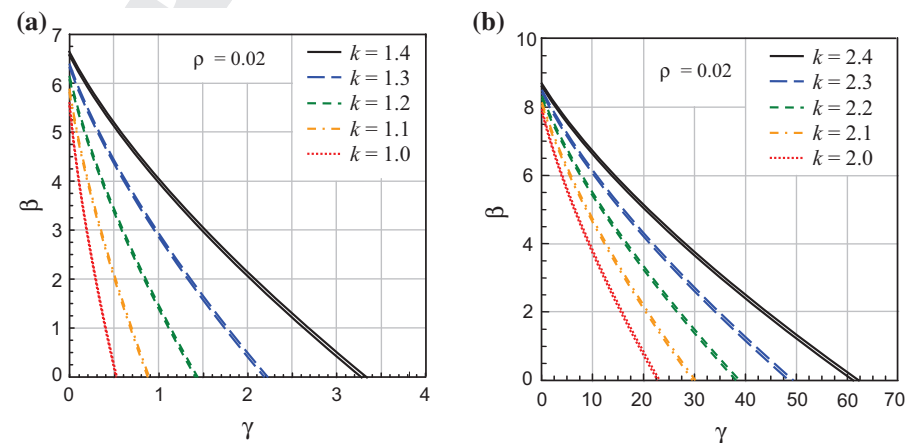


Fig. 7 Normalized variations of the pull-in voltage β with the van der Waals parameter γ , for small (a) and large (b) values of the geometric ratios k



560 Waals parameter. Indeed, according to Eq. (6) ρ
 561 affects the concentrated tip load only, which is
 562 vanishing for $\beta = 0$.

On the basis of the performed investigations a
 simple closed-form relation is proposed here for the

563
 564

565 pull-in value of the van der Waals parameter for a
 566 freestanding CNT cantilever, namely

$$\gamma_{PI}^* = \frac{1}{2}k^{11/2}. \quad (49)$$

568 The variations of γ_{PI}^* with k are plotted in Fig. 6a, b
 569 together with the upper and lower bounds γ_U and γ_L
 570 provided by the present analysis. Values of γ_{PI}^* for
 571 some specific value of k have been reported in Table 3
 572 also. In Fig. 6a, it can be noted that relation (49) fits
 573 very well the lower bounds γ_U for small values of k ,
 574 namely for $k < 2.8$, and thus it can be conveniently
 575 used for the safe design of CNT switches with a small
 576 gap spacing. Equation (49) provides accurate predic-
 577 tions also for $k > 2.8$, as it can be observed in Fig. 6b,
 578 but in this case γ_{PI}^* may result a bit larger than γ_U , as it
 579 can be noted in Table 3 for $k = 3 \div 5$. Relation (49)
 580 actually defines a minimum gap spacing H_{min} or,

equivalently, a maximum CNT length L_{max} for
 preventing the pull-in collapse of a CNT in the
 absence of electrostatic loading, namely

$$H_{min} = \left(C_6 \sigma^2 \frac{\pi^2 L^4}{EI} \right)^{2/11} R^{1/11}, \quad (50)$$

$$L_{max} = \left(\frac{EI}{C_6 \sigma^2 \pi^2} \right)^{1/4} \left(\frac{H^{11}}{R} \right)^{1/8}.$$

The variations of β_U and β_L with γ are plotted in Fig. 7
 for various value of the geometric ratio k . These
 estimates are very close each other and, thus,
 extremely accurate, for every value of the van der
 Waals parameter. Both the pull-in voltage β and the
 limit value of the coefficients γ increase as the
 parameter k is increased. In general, for assigned
 geometry, namely for given values of ρ and k , the pull-
 in voltage decreases as the strength of the van der
 Waals attractions increases. The pull-in voltage

Fig. 8 Variations of lower and upper bounds β_L and β_U of the pull-in voltage with the geometric ratio k , for $\gamma = 0$ and for various geometric ratios ρ

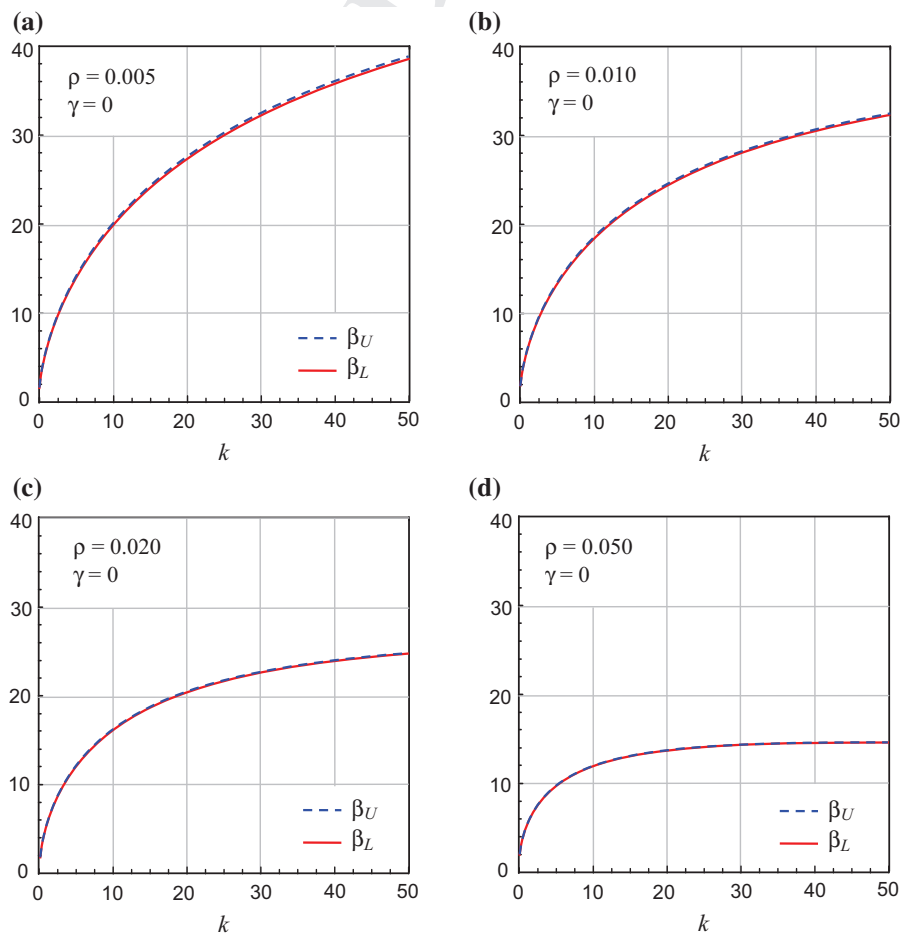


Fig. 9 Variations of δ_L and δ_U with the geometric ratio k , for $\gamma = 0$ and for two different geometric ratios ρ

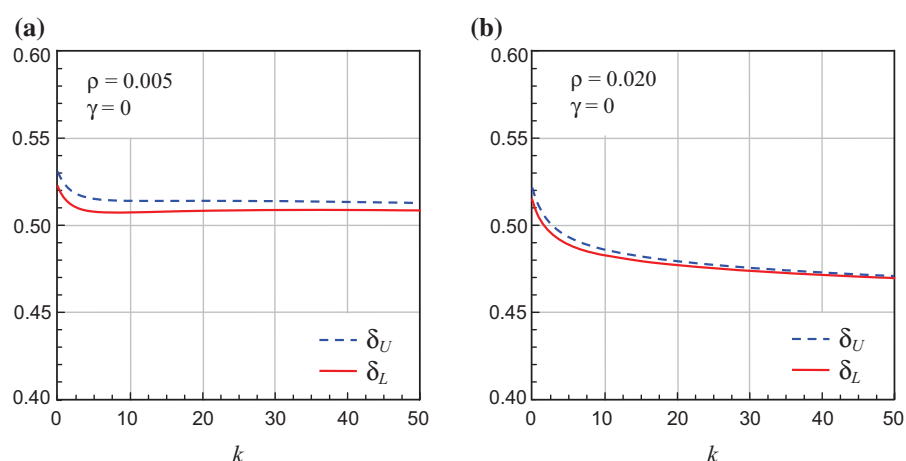
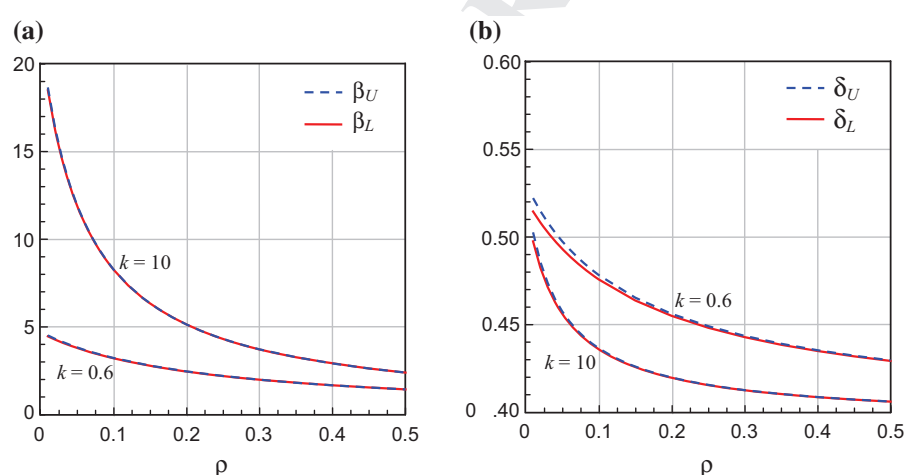


Fig. 10 Variations of β_L and β_U (a), and δ_L and δ_U (b) with the geometric ratio ρ for two different values of the geometric ratio k and $\gamma = 0$

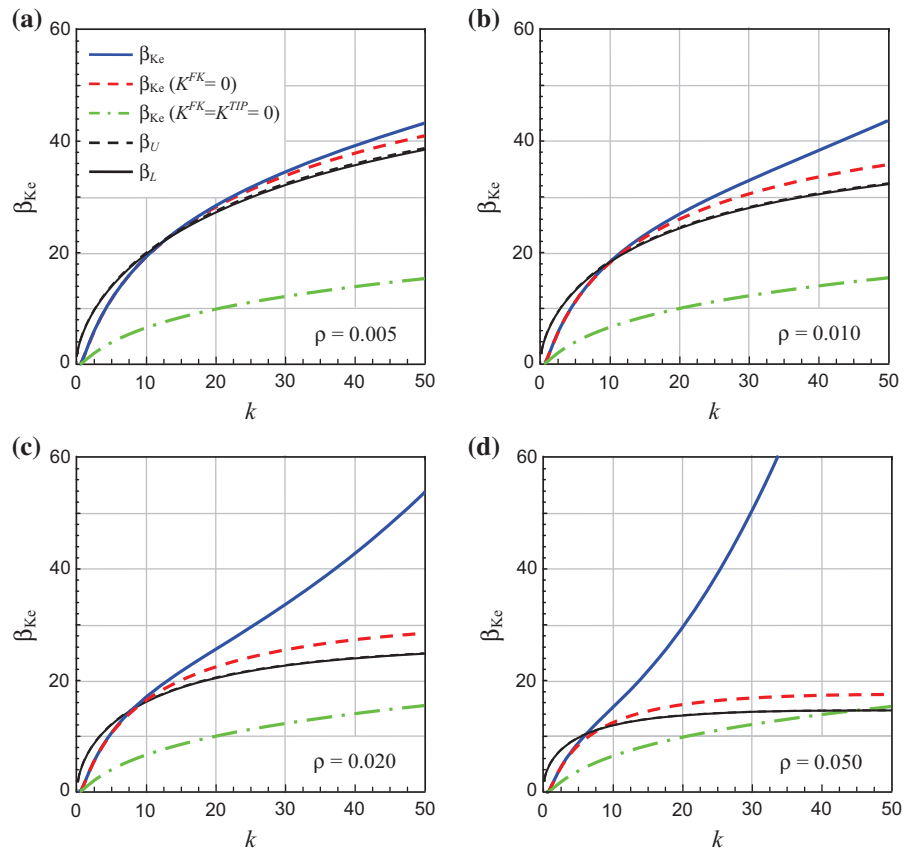


595 vanishes when the van der Waals parameter attains its
 596 critical values γ_{PI} . Negative values of β then imply that
 597 a repulsive electrostatic force is required to prevent
 598 pull-in instability induced by the van der Waals
 599 attraction when it overcomes the elastic restoring
 600 force. In this case, the CNT collapses onto and adheres
 601 to the ground plane in the absence of electrostatic
 602 actuation, due only to the van der Waals attraction that
 603 is responsible of the occurring of stiction [31]. This
 604 phenomenon is exploited in non-volatile memory
 605 cells, where the switch is hold in the closed state with
 606 no need of continued power input. The occurrence of
 607 stiction in applications such as nanoactuators,
 608 nanoresonators and nano-tweezers may instead limit
 609 the range of operability of the device and lead to
 610 undesirable consequences.

611 The variations of lower and upper bounds of the
 612 pull-in parameters β_{PI} and δ_{PI} with the geometric ratio

613 k are plotted in Figs. 8 and 9, respectively, for
 614 vanishing contribution of the van der Waals force
 615 ($\gamma = 0$). The lower and upper analytical bounds turn
 616 out to be very close each other for every value of the
 617 geometric ratio k , thus ensuring accurate estimates of
 618 the pull-in parameters. Moreover, the pull-in voltage is
 619 found to increase with the gap spacing parameter k , as
 620 expected, and it seems to approach an almost constant
 621 limit values for large k . Note that the pull-in deflection
 622 δ_{PI} display a limited variation with k so that the range
 623 of variation of the plots in Fig. 9 has been restricted
 624 between 0.4 and 0.6 to make the gap more visible. Due
 625 to the adopted graphical representation, it may seem
 626 that the predicted upper and lower pull-in deflections
 627 δ_L and δ_U in Fig. 9 are more separated than the upper
 628 and lower pull-in voltages plotted in Fig. 8, but
 629 actually the former are as close as the latter. 629

Fig. 11 Variation of approximated normalized pull-in voltage β_{Kc} with k for four values of slenderness ratio ρ : considering both effects of concentrated charge and finite kinematics (solid line), considering only the effect of concentrated charge (dashed line), neglecting both effects (dash-dotted line). The analytical predictions of the lower and upper bounds are plotted by solid and dashed black lines, respectively



630 The variations of lower and upper bounds of the
 631 pull-in parameters β_{PI} and δ_{PI} with the geometric ratio
 632 ρ are plotted in Fig. 10a, b, respectively, neglecting
 633 the contribution of the van der Waals attractions
 634 ($\gamma = 0$). It can be observed that increasing the
 635 geometric ratio ρ results in decreasing the pull-in
 636 voltage (Fig. 10a) and the normalized pull-in dis-
 637 placement (Fig. 10b). The rapid variation observed for
 638 $k = 10$, namely for large gap spacing, proves that the
 639 pull-in parameters are very sensitive to the geometric
 640 ratio ρ , especially when it is small, namely for very
 641 slender CNTs. Note the reduced range of variation
 642 considered for δ_L and δ_U in Fig. 10b.

643 According to Eq. (48), the variations of the approx-
 644 imated pull-in voltage β_{Kc} proposed by Ke et al. [14]
 645 with k are plotted in Fig. 11 for four values of ρ . In
 646 particular, the blue solid lines take into consideration
 647 both the effects of concentrated charge and finite
 648 kinematics, the red dashed lines take into considera-
 649 tion the effect of concentrated charge only, and the
 650 green dash-dotted lines neglect both effects. The

analytical predictions of the lower and upper bounds
 proposed here are plotted in the same figures by solid
 and dashed black lines, respectively. From Fig. 11 it
 can be observed that the effect of finite kinematics,
 namely the term K^{FK} , is negligible for $k < 20$ if
 $\rho = 0.005$, for $k < 15$ if $\rho = 0.01$, for $k < 10$ if
 $\rho = 0.02$, and for $k < 5$ if $\rho = 0.05$, whereas the effect
 of concentrated charge K^{TIP} can never be neglected.
 Moreover, if the effects of finite kinematics are
 neglected, relation (48) roughly approximates the
 estimates of the pull-in voltage obtained by the present
 approach. However, Eq. (48) provides estimates of the
 pull-in voltage smaller than the lower bound β_L
 for small values of k and larger than the upper bound
 β_U for large values of k .

6 Conclusions

Analytical lower and upper bounds for the pull-in
 voltage and deflection of an electro-statically actuated

669 CNT cantilever switch are proposed and then vali-
 670 dated by comparison with the results obtained from a
 671 numerical integration procedure of the governing
 672 nonlinear BVP. The combined effects of tip charge
 673 concentration and van der Waals attractions are found
 674 to reduce the pull-in voltage considerably. The upper
 675 and lower bounds are very close to the exact values,
 676 for every set of material and loading parameter
 677 considered here, thus proving the efficiency of the
 678 proposed approach. Moreover, they are found to
 679 improve the accuracy with respect to approximated
 680 relations proposed in the literature for the fast estimate
 681 of the pull-in voltage of CNT switches.

682 In conclusion, the present study can be regarded as
 683 a useful tools for the safe design of NEMS devices
 684 exploiting the smart properties of CNTs. It allows
 685 indeed for preventing unpredicted structural damage
 686 during operation, thus assuring robust and consistent
 687 performance over many actuation cycles.

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 690 acknowledged.

691 **Compliance with ethical standards**

692 **Conflict of interest** The authors declare that they have no
 693 conflict of interest.

694 **Appendix**

695 The proofs of the two lemmas used in Sect. 3 for
 696 obtaining the upper and lower bounds to the CNT
 697 deflection are given in the following. These proofs
 698 were also given in [16, 17] and are reported here for
 699 the sake of convenience.

700 **Lemma A** *Let the function $h(x)$ be continuous up to*
 701 *the third derivative for $x \in [0, 1]$ and satisfy the*
 702 *following conditions*

$$h(0) = 0, \quad h(1) = 0, \quad h'(0) = 0, \quad h''(1) = 0, \\ h'''(1) = 0, \tag{51}$$

704 *and*

$$h^V(x) \leq 0, \quad \text{for } x \in [0, 1] \tag{52}$$

706 *then*

$$h(x) \geq 0, \quad \text{for } x \in [0, 1] \tag{53}$$

Proof By using the mean value theorem, from con-
 708 tinuity and conditions (51)_{1,2} it follows that there
 709 exists $x_1 \in [0, 1]$ such that $h'(x_1) = 0$. Then, by using
 710 conditions (51)_{3,4} there exist $x_2 \in [0, x_1]$ and $x_3 \in [x_2,$
 711 $1]$ such that $h''(x_2) = 0$ and $h'''(x_3) = 0$. Since the
 712 function $h'''(x)$ is concave for $x \in [0, 1]$ according to
 713 (52), it follows that $h''(x) \leq 0$ for $x \in [x_2, 1]$ and
 714 $h''(x) \geq 0$ for $x \in [0, x_2]$. Therefore, $h'(x) \geq 0$ for
 715 $x \in [0, x_1]$ and $h'(x) \leq 0$ for $x \in [x_1, 1]$. Since
 716 $h(0) = h(1) = 0$ according to Eq. (51)_{1,2}, then it nec-
 717 essarily follows that $h(x) \geq 0$ for $x \in [0, 1]$, so that
 718 condition (53) holds true. \square 719

720 **Lemma B** *Let the function $g(x)$ be continuous up to*
 721 *the third derivative for $x \in [0, 1]$ and satisfy the fol-*
 722 *lowing conditions*

$$g(0) = 0, \quad g(1) = 0, \quad g'(0) = 0, \quad g''(1) = 0. \tag{54}$$

724 *and*

$$g^{IV}(x) \geq 0, \quad \text{for } x \in [0, 1] \tag{55}$$

726 *then*

$$g(x) \geq 0, \quad \text{for } x \in [0, 1] \tag{56}$$

Proof By using the mean value theorem, from con-
 727 ditions (54)_{1,2} it follows that there exists $x_1 \in [0, 1]$
 729 such that $g'(x_1) = 0$. Moreover, by using conditions
 730 (54)_{3,4} there exists $x_2 \in [0, x_1]$ such that $g''(x_2) = 0$.
 731 Condition (55) then implies that $g''(x)$ is convex. It
 732 follows that $g''(x) \leq 0$ for $x \in [x_2, 1]$ and $g''(x) \geq 0$
 733 for $x \in [0, x_2]$, and thus $g'(x) \geq 0$ for $x \in [0, x_1]$ and
 734 $g'(x) \leq 0$ for $x \in [x_1, 1]$. Since $g(0) = g(1) = 0$
 735 according to conditions (54)_{1,2}, then it necessarily
 736 follows that inequality (56) holds true. \square 737
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