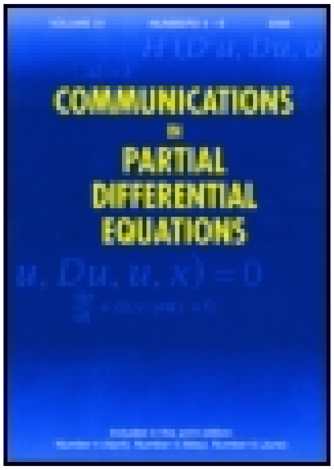

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Nader Masmoudi[a] & Federica Sani[b]
[a] Courant Institute, New York, NY, USA
[b] Università degli Studi di Milano, Milano, Italy
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Trudinger-Moser Inequalities with the Exact Growth Condition in $\mathbb{R}^N$ and Applications

Nader Masmoudi[1] and Federica Sani[2]

[1]Courant Institute, New York, NY, USA

[2]Università degli Studi di Milano, Milano, Italy

Abstract

In a recent paper (19), the authors obtained a sharp version of the Trudinger-Moser inequality in the whole space $\mathbb{R}^2$, giving necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in $W^{1,2}(\mathbb{R}^2)$. We complete this study showing that an analogue of the result in (19) holds in arbitrary dimensions $N \geq 2$. We also provide an application to the study of the existence of ground state solutions for quasilinear elliptic equations in $\mathbb{R}^N$.

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Federica Sani, Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy;

E-mail: federica.sani@unimi.it

1. INTRODUCTION AND MAIN RESULTS

Trudinger-Moser Inequalities

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $W_0^{1,p}(\Omega)$ be the usual Sobolev space obtained as the closure of $\mathscr{C}_0^\infty(\Omega)$ with respect to the L^p-Dirichlet norm, i.e.

$$\|\nabla u\|_p^p := \int_\Omega |\nabla u|^p \, dx.$$

The Sobolev embedding theorem reads as follows:

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^{p^*}(\Omega) & \text{if } 1 \leq p < N, \\ L^\infty(\Omega) & \text{if } p > N, \end{cases}$$

where $p^* := Np/(N-p)$ is the critical Sobolev exponent. In the so-called limiting Sobolev case, which occurs when $p = N$, $W_0^{1,N}(\Omega) \subset L^q(\Omega)$ for any $q \geq 1$ but, it is well know that $W_0^{1,N}(\Omega) \not\subseteq L^\infty(\Omega)$. Actually, the celebrated Trudinger-Moser inequality (proved independently by V. I Yudovich (33), S. I. Pohozaev (27) and N. S. Trudinger (30) and, later refined by J. Moser (26)) states that

$$\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leq 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} \, dx = C(\Omega, \alpha) \begin{cases} < +\infty & \text{if } \alpha \leq \alpha_N, \\ = +\infty & \text{if } \alpha > \alpha_N, \end{cases} \tag{1.1}$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface measure of the unit sphere $S^{N-1} \subset \mathbb{R}^N$. A remarkable phenomenon is that inequality (1.1) still holds for the critical value α_N itself.

The supremum in (1.1) becomes infinite, even in the case $\alpha \leq \alpha_N$, for domains with infinite measure. Therefore an interesting extension is to construct Trudinger-Moser type inequalities in the whole space $\mathbb{R}^N$. A weaker result in this direction is due to S. Adachi and K. Tanaka (1). Let

$$\phi_N(t) := e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

Theorem 1.1 ((1)). *If $N \geq 2$ then for any $\alpha \in (0, \alpha_N)$ there is a constant $C(\alpha, N) > 0$ such that*

$$\int_{\mathbb{R}^N} \phi_N(\alpha |u|^{\frac{N}{N-1}})\, dx \leq C(\alpha, N) \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N \leq 1 \tag{1.2}$$

and, this inequality is false for $\alpha \geq \alpha_N$.

We point out that the critical exponent α_N is excluded in (1.2) and the necessity of $\alpha < \alpha_N$ was proved in (1) using the sequence of test functions introduced by Moser. This is quite different from the Trudinger-Moser inequality in its original form. However, B. Ruf (28) (in the case $N = 2$) and Y. Li, B. Ruf (22) (in the case $N \geq 3$) showed that if the Dirichlet norm is replaced by the standard Sobolev norm, i.e.

$$\|u\|_{W^{1,N}}^N := \|\nabla u\|_N^N + \|u\|_N^N,$$

then the critical exponent $\alpha = \alpha_N$ becomes admissible and Moser's result can be fully extended to the whole space $\mathbb{R}^N$.

Theorem 1.2 ((28), (22)). *Let $N \geq 2$, then*

$$\sup_{u \in W^{1,N}(\mathbb{R}^N),\ \|u\|_{W^{1,N}} \leq 1} \int_{\mathbb{R}^N} \phi_N(\alpha |u|^{\frac{N}{N-1}})\, dx = C(N, \alpha) \begin{cases} < +\infty & \text{if } \alpha \leq \alpha_N, \\ = +\infty & \text{if } \alpha > \alpha_N. \end{cases} \tag{1.3}$$

In view of Theorem 1.1 and Theorem 1.2, we can say that the failure of the Trudinger-Moser inequality (1.1) in the whole space $\mathbb{R}^N$ can be recovered either by weakening the exponent α_N or by strengthening the Dirichlet norm. In (19), the authors, dealing with the 2-dimensional case, remarked that, weakening slightly the growth of the exponential nonlinearity, it is possible to preserve both the sharp exponent $\alpha_2 = 4\pi$ and the Dirichlet norm.

Theorem 1.3 ((19), Proposition 1.4). *There is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2} - 1}{(1 + |u|)^2}\, dx \leq C\|u\|_2^2 \quad \forall u \in W^{1,2}(\mathbb{R}^2) \text{with } \|\nabla u\|_2 \leq 1$$

and, this inequality fails if the power 2 in the denominator is replaced by any $p < 2$.

Obviously this last inequality implies (1.2), but the interesting fact is that it also implies (1.3). This result raised the following open question:

Does an analogue of Theorem 1.3 hold in higher dimensions $N > 2$?

As expected the answer to this question is affirmative and one of the main purposes of this paper is to show that, actually, if we keep both the conditions $\alpha = \alpha_N$ and $\|\nabla u\|_N \leq 1$ then we have

Theorem 1.4. *Let $N \geq 2$ then there is a constant $C_N > 0$ such that*

$$\int_{R^N} \frac{\phi_N(\alpha_N |u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\, dx \leq C_N \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N \leq 1. \tag{1.4}$$

Moreover, this inequality fails if the power $\frac{N}{N-1}$ in the denominator is replaced by any $p < \frac{N}{N-1}$.

As in the 2-dimensional case, this last inequality implies both (1.2) and (1.3).

Remark 1.1. We mention that the extension of Theorem 1.3 to the higher order Sobolev space $W^{2,2}(\mathbb{R}^4)$ has been obtained in (25). It will be clear from the proof of (1.4) that, a suitable combination of the arguments proposed in the present paper together with the ideas used in (25) enables to obtain an analogue inequality also in the space $W^{2,\frac{N}{2}}(\mathbb{R}^N)$ with $N > 2$.

We point out that in (19) the authors obtained not only a precised version of the Trudinger-Moser inequality in the whole plane $\mathbb{R}^2$, but necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in $W^{1,2}(\mathbb{R}^2)$. In this way, they completely determined the growth order, not only among exponential growth functionals. Arguing as in (19), we obtain the complete generalization to the higher-dimensional case of the above mentioned result (see (19), Theorem 1.5).

Theorem 1.5 (Boundedness). *Let* $N \geq 2$. *Let* $g : \mathbb{R} \to [0, +\infty)$ *be any Borel function and define the functional* G *as*

$$G(u) := \int_{\mathbb{R}^N} g(u(x))\, dx.$$

Then for any $K > 0$ *the following conditions are equivalent*:

(1) $\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) < +\infty$ *and* $\limsup_{|t|\to 0} |t|^{-N} g(t) < +\infty$.

(2) *There exists a constant* $C_{N,g,K} > 0$ *such that*

$$G(u) \leq C_{N,g,K} \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N^N \leq \omega_{N-1} K.$$

Theorem 1.6 (Compactness). *Let* $N \geq 2$. *Let* $g : \mathbb{R} \to [0, +\infty)$ *be any a.e.-continuous function and define the functional* G *as*

$$G(u) := \int_{\mathbb{R}^N} g(u(x))\, dx.$$

Then for any $K > 0$ *the following conditions are equivalent*:

(3) $\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) = 0$ *and* $\limsup_{|t|\to 0} |t|^{-N} g(t) = 0$.

(4) *For any sequence* $\{u_n\}_{n\geq 1} \subset W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ *satisfying* $\|\nabla u_n\|_N^N \leq \omega_{N-1} K$ *and weakly converging to some* $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)$, *we have that* $G(u_n) \to G(u)$.

Theorem 1.5 includes Theorem 1.4 as a particular case, in fact it suffices to take $\omega_{N-1}K = 1$, moreover Theorem 1.6 shows that for functionals behaving like the one appearing in (1.4) we have a loss of compactness.

The proof of Theorem 1.5 and Theorem 1.6 follows the arguments introduced in (19). In Section 2, we explicitly exhibits sequences of test functions constructed to prove the necessity of conditions (1) and (3). More precisely, we show that if (1) fails then there exists a sequence $\{u_n\}_{n\geq 1} \subset W^{1,N}(\mathbb{R}^N)$ such that

$$\|\nabla u_n\|_N^N \leq \omega_{N-1}K \quad \forall n \geq 1 \quad \text{and} \quad \|u_n\|_N \to 0,\ G(u_n) \to +\infty \quad \text{as } n \to +\infty,$$

while if (3) fails then there exists a sequence $\{u_n\}_{n\geq 1} \subset W^{1,N}_{\text{rad}}(\mathbb{R}^N)$, satisfying $\|\nabla u_n\|_N^N \leq \omega_{N-1}K$ and weakly converging to 0 in $W^{1,N}(\mathbb{R}^N)$, such that $G(u_n) > \delta$ for some $\delta > 0$. In Section 3, we obtain an exponential version of the radial Sobolev inequality expressing the optimal growth of radial functions in the exterior of balls when the L^N-norm and the Dirichlet norm are given (see Theorem 3.2). In order to obtain this optimal descending growth condition, we argue as in (19) reducing the problem to a discrete version. Section 4 and Section 5 are devoted to the proof of the sufficiency of conditions (1) and (3) respectively. While the proof of the sufficiency of condition (3) again follows (19), when we show the sufficiency of (1) we avoid the reduction of the problem to a discrete version and we propose an alternative proof. In fact, exploiting the arguments introduced by B. Ruf in (28) and applying the optimal descending growth condition obtained in Section 3, we show that the problem can be solved using the classical Trudinger-Moser inequality (1.1) for bounded domains in $\mathbb{R}^N$. Finally, in Section 6, we prove that inequality (1.4) implies (1.3).

Ground State Solutions for Quasilinear Equations in $\mathbb{R}^N$

The above Trudinger-Moser inequalities play an important role in nonlinear analysis. Let us consider the following quasilinear equation

$$-\Delta_N u + c|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N,\ N \geq 2, \tag{1.5}$$

where Δ_N is the N-Laplacian operator, i.e. $\Delta_N u := \text{div}\,(|\nabla u|^{N-2}\nabla u)$, and $c > 0$ is a positive constant. The research of ground state solutions for problems of the form (1.5) is crucial in several applications to the study of evolution equations of N-Laplacian type that appear in non-Newtonian fluids, turbulent flows in porus media and other contexts.

In view of (1.3), the maximal growth on the nonlinear term f which allows to treat equation (1.5) variationally in $W^{1,N}(\mathbb{R}^N)$ is of exponential type and is given by functions $f : \mathbb{R} \to \mathbb{R}$ behaving as $e^{\alpha_0|u|^{N/(N-1)}}$ at infinity, more precisely

$$\lim_{|t|\to+\infty} \frac{f(t)}{e^{\alpha|t|^{N/(N-1)}}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0, \end{cases}$$

for some $\alpha_0 > 0$. Differently from the case of quasilinear problems on bounded domains (see (9; 2; 3; 12; 11)), in the study of quasilinear problems on the whole space $\mathbb{R}^N$ one should be warned that the loss of compactness in $W^{1,N}(\mathbb{R}^N)$ can be produced not only by concentration phenomena but also by vanishing phenomena. In fact, in the case when the nonlinearity f has exponential growth, the functional associated to a variational approach of problem (1.5) reveals a lack of compactness due to the critical behavior of the nonlinearity and to the unboundedness of the domain $\mathbb{R}^N$: at certain levels the Palais-Smale compactness condition fails due to concentration phenomena and to the leak of the L^N-norm to infinity, i.e. to vanishing phenomena.

Recently, there has been considerable interest in the study of existence results for equations of the form

$$-\Delta_N u + V(x)|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N,\ N \geq 2, \tag{1.6}$$

where the nonlinear term f has an exponential behavior at infinity and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is bounded away from zero, i.e.

$$V(x) \geq c > 0 \quad x \in \mathbb{R}^N.$$

If V is large at infinity in some suitable sense, then the loss of compactness due to the unboundedness of the domain $\mathbb{R}^N$ can be overcome and vanishing phenomena can be ruled out. In fact, a natural framework for the function space setting of problem (1.6) is given by the subspace E of $W^{1,N}(\mathbb{R}^N)$ defined as

$$E := \left\{ u \in W^{1,N}(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} V(x)|u|^N\,dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_E := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^N + V(x)|u|^N \right) dx \right)^{\frac{1}{N}} \quad \forall u \in E.$$

Under appropriate assumptions on the potential V, the embedding

$$E \hookrightarrow L^p(\mathbb{R}^N) \tag{1.7}$$

turns out to be compact. For instance, if

$$V^{-1} \in L^{\frac{1}{N-1}}(\mathbb{R}^N) \tag{1.8}$$

then the embedding (1.7) is compact for any $p \geq 1$ (see e.g. (31, Lemma2.4)), while assuming the weaker condition

$$V^{-1} \in L^1(\mathbb{R}^N) \tag{1.9}$$

the embedding (1.7) is compact only for any $p \geq N$ (see (10)).

The authors of (13; 16; 17; 5; 4; 14; 31; 21), considering a potential V satisfying (1.8) or (1.9), obtained existence results for equations of the form (1.6) and even more general equations. However, the arguments of their proofs depend crucially on the compact embeddings (1.7) given by (1.8) and (1.9), and in particular on the compact embedding of E into $L^N(\mathbb{R}^N)$.

In the case when the potential V is constant, i.e. $V(x) = c$ for any $x \in \mathbb{R}^N$, there is a long way to go yet. The natural space for a variational treatment of (1.5) is the whole space $W^{1,N}(\mathbb{R}^N)$ and it is well known that the embedding

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N)$$

is continuous but not compact, even if we restrict out attention to the radial case. In the literature, up to our knowledge, there are only few existence results concerning the study of problem (1.5) by means of variational methods. We refer the reader to the papers (8), (6), (29), (18), (19) and the references therein for the semilinear case $N = 2$. In order to overcome the possible failure of the Palais-Smale compactness condition, there is a common approach that relate (6), (29), (18) and (19), and it involves a constrained minimization problem through the Pohozaev identity. Combining the Trudinger-Moser inequality with the exact growth (Theorem 1.5 and Theorem 1.6) with the arguments in (18) and (19), our aim is to obtain the existence of ground state solutions for equations of the form (1.5) in the general case $N \geq 3$. This will be done in Section 7, where we will prove the following result.

Theorem 1.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* $f(0) = 0$ *and*

$$\exists \mu > N \quad \text{such that } 0 < \mu F(t) := \mu \int_0^t f(s)\, ds \leq t f(t) \quad \forall t \in \mathbb{R}\backslash\{0\}, \tag{f_1}$$

$$\exists t_0, M_0 > 0 \quad \text{such that } F(t) \leq M_0 f(t) \quad \forall t \geq t_0, \tag{f_2}$$

$$\lim_{t\to+\infty} \frac{f(t)}{e^{\alpha t^{N/(N-1)}}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0. \end{cases} \tag{f_2}$$

If

$$\lim_{t\to+\infty} \frac{t^{N/(N-1)} F(t)}{e^{\alpha_0 t^{N/(N-1)}}} = +\infty$$

then, for each $c > 0$, equation (1.5) *admits a positive radial solution $u \in W^{1,N}(\mathbb{R}^N)$ which has the least energy among all the solutions of* (1.5).

We complete this paper proposing, in Section 8 (see also Theorem 7.4), other sufficient conditions for the existence of ground states solutions for equation (1.5).

Notations

We will write $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C > 0$ depending only on the dimension N. We will also write $A \sim B$ to denote that $A \lesssim B$ and that $B \lesssim A$. Finally, we will always denote by $B_R \subset \mathbb{R}^N$ the ball of radius $R > 0$ centered at 0, i.e.

$$B_R := \{x \in \mathbb{R}^N \mid |x| \leq R\}.$$

2. NECESSITY OF (1) AND (3): COUNTEREXAMPLES

In order to prove the necessity of (1), we show that if (1) fails then there exists a sequence $\{u_n\}_{n\geq 1} \subset W^{1,N}(\mathbb{R}^N)$ such that

$$\|\nabla u_n\|_N^N \leq \omega_{N-1} K \ \forall n \geq 1 \quad \text{and} \quad \|u_n\|_N \to 0,\ G(u_n) \to +\infty \text{as } n \to +\infty.$$

Similarly, in order to prove the necessity of (3), we show that if (3) fails then there exists a sequence $\{u_n\}_{n\geq 1} \subset W^{1,N}_{\text{rad}}(\mathbb{R}^N)$, satisfying $\|\nabla u_n\|_N^N \leq \omega_{N-1} K$ and weakly converging to 0 in $W^{1,N}(\mathbb{R}^N)$, such that $G(u_n) > \delta$ for some $\delta > 0$.

First we consider the much easier case with the failure of the conditions (1) and (3) at the origin. Let $\{u_n\}_{n\geq 1} \subset W^{1,N}(\mathbb{R}^N)$ be a sequence of spherically symmetric functions defined by

$$u_n(x) := \begin{cases} a_n & \text{if } 0 \leq |x| \leq R_n, \\ a_n(1 - |x| + R_n) & \text{if } R_n < |x| \leq R_n + 1, \\ 0 & \text{if } |x| > R_n + 1, \end{cases}$$

where $\{a_n\}_{n\geq 1}$, $\{R_n\}_{n\geq 1}$ are sequences of positive real numbers to be chosen and satisfying $a_n \to 0$, $R_n \to +\infty$ as $n \to +\infty$. We have that

$$\|u_n\|_N^N \lesssim a_n^N R_n^N, \quad \|\nabla u_n\|_N^N \lesssim a_n^N R_n^{N-1} \quad \text{and} \quad G(u_n) \geq \frac{\omega_{N-1}}{N} g(a_n) R_n^N.$$

If (1) is violated by

$$\limsup_{|t|\to 0} |t|^{-N} g(t) = +\infty,$$

then there exists a sequence $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, $a_n \to 0$, such that $a_n^{-N} g(a_n) \to +\infty$. Let $\{b_n\}_{n\geq 1} \subset \mathbb{R}^+$, $b_n \to +\infty$, be such that $a_n^{-N} g(a_n) \geq b_n$ and choose

$$R_n := a_n^{-\frac{1}{N}} + a_n^{-1} b_n^{-\frac{1}{2N}}.$$

Then $R_n \to +\infty$, $a_n R_n \to 0$, $a_n^N R_n^{N-1} \to 0$ and

$$G(u_n) \geq \frac{\omega_{N-1}}{N} b_n a_n^N R_n^N \to +\infty.$$

If (3) is violated by

$$\limsup_{|t|\to 0} |t|^{-N} g(t) > 0,$$

then there exist a sequence $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, $a_n \to 0$ and a constant $\delta > 0$ such that $g(a_n) \geq \delta a_n^N$. Choosing $R_n := a_n^{-1}$, we have that $R_n \to +\infty$, $a_n R_n = 1$, $a_n^N R_n^{N-1} \to 0$ and

$$G(u_n) \geq \frac{\omega_{N-1}}{N} \delta a_n^N R_n^N = \frac{\omega_{N-1}}{N} \delta > 0.$$

It remains to consider the case when the conditions (1) and (3) fail at infinity. Let $\{b_n\}_{n\geq 1} \subset \mathbb{R}^+$, $b_n \to +\infty$, and $\{K_n\}_{n\geq 1} \subset \mathbb{R}^+$, $K_n \uparrow K$, be such that

$$\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) = \limsup_{n\to+\infty} c_n$$

where

$$c_n := b_n^{\frac{N}{N-1}} e^{-\frac{N}{K_n^{1/(N-1)}} b_n^{\frac{N}{N-1}}} g(b_n).$$

Also, define

$$R_n := e^{-\frac{1}{K_n^{1/(N-1)}} b_n^{\frac{N}{N-1}}},$$

so that $c_n = b_n^{\frac{N}{N-1}} R_n^N g(b_n)$. Now, we consider the so-called *Moser's sequence* $\{\phi_n\}_{n\geq 1} \subset W^{1,N}(\mathbb{R}^N)$ consisting of spherically symmetric functions defined by

$$\phi_n(x) := \begin{cases} b_n & \text{if } 0 \leq |x| \leq R_n, \\ b_n \dfrac{|\log|x||}{|\log R_n|} & \text{if } R_n < |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

We have that

$$\|\phi_n\|_N^N \lesssim \frac{b_n^N}{|\log R_n|^N} = \frac{K_n^{\frac{N}{N-1}}}{b_n^{\frac{N}{N-1}}}, \quad \|\nabla\phi_n\|_N^N = \omega_{N-1} K_n \leq \omega_{N-1} K$$

and

$$G(\phi_n) \geq \frac{\omega_{N-1}}{N} g(b_n) R_n^N = \frac{\omega_{N-1}}{N} \frac{c_n}{b_n^{\frac{N}{N-1}}}.$$

Moreover, if $\{S_n\}_{n\geq 1} \subset \mathbb{R}^+$ and we consider the sequence $\{u_n\}_{n\geq 1} \subset W^{1,N}(\mathbb{R}^N)$ with $u_n(x) := \phi_n(x/S_n)$ then

$$\|u_n\|_N^N = S_n^N \|\phi_n\|_N^N \lesssim \frac{S_n^N K_n^{\frac{N}{N-1}}}{b_n^{\frac{N}{N-1}}}, \quad \|\nabla u_n\|_N^N = \|\nabla \phi_n\|_N^N \leq \omega_{N-1} K$$

and

$$G(u_n) = S_n^N G(\phi_n) \geq \frac{\omega_{N-1}}{N} \frac{S_n^N c_n}{b_n^{\frac{N}{N-1}}}.$$

Assume that the condition (1) fails at infinity, namely

$$\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) = \limsup_{n\to+\infty} c_n = +\infty,$$

and let $\{S_n\}_{n\geq 1} \subset \mathbb{R}^+$ be such that

$$S_n = o\big(b_n^{\frac{1}{N-1}}\big) \quad \text{and} \quad \frac{S_n^N c_n}{b_n^{\frac{N}{N-1}}} \to +\infty.$$

Then by construction

$$\|\nabla u_n\|_N^N \leq \omega_{N-1} K, \quad \forall n \geq 1 \quad \text{and} \quad \|u_n\|_N \to 0,\ G(u_n) \to +\infty \quad \text{as } n \to +\infty.$$

Now, assume that the condition (3) fails at infinity, namely

$$\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) = \limsup_{n\to+\infty} c_n > 0.$$

Choosing $S_n = b_n^{\frac{1}{N-1}}$, the corresponding sequence $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,N}(\mathbb{R}^N)$, in fact

$$\|u_n\|_N^N \lesssim K_n^{\frac{N}{N-1}} \leq K^{\frac{N}{N-1}} \quad \text{and} \quad \|\nabla u_n\|_N^N \leq \omega_{N-1}K.$$

Moreover $u_n \to 0$ a.e. in $\mathbb{R}^N$ and

$$G(u_n) \geq \frac{\omega_{N-1}}{N} c_n \geq \frac{\omega_{N-1}}{N}\delta$$

for some $\delta > 0$.

3. OPTIMAL DESCENDING GROWTH CONDITION

In (19), the authors obtained the following exponential version of the radial Sobolev inequality expressing the optimal growth of radial functions in the exterior of balls when the L^2-norm and the Dirichlet norm are given.

Theorem 3.1 ((19), Theorem 3.1)**.** *There exists a constant $C > 0$ such that for any $u \in W^{1,2}_{\mathrm{rad}}(\mathbb{R}^2)$ satisfying $u_r \leq 0 \leq u$, $u(R) > 1$ and*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2\setminus B_R)} \leq 2\pi K,$$

for some R, $K > 0$, we have

$$\frac{e^{\frac{2}{K}u^2(R)}}{u^2(R)} K^2 R^2 \leq C\|u\|^2_{L^2(\mathbb{R}^2\setminus B_R)}.$$

In this section, we show that an analogue Theorem 3.1 still holds in any higher dimension $N > 2$. This optimal descending growth condition will enable us to reduce the proof of the inequality expressed by condition (2) of Theorem 1.5 to a simple application of the Trudinger-Moser inequality for bounded domains in $\mathbb{R}^N$.

Theorem 3.2. *There exists a constant* $C_N > 0$ *such that for any* $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ *satisfying* $u_r \leq 0 \leq u$, $u(R) > 1$ *and*

$$\|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_R)} \leq \omega_{N-1} K,$$

for some $R, K > 0$, *we have*

$$\frac{e^{\frac{N}{K^{1/(N-1)}} u^{\frac{N}{N-1}}(R)}}{u^{\frac{N}{N-1}}(R)} K^{\frac{N}{N-1}} R^N \leq C_N \|u\|^N_{L^N(\mathbb{R}^N \setminus B_R)}.$$

In the above estimate, the function

$$\frac{e^{u^{\frac{N}{N-1}}}}{u^{\frac{1}{N-1}}}$$

is optimal. In fact,

Theorem 3.3. *Let*

$$\mu(h) := \inf\left\{ \|u\|_{L^N(\mathbb{R}^N \setminus B_1)} \,\middle|\, u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N),\ u_r \leq 0 \leq u,\ u(1) = h,\ \|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_1)} \leq \omega_{N-1} \right\}.$$

For any $h > 1$, *we have*

$$\mu(h) \sim \frac{e^{h^{\frac{N}{N-1}}}}{h^{\frac{1}{N-1}}}.$$

We can notice that Theorem 3.2 follows from Theorem 3.3 by rescaling. In order to prove Theorem 3.3, we consider the discrete version and, following the arguments introduced in (19), we define

$$\mu_d(h) := \inf\big\{\|a\|_{(e)} \;\big|\; \|a\|_1 = h,\ \|a\|_N \le 1\big\},$$

where for any sequence $a := \{a_k\}_{k\ge 0}$

$$\|a\|_p^p := \sum_{k=0}^{+\infty} |a_k|^p \quad \text{and} \quad \|a\|_{(e)}^N := \sum_{k=0}^{+\infty} |a_k|^N e^{Nk}.$$

Lemma 3.4. *For any* $h > 1$, *we have that*

$$\mu_d(h) \sim \frac{e^{h^{\frac{N}{N-1}}}}{h^{\frac{1}{N-1}}}.$$

Proof. Since $\mu_d(h)$ is increasing in h, it suffices to show that

$$\mu_d(k^{\frac{N-1}{N}}) \sim \frac{e^k}{k^{\frac{1}{N}}}$$

for any positive integer k.

If we consider the sequence $a := \{a_j\}_{j\geq 0}$ defined by

$$a_j := \begin{cases} \frac{1}{k^{\frac{1}{N}}} & \text{if } j \in \{0, 1, \ldots, k-1\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

then it is easy to see that

$$\mu_d(k^{\frac{N-1}{N}}) \lesssim \frac{e^k}{k^{\frac{1}{N}}}.$$

Therefore, to complete the proof, it remains to show that, for any positive integer k,

$$\mu_d(k^{\frac{N-1}{N}}) \gtrsim \frac{e^k}{k^{\frac{1}{N}}}.$$

We argue by contradiction assuming that for any $0 < \varepsilon << 1$ there exist a positive integer k and a sequence $a := \{a_j\}_{j\geq 0}$ satisfying

$$\|a\|_N \leq 1, \quad \|a\|_1 = k^{\frac{N-1}{N}} \quad \text{and} \quad \|a\|_{(e)} \leq \frac{\varepsilon e^k}{k^{\frac{1}{N}}}.$$

Using the upper bound of $\|a\|_{(e)}$, we can estimate each term $|a_j|$ as follows:

$$|a_j| \leq \frac{\varepsilon e^{k-j}}{k^{\frac{1}{N}}} \quad \forall j \geq 0. \tag{3.2}$$

Now the idea is to consider the truncated sequence $a' := \{a'_j\}_{j\geq 0}$ defined as

$$a'_j := \begin{cases} a_j & \text{if } j \in \{0, 1, \ldots, k-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

From (3.2), we deduce the following lower bound for $\|a'\|_1$

$$\|a'\|_1 = \|a\|_1 - \sum_{j=k}^{+\infty} |a_j| \geq k^{\frac{N-1}{N}} - \frac{\varepsilon}{k^{\frac{1}{N}}} \sum_{j=k}^{+\infty} e^{k-j} \geq k^{\frac{N-1}{N}} - C_1 \frac{\varepsilon}{k^{\frac{1}{N}}} \tag{3.3}$$

where C_1 is a positive constant independent of ε, k and a.

In order to obtain an upper bound for $\|a'\|_1$, we recall the following version of Hölder's inequality for sums which takes into account a difference defect (see (23), Inequality (3.3), see also (24), Example 7)

$$\left(\left|\sum_{j=0}^{k-1} c_j\right|^2 + \sum_{0\le i<j\le k-1} |c_i - c_j|^2\right)^{\frac{1}{2}} \le k^{\frac{N-1}{N}}\left(\sum_{j=0}^{k-1} |c_j|^N\right)^{\frac{1}{N}} \tag{3.4}$$

where $c_0,\ c_1,\ \ldots,\ c_{k-1} \in \mathbb{R}$. Applying (3.4) and recalling that $\|a\|_N \le 1$, we get

$$\begin{aligned}\|a'\|_1^2 = \left(\sum_{j=0}^{k-1} |a_j|\right)^2 &\le k^{2\frac{N-1}{N}}\left(\sum_{j=0}^{k-1} |a_j|^N\right)^{\frac{2}{N}} - \sum_{0\le i<j\le k-1} \big||a_i| - |a_j|\big|^2 \\ &\le k^{2\frac{N-1}{N}} - \sum_{0\le i<j\le k-1} \big||a_i| - |a_j|\big|^2.\end{aligned} \tag{3.5}$$

In particular, combining (3.3) with (3.5), we can deduce the following estimate of the defect

$$\sum_{0\le i<j\le k-1} \big||a_i| - |a_j|\big|^2 \le 2C_1\varepsilon k^{\frac{N-2}{N}}. \tag{3.6}$$

Now, let $0 \le m \le k-1$ be such that $|a_m| := \min_{0\le j\le k-1} |a_j|$, using Hölder's inequality for sums and (3.6) we get

$$\|a'\|_1 - k|a_m| = \sum_{j=0}^{k-1}(|a_j| - |a_m|) \le \sqrt{k}\left(\sum_{j=0}^{k-1}\big||a_j| - |a_m|\big|^2\right)^{\frac{1}{2}} \le \sqrt{2C_1\varepsilon}\,k^{\frac{N-1}{N}}.$$

From this last inequality and from (3.3), it follows that

$$|a_m| \ge \frac{1}{k}\|a'\|_1 - \frac{\sqrt{2C_1\varepsilon}}{k^{\frac{1}{N}}} \ge \frac{1}{k^{\frac{1}{N}}}\left(1 - C_1\frac{\varepsilon}{k} - \sqrt{2C_1\varepsilon}\right) \ge \frac{1}{k^{\frac{1}{N}}}\left(1 - C_1\varepsilon - \sqrt{2C_1\varepsilon}\right) \gtrsim \frac{1}{k^{\frac{1}{N}}}$$

provided that $\varepsilon > 0$ is sufficiently small. Consequently,

$$\|a\|_{(e)} \geq (|a_{k-1}|^N e^{N(k-1)})^{\frac{1}{N}} \geq \frac{|a_m|e^k}{e} \gtrsim \frac{e^k}{k^{\frac{1}{N}}}$$

which is a contradiction. □

In view of Lemma 3.4, we can notice that Theorem 3.3 (and hence Theorem 3.2) follows from the following result.

Lemma 3.5. *For any $h > 1$, we have $\mu(h) \sim \mu_d(h)$.*

Proof. Let $h > 1$.

In order to show that $\mu_d(h) \lesssim \mu(h)$, let $u \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ be such that $u_r \leq 0 \leq u$, $u(1) = h$ and

$$\|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_1)} \leq \omega_{N-1}.$$

Let $h_k := u(e^k)$ and $a_k := h_k - h_{k+1} \geq 0$. Then by construction $\|a\|_1 = h_0 = u(1) = h$ and, applying Hölder's inequality, we get

$$\begin{aligned}\|a\|_N^N &= \sum_{k=0}^{+\infty}[u(e^k) - u(e^{k+1})]^N \\ &= \sum_{k=0}^{+\infty}\Big(\int_{e^k}^{e^{k+1}} -u_r\,dr\Big)^N \leq \sum_{k=0}^{+\infty}\int_{e^k}^{e^{k+1}} |u_r|^N r^{N-1}\,dr \\ &= \int_1^{+\infty} |u_r|^N r^{N-1}\,dr = \frac{1}{\omega_{N-1}}\|\nabla u\|^N_{L^N(\mathbb{R}^N\setminus B_1)} \leq 1. \qquad (3.7)\end{aligned}$$

Moreover

$$\|u\|_{L^N(\mathbb{R}^N\setminus B_1)}^N = \omega_{N-1}\sum_{k=0}^{+\infty}\int_{e^k}^{e^{k+1}} u^N r^{N-1}\,dr = \omega_{N-1}\frac{1-e^{-1}}{N}\sum_{k=0}^{+\infty} u^N(e^{k+1})e^{N(k+1)}$$
$$= \omega_{N-1}\frac{1-e^{-1}}{N}\sum_{k=1}^{+\infty} h_k^N e^{Nk} \geq \omega_{N-1}\frac{1-e^{-1}}{N}\sum_{k=1}^{+\infty} a_k^N e^{Nk},$$

from which we deduce that

$$\|a\|_{(e)}^N = a_0^N + \sum_{k=1}^{+\infty} a_k^N e^{Nk} \lesssim h_0^N + \|u\|_{L^N(\mathbb{R}^N\setminus B_1)}^N.$$

If we prove that

$$h_0^N \lesssim \|u\|_{L^N(\mathbb{R}^N\setminus B_1)}^N \tag{3.8}$$

then we can conclude that $\mu_d(h) \lesssim \mu(h)$. In order to show that (3.8) holds, it suffices to notice that for $1 < s < e^\alpha$ with $\alpha := 2^{\frac{1-N}{N}}$ we have

$$h_0 - u(s) = \int_1^s -u_r\,dr \leq \left(\int_1^{e^\alpha} |u_r|^N r^{N-1}\,dr\right)^{\frac{1}{N}}\left(\int_1^{e^\alpha}\frac{1}{r}\,dr\right)^{\frac{N-1}{N}} \leq \alpha^{\frac{N}{N-1}} = \frac{1}{2} < \frac{h_0}{2},$$

namely $h_0/2 \leq u(s)$ for $1 < s < e^\alpha$. Consequently

$$\|u\|_{L^N(\mathbb{R}^N\setminus B_1)}^N \geq \omega_{N-1}\int_1^{e^\alpha} u^N r^{N-1}\,dr \geq \omega_{N-1}\frac{e^{\alpha N}-1}{N}\frac{h_0^N}{2^N},$$

which is the desired estimate.

To complete the proof, it remains to show that $\mu(h) \lesssim \mu_d(h)$. Normally, we would like to prove that given a sequence $a = \{a_k\}_{k\geq 0}$ such that $\|a\|_1 = h$, $\|a\|_N \leq 1$, we can find a $u \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_r \leq 0 \leq u$, $u(1) = h$, $\|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_1)} \leq \omega_{N-1}$ and $\|u\|_{L^N(\mathbb{R}^N \setminus B_1)} \leq C\|a\|_{(e)}$. Actually, since from Lemma 3.4, we know that

$$\mu_d(h) \sim \frac{e^{h^{\frac{N}{N-1}}}}{h^{\frac{1}{N-1}}},$$

it is enough to consider a sequence a which is close to the infimum $\mu_d(h)$, as in (3.1), and optimize the energy. So let k be a positive integer satisfying

$$(k-1)^{\frac{N-1}{N}} \leq h \leq k^{\frac{N-1}{N}},$$

we consider the sequence $a := \{a_j\}_{j \geq 0}$ with

$$a_j := \begin{cases} \dfrac{h}{k} & \text{if } j \in \{0, 1, \ldots, k-1\}, \\ 0 & \text{otherwise .} \end{cases}$$

We will first define $u(e^j)$ for any integer $j \geq 0$. Let $u(1) = h_0 := h$ and for $j \geq 0$, we define $u(e^{j+1}) = h_{j+1} := h_j - a_j$. It is clear that the sequence h_j is nonincreasing and that h_j goes to zero when j goes to infinity. Optimizing $\|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_1)}$, we see that we should take

$$u(r) = a_j|\log(\mathrm{e}^{-j-1}r)| + h_{j+1} \quad (\mathrm{e}^j \leq r \leq \mathrm{e}^{j+1}). \tag{3.9}$$

In particular this yields an equality in the first inequality in (3.7) and hence we deduce that $\|\nabla u\|^N_{L^N(\mathbb{R}^N \setminus B_1)} \leq \omega_{N-1}$.

Finally, we have

$$\|u\|^N_{L^N(\mathbb{R}^N \setminus B_1)} = \omega_{N-1}\frac{e^N - 1}{N}\sum_{j=0}^{k-1} h_j^N e^{Nj} = \omega_{N-1}\frac{e^N - 1}{N}\sum_{j=0}^{k-1}\frac{h^N}{k^N}(k-j)^N e^{Nj}$$

$$\sim \frac{h^N}{k^N} e^{Nk} \sum_{l=1}^{k} l^N e^{-Nl} \lesssim \frac{h^N}{k^N} e^{Nk} \lesssim \frac{e^{h^{\frac{N}{N-1}}}}{h^{\frac{1}{N-1}}} \lesssim \mu_d(h).$$

□

4. PROOF OF THEOREM ??: SUFFICIENCY OF (1)

In order to prove that (1) of Theorem 1.5 implies (2), we will only consider the case $K = \omega_{N-1}^{-1}$. Then the general case follows easily by rescaling.

So, let $g : \mathbb{R} \to [0, +\infty)$ be a Borel function such that

$$\limsup_{|t|\to+\infty} |t|^{\frac{N}{N-1}} e^{-\alpha_N |t|^{\frac{N}{N-1}}} g(t) < +\infty, \quad \text{where } \alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}, \tag{4.1}$$

and

$$\limsup_{|t|\to 0} |t|^{-N} g(t) < +\infty. \tag{4.2}$$

The aim of this Section is to show that there exists a constant $C_{N,g} > 0$ such that

$$G(u) := \int_{\mathbb{R}^N} g(u(x))\, dx \le C_{N,g} \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N \le 1. \tag{4.3}$$

Recalling the notion of symmetric decreasing rearrangement of functions, it is sufficient to prove that (4.3) holds for non-negative and radially symmetric non-increasing functions $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ satisfying $\|\nabla u\|_N \le 1$.

Given such a function u, we can notice that the proof of inequality (4.3) reduces to

$$\int_{\mathbb{R}^N} \varphi(u)\, dx \le C_{N,g} \|u\|_N^N, \quad \text{where } \varphi(u) := \min\left(|u|^N,\ |u|^{-\frac{N}{N-1}}\right) e^{\alpha_N |u|^{\frac{N}{N-1}}},$$

as a consequence of the assumptions (4.1) and (4.2) on g.

Let $R_0 = R_0(u) > 0$ be such that

$$R_0 := \inf\{r > 0 \mid u(r) \le 1\} \in [0, +\infty),$$

the idea is to split the integral we are interested in into two parts:

$$\int_{\mathbb{R}^N} \varphi(u)\, dx = \int_{\mathbb{R}^N \setminus B_{R_0}} + \int_{B_{R_0}} \varphi(u)\, dx = \int_{\mathbb{R}^N \setminus B_{R_0}} |u|^N e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx + \int_{B_{R_0}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\, dx,$$

here we have also exploited the definition of φ.

The estimate of the integral on $\mathbb{R}^N \setminus B_{R_0}$ is trivial. In fact, by construction $u \le 1$ on $\mathbb{R}^N \setminus B_{R_0}$, consequently

$$\int_{\mathbb{R}^N \setminus B_{R_0}} |u|^N e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx \le e^{\alpha_N} \int_{\mathbb{R}^N \setminus B_{R_0}} |u|^N\, dx.$$

Therefore, from now on, we will always assume that $R_0 > 0$.

Let $\sigma_0 \in (0, 1)$ be arbitrarily fixed. Then there exists $R_1 = R_1(u) > 0$ such that

$$\int_{B_{R_1}} |\nabla u|^N\, dx \le \sigma_0 \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_{R_1}} |\nabla u|^N\, dx \le 1 - \sigma_0.$$

We remark that the choice of σ_0 is independent of u, whereas the choice of R_0 and R_1 depends on u and hence our goal is to obtain an estimate which depends only on σ_0 and N.

In order to estimate the integral on B_{R_0}, we will distinguish between the case $0 < R_0 \le R_1$ and the case $0 < R_1 \le R_0$. The reason is that the first case, namely $0 < R_0 \le R_1$ is subcritical and easily estimated. In fact, if $0 < R_0 \le R_1$ then for $0 \le r \le R_0$ we have

$$
\begin{aligned}
u(r) = u(R_0) + \int_r^{R_0} -u_s(s)\,ds &\le 1 + \Big(\int_r^{R_0} |u_s(s)|^N s^{N-1}\,ds\Big)^{\frac{1}{N}} \Big(\log \frac{R_0}{r}\Big)^{\frac{N-1}{N}} \\
&\le 1 + \frac{1}{\omega_{N-1}^{1/N}} \sigma_0^{\frac{1}{N}} \Big(\log \frac{R_0}{r}\Big)^{\frac{N-1}{N}}. \qquad (4.4)
\end{aligned}
$$

We recall that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$
1 + s^{\frac{N-1}{N}} \le [(1+\varepsilon)s + C_\varepsilon]^{\frac{N-1}{N}} \quad \forall s \ge 0, \qquad (4.5)
$$

hence, for $0 \le r \le R_0$, we have

$$
u^{\frac{N}{N-1}}(r) \le \frac{1}{\omega_{N-1}^{1/(N-1)}} (1+\varepsilon) \sigma_0^{\frac{1}{N-1}} \log \frac{R_0}{r} + C_\varepsilon.
$$

Since $\sigma_0 \in (0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(\sigma_0) > 0$ satisfying $\sigma_0^{\frac{1}{N-1}} = 1 - \varepsilon_0$ and, choosing $\varepsilon = \varepsilon_0$ in the above estimate, we get

$$
u^{\frac{N}{N-1}}(r) \le \frac{1}{\omega_{N-1}^{1/(N-1)}} (1-\varepsilon_0^2) \log \frac{R_0}{r} + C_{\sigma_0}.
$$

Therefore

$$
\begin{aligned}
\int_{B_{R_0}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\,dx \le \int_{B_{R_0}} e^{\alpha_N |u|^{\frac{N}{N-1}}}\,dx &\le \omega_{N-1} e^{\alpha_N C_{\sigma_0}} \int_0^{R_0} \Big(\frac{R_0}{r}\Big)^{N(1-\varepsilon_0^2)} r^{N-1}\,dr \\
&= \frac{\omega_{N-1}}{N\varepsilon_0^2} e^{\alpha_N C_{\sigma_0}} R_0^N \le \frac{e^{\alpha_N C_{\sigma_0}}}{\varepsilon_0^2} \int_{B_{R_0}} |u|^N\,dx
\end{aligned}
$$

and the proof is completed.

So, from now on, we will focus our attention on the case $0 < R_1 \le R_0$. Again, we split the integral on B_{R_0} into two parts:

$$\int_{B_{R_0}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\, dx = \int_{B_{R_0}\setminus B_{R_1}} + \int_{B_{R_1}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\, dx.$$

If $R_1 < r < R_0$ then, arguing as in (4.4), we can estimate

$$u(r) \le 1 + \frac{1}{\omega_{N-1}^{1/N}}(1-\sigma_0)^{\frac{1}{N}}\Big(\log\frac{R_0}{r}\Big)^{\frac{N-1}{N}}$$

and, applying (4.5), we get for any $\varepsilon > 0$

$$\begin{aligned} u^{\frac{N}{N-1}}(r) &\le \frac{1}{\omega_{N-1}^{1/(N-1)}}(1+\varepsilon)(1-\sigma_0)^{\frac{1}{N-1}}\log\frac{R_0}{r} + C_\varepsilon \\ &\le \frac{1}{\omega_{N-1}^{1/(N-1)}}(1+\varepsilon)\Big(1-\frac{\sigma_0}{N-1}\Big)\log\frac{R_0}{r} + C_\varepsilon, \end{aligned}$$

where we also used the inequality $(1-A)^q \le 1 - qA$ which holds for all $A \in [0,1]$ and $q \in [0,1]$. In particular, choosing $\varepsilon = \frac{\sigma_0}{N-1}$, we have

$$u^{\frac{N}{N-1}}(r) \le \frac{1}{\omega_{N-1}^{1/(N-1)}}\Big(1-\frac{\sigma_0^2}{(N-1)^2}\Big)\log\frac{R_0}{r} + C_{\sigma_0,N} \quad \text{for } R_1 < r < R_0,$$

and hence

$$\begin{aligned} \int_{B_{R_0}\setminus B_{R_1}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\, dx &\le \omega_{N-1} e^{\alpha_N C_{\sigma_0,N}} \int_{R_1}^{R_0} \Big(\frac{R_0}{r}\Big)^{N\left(1-\frac{\sigma_0^2}{(N-1)^2}\right)} r^{N-1}\, dr \\ &\le \frac{(N-1)^2}{\sigma_0^2}\frac{e^{\alpha_N C_{\sigma_0,N}}}{\varepsilon_0^2}\int_{B_{R_0}} |u|^N\, dx. \end{aligned}$$

Therefore, the proof is complete if we show that in the case $0 < R_1 \le R_0$ we have

$$\int_{B_{R_1}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\, dx \le C\|u\|_N^N \tag{4.6}$$

where the constant $C > 0$ depends only on N and σ_0.

In order to prove (4.6), following the arguments introduced by B. Ruf in (28), we define

$$v(r) := u(r) - u(R_1) \quad \text{for } 0 \le r \le R_1.$$

By construction $v \in W_0^{1,N}(B_{R_1})$ and $\nabla v = \nabla u$ in B_{R_1}. In the same spirit as in (28), the motivation behind the introduction of such a function v is that we would reduce the proof of (4.6) to an application of the Trudinger-Moser inequality (1.1) for bounded domains of $\mathbb{R}^N$. Since we deal with the Dirichlet norm on the whole space $\mathbb{R}^N$ and we don't have any estimate of the complete Sobolev norm of u, the above mentioned argument will not enable us to complete the proof until we provide some additional information about u. In this sense, the optimal descending growth condition expressed by Theorem 3.2 will be a crucial tool. In fact, since

$$u(R_1) > 1 \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_{R_1}} |\nabla u|^N \, dx \le 1 - \sigma_0$$

in view of Theorem 3.2 we know that

$$\frac{e^{\frac{\alpha_N}{(1-\sigma_0)^{1/(N-1)}} u^{\frac{N}{N-1}}(R_1)}}{u^{\frac{N}{N-1}}(R_1)} \le C_{N,\sigma_0} \frac{1}{R_1^N} \|u\|_{L^N(\mathbb{R}^N \setminus B_{R_1})}^N. \tag{4.7}$$

Applying the following one-dimensional calculus inequality

$$(1+a)^q \le (1+\varepsilon)a^q + \left(1 - \frac{1}{(1+\varepsilon)^{1/(q-1)}}\right)^{1-q} \quad \forall a \ge 0,\ \forall \varepsilon > 0 \text{and } \forall q > 1,$$

we can estimate u on B_{R_1} as follows

$$u^{\frac{N}{N-1}} = u^{\frac{N}{N-1}}(R_1)\Big(1+\frac{v}{u(R_1)}\Big)^{\frac{N}{N-1}} \le (1+\varepsilon)v^{\frac{N}{N-1}} + \Big(1-\frac{1}{(1+\varepsilon)^{N-1}}\Big)^{-\frac{1}{N-1}} u^{\frac{N}{N-1}}(R_1),$$

for any $\varepsilon > 0$.

Consequently,

$$\begin{aligned}\int_{B_{R_1}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\,dx &\le \frac{1}{u^{\frac{N}{N-1}}(R_1)}\int_{B_{R_1}} e^{\alpha_N u^{\frac{N}{N-1}}}\,dx \\ &\le \frac{e^{\alpha_N\left(1-\frac{1}{(1+\varepsilon)^{N-1}}\right)^{-\frac{1}{N-1}} u^{\frac{N}{N-1}}(R_1)}}{u^{\frac{N}{N-1}}(R_1)}\int_{B_{R_1}} e^{\alpha_N(1+\varepsilon)v^{\frac{N}{N-1}}}\,dx\end{aligned}$$

and, if we choose $\varepsilon := \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(\sigma_0) > 0$ satisfying

$$\Big(1-\frac{1}{(1+\varepsilon_0)^{N-1}}\Big)^{-\frac{1}{N-1}} \le \frac{1}{(1-\sigma_0)^{\frac{1}{N-1}}}\,, \quad \text{namely} \;\; \varepsilon_0 \ge \frac{1-\sigma_0^{1/(N-1)}}{\sigma_0^{1/(N-1)}}, \tag{4.8}$$

then we can apply (4.7), obtaining

$$\begin{aligned}\int_{B_{R_1}} \frac{e^{\alpha_N |u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\,dx &\le \frac{e^{\alpha_N\left(1-\frac{1}{(1+\varepsilon_0)^{N-1}}\right)^{-\frac{1}{N-1}} u^{\frac{N}{N-1}}(R_1)}}{u^{\frac{N}{N-1}}(R_1)}\int_{B_{R_1}} e^{\alpha_N(1+\varepsilon_0)v^{\frac{N}{N-1}}}\,dx \\ &\le C_{N,\sigma_0}\frac{1}{R_1^N}\|u\|^N_{L^N(\mathbb{R}^N\setminus B_{R_1})}\int_{B_{R_1}} e^{\alpha_N(1+\varepsilon_0)v^{\frac{N}{N-1}}}\,dx.\end{aligned}$$

In conclusion, if we show the existence of $\varepsilon_0 > 0$ satisfying (**??**) and such that

$$\int_{B_{R_1}} e^{\alpha_N(1+\varepsilon_0)v^{\frac{N}{N-1}}}\,dx \le C_{N,\sigma_0}R_1^N \tag{4.9}$$

for some constant $C_{N,\sigma_0} > 0$ depending only on N and σ_0, then (4.6) follows. In order to achieve (4.9), we set

$$w := (1+\varepsilon_0)^{\frac{N-1}{N}} v \quad \text{on } B_{R_1}$$

Since $w \in W_0^{1,N}(B_{R_1})$, if we prove that

$$\|\nabla w\|_{L^N(B_{R_1})} \leq 1$$

then (4.9) is nothing but a direct consequence of the Trudinger-Moser inequality (1.1) for bounded domains of $\mathbb{R}^N$. We have

$$\|\nabla w\|^N_{L^N(B_{R_1})} = (1+\varepsilon_0)^{N-1}\|\nabla v\|^N_{L^N(B_{R_1})} = (1+\varepsilon_0)^{N-1}\|\nabla u\|^N_{L^N(B_{R_1})} \leq (1+\varepsilon_0)^{N-1}\sigma_0 \leq 1$$

provided $\varepsilon_0 > 0$ satisfies

$$\varepsilon_0 \leq \frac{1-\sigma_0^{1/(N-1)}}{\sigma_0^{1/(N-1)}}$$

Therefore, the choice

$$\varepsilon_0 = \frac{1-\sigma_0^{1/(N-1)}}{\sigma_0^{1/(N-1)}},$$

which is independent of u, yields the desired estimate (4.6).

5. PROOF OF THEOREM ??: SUFFICIENCY OF (3)

Let $g : \mathbb{R} \to [0, +\infty)$ be any a.e.-continuous function satisfying

$$\limsup_{|t|\to 0} |t|^{-N} g(t) = 0 \tag{5.1}$$

and

$$\limsup_{|t|\to +\infty} |t|^{\frac{N}{N-1}} e^{-\frac{N}{K^{1/(N-1)}}|t|^{\frac{N}{N-1}}} g(t) = 0 \tag{5.2}$$

for some $K > 0$. Let $\{u_n\}_{n\geq 1} \subset W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ be such that $\|\nabla u_n\|_N^N \leq \omega_{N-1} K$ and assume that $u_n \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$. The aim of this Section is to show that

$$G(u_n) - G(u) := \int_{\mathbb{R}^N} [g(u_n) - g(u)]\, dx \to 0 \quad \text{as } n \to +\infty.$$

First we remark that, from the assumptions on $\{u_n\}_{n\geq 1}$, it follows that $u_n \to u$ a.e. in $\mathbb{R}^N$ and that $\|u_n\|_{W^{1,N}} \leq C$ for some constant $C > 0$ independent of n.

Now, from the *radial Sobolev lemma*,

$$|v(r)| \leq \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N}} \frac{1}{r^{(N-1)/N}} \|v\|_{W^{1,N}} \quad \text{for a.e. } r > 0$$

which holds for any $v \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, we deduce that $u_n(r) \to 0$ as $r \to +\infty$ uniformly with respect to n. This together with (5.1) leads to conclude that for any $\varepsilon > 0$ there exists $R > 0$ independent of n such that

$$\int_{\mathbb{R}^N \setminus B_R} g(u_n)\, dx \leq \varepsilon \int_{\mathbb{R}^N \setminus B_R} |u_n|^N\, dx \lesssim \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R} g(u)\, dx \lesssim \varepsilon. \tag{5.3}$$

On the other hand, from (5.2), we deduce that for any $\varepsilon > 0$ there exists $L > 0$ independent of n such that

$$\int_{\{|u_n|>L\}} g(u_n)\,dx \le \varepsilon \int_{\{|u_n|>L\}} \frac{e^{\frac{N}{\kappa^{1/(N-1)}}|u_n|^{\frac{N}{N-1}}}}{|u_n|^{\frac{N}{N-1}}}\,dx$$

and

$$\int_{\{|u_n|>L\}} g(u)\,dx \le \varepsilon \int_{\{|u_n|>L\}} \frac{e^{\frac{N}{\kappa^{1/(N-1)}}|u|^{\frac{N}{N-1}}}}{|u|^{\frac{N}{N-1}}}\,dx.$$

Therefore, applying Theorem 1.5, we get

$$\int_{\{|u_n|>L\}} g(u_n)\,dx \lesssim \varepsilon \|u_n\|_N^N \lesssim \varepsilon \quad \text{and} \quad \int_{\{|u_n|>L\}} g(u)\,dx \lesssim \varepsilon. \tag{5.4}$$

Hence, combining (5.3) with (5.4), we have

$$\begin{aligned}|G(u_n) - G(u)| &\le \int_{\mathbb{R}^N\setminus B_R} + \int_{B_R} |g(u_n) - g(u)|\,dx \lesssim \varepsilon + \int_{B_R} |g(u_n) - g(u)|\,dx \\ &= \varepsilon + \int_{B_R\cap\{|u_n|>L\}} + \int_{B_R\cap\{|u_n|\le L\}} |g(u_n) - g(u)|\,dx \lesssim \\ &\lesssim \varepsilon + \int_{B_R\cap\{|u_n|\le L\}} |g(u_n) - g(u)|\,dx.\end{aligned}$$

If we define

$$g^L(t) := \begin{cases} g(t) & \text{if } |t| \le L, \\ g(L) & \text{if } |t| > L, \end{cases}$$

then

$$\lim_{n\to+\infty} |G(u_n) - G(u)| \lesssim \varepsilon + \lim_{n\to+\infty} \int_{B_R} |g^L(u_n) - g^L(u)|\,dx \lesssim \varepsilon,$$

as a consequence of the Lebesgue dominated convergence theorem. Since $\varepsilon > 0$ is arbitrary fixed, the proof is complete.

6. FROM TRUDINGER-MOSER INEQUALITY WITH THE EXACT GROWTH TO TRUDINGER-MOSER INQUALITY IN $W^{1,N}(\mathbb{R}^N)$

In this Section we show that Trudinger-Moser inequality with the exact growth condition (1.4) implies Trudinger-Moser inequality in $W^{1,N}(\mathbb{R}^N)$ (1.3).

Before proceeding with the proof, we point out that Adachi-Tanaka inequality (1.2) can be deduced as a direct consequence of the Trudinger-Moser inequality with the exact growth condition (1.4) and in particular this inequality tells us that

$$\int_{\mathbb{R}^N} \phi_N(|u|^{\frac{N}{N-1}})\,dx \leq C_N \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N \leq 1. \tag{6.1}$$

Using the power series expansion of the exponential function together with Stirling's formula, it is easy to see that (6.1) implies that

$$\||u|^{\frac{N}{N-1}}\|_p \lesssim p\|u\|_N^{\frac{N}{p}} \quad \forall u \in W^{1,N}(\mathbb{R}^N) \text{with } \|\nabla u\|_N \leq 1 \tag{6.2}$$

for any integer $p \geq N-1$, then (6.2) can be extended to non-integers $p \geq N-1$, simply by interpolation (see (7), Chapitre IV.2, Remarque 2).

Now let $u \in W^{1,N}(\mathbb{R}^N)\backslash\{0\}$ be such that $\|u\|_{W^{1,N}} \leq 1$, our aim is to prove that

$$\int_{\mathbb{R}^N} \phi_N(\alpha_N |u|^{\frac{N}{N-1}})\,dx \leq C_N \tag{6.3}$$

for some constant $C_N > 0$ independent of u. Let $\theta \in (0,\,1)$ be such that $\|u\|_N^N = \theta$, so that $\|\nabla u\|_N^N \leq 1 - \theta$. We distinguish two cases,

1st case $\theta \geq \frac{N-1}{N}$;

2nd case $\theta < \frac{N-1}{N}$.

If $\theta \geq \frac{N-1}{N}$, we define $\tilde{u} := N^{\frac{1}{N}} u$, so that

$$\|\tilde{u}\|_N^N = N\|u\|_N^N \leq N \quad \text{and} \quad \|\nabla\tilde{u}\|_N^N = N\|\nabla u\|_N^N \leq N(1-\theta) \leq 1.$$

Applying Adachi-Tanaka inequality (1.2) to $\tilde{u}$, we get for any $\alpha \in (0,\,\alpha_N)$

$$\int_{\mathbb{R}^N} \phi_N(\alpha N^{\frac{1}{N-1}}|u|^{\frac{N}{N-1}})\,dx \leq C(\alpha,\,N)\|\tilde{u}\|_N^N \lesssim C(\alpha,\,N)$$

and, in particular, choosing α so that

$$\alpha N^{\frac{1}{N-1}} = \alpha_N, \quad \text{i.e. } \alpha = N^{\frac{N-2}{N-1}}\omega_{N-1}^{\frac{1}{N-1}} < N\omega_{N-1}^{\frac{1}{N-1}} = \alpha_N,$$

we obtain the desired estimate (**??**).

Therefore, from now on, we will focus our attention to the case $\theta < \frac{N-1}{N}$. Let

$$A := \{x \in \mathbb{R}^N \mid |u(x)| \geq 1\}.$$

By construction $|u| < 1$ on $\mathbb{R}^N \setminus A$ and it is easy to see that

$$\phi_N(t) \leq C_N t^{N-1} \quad \forall t \in [0,\,\alpha_N]$$

for some constant $C_N > 0$. Consequently,

$$\int_{\mathbb{R}^N\setminus A} \phi_N(\alpha_N|u|^{\frac{N}{N-1}})\,dx \leq C_N\alpha_N^{N-1}\int_{\mathbb{R}^N\setminus A}|u|^N\,dx \leq C_N\alpha_N^{N-1}.$$

Thus, to complete the proof, it remains only to show that

$$\int_A \phi_N(\alpha_N|u|^{\frac{N}{N-1}})\,dx \leq C_N.$$

Since

$$[\phi_N(t)]^p \leq \phi_N(pt) \quad \forall t \geq 0,\ \forall p \geq 1,$$

applying Hölder's inequality, with $1 < p < +\infty$ to be suitably chosen, we get

$$\begin{aligned}\int_A \phi_N(\alpha_N|u|^{\frac{N}{N-1}})\,dx &\leq \left(\int_A \frac{\phi_N(\alpha_N p|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\,dx\right)^{\frac{1}{p}}\left(\int_A (1+|u|)^{\frac{N}{(N-1)(p-1)}}\right)^{\frac{p-1}{p}}\\ &\leq 2^{\frac{N}{N-1}}\left(\int_A \frac{\phi_N(\alpha_N p|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\,dx\right)^{\frac{1}{p}}\||u|^{\frac{N}{N-1}}\|_{\frac{1}{p-1}}^{\frac{1}{p}}.\end{aligned}$$

Our choice of p is the following

$$p := \frac{N-1}{(N-1)-\theta} > 1.$$

In this way

$$\frac{1}{p-1} = \frac{(N-1)-\theta}{\theta} > N-1$$

and, in view of (6.2), we have

$$\||u|^{\frac{N}{N-1}}\|_{\frac{1}{p-1}}^{\frac{1}{p}} \lesssim \left(\frac{1}{p-1}\right)^{\frac{1}{p}}\|u\|_N^{N\frac{p-1}{p}}. \tag{6.4}$$

Now, we let $\tilde{u} := p^{\frac{N-1}{N}} u$. By construction we have

$$\|\nabla \tilde{u}\|_N^N = p^{N-1} \|\nabla u\|_N^N \leq \Big(\frac{N-1}{(N-1)-\theta}\Big)^{N-1} (1-\theta) = \Big[\frac{(1-\theta)^{\frac{1}{N-1}}}{1-\theta/(N-1)}\Big]^{N-1} \leq 1$$

since $(1-A)^q \leq 1 - qA$ for any q, $A \in [0, 1]$. Consequently, we can apply the Trudinger-Moser inequality with the exact growth condition (1.4) to $\tilde{u}$ and this leads to

$$\left(\int_A \frac{\phi_N(\alpha_N p |u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\, dx\right)^{\frac{1}{p}} \leq \left(p \int_A \frac{\phi_N(\alpha_N |\tilde{u}|^{\frac{N}{N-1}})}{(1+|\tilde{u}|)^{\frac{N}{N-1}}}\, dx\right)^{\frac{1}{p}} \lesssim (p \|\tilde{u}\|_N^N)^{\frac{1}{p}} = p^{\frac{N}{p}} \|u\|_N^{\frac{N}{p}}. \tag{6.5}$$

In conclusion, combining (6.4) and (6.5), we obtain

$$\int_A \phi_N(\alpha_N |u|^{\frac{N}{N-1}})\, dx \lesssim p^{\frac{N-1}{p}} \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \|u\|_N^N \lesssim \frac{\theta^{N-1+\frac{\theta}{N-1}}}{(1-\theta/(N-1))^{N-1-\theta}},$$

and the right hand side of this last inequality is bounded by a constant depending only on the dimension N, since $0 < \theta < \frac{N-1}{N}$.

7. EXISTENCE OF GROUND STATES

In this section, we study the following quasilinear elliptic equation

$$\begin{cases} -\Delta_N u + c|u|^{N-2} u = f(u) & \text{in } \mathbb{R}^N,\ N \geq 2, \\ u \in W^{1,N}(\mathbb{R}^N),\ u > 0 \text{in } \mathbb{R}^N \end{cases} \tag{7.1}$$

where $c > 0$ and the nonlinear term f satisfies the assumptions (f_1), (f_2) and (f_2) of Theorem 1.7. Since we look for positive solutions of (7.1), we may assume without loss of generality that $f = 0$ on $(-\infty, 0)$.

The natural functional associated to a variational approach to problem (7.1) is

$$I_c(u) := \frac{1}{N}\big(\|\nabla u\|_N^N + c\|u\|_N^N\big) - \int_{\mathbb{R}^N} F(u)\,dx \quad \forall u \in W^{1,N}(\mathbb{R}^N),$$

which is well defined and of class $\mathscr{C}^1$ on $W^{1,N}(\mathbb{R}^N)$. Our goal is to prove the existence of ground state solutions for (7.1) and we recall that a solution u of (7.1) is a ground state if $I_c(u) = m_c$ where

$$m_c := \inf\big\{\, I_c(u) \;\big|\; u \in W^{1,N}(\mathbb{R}^N)\backslash\{0\} \text{is a solution of (7.1)} \,\big\}$$

To this aim, motivated by the Pohozaev identity for equation (7.1), we introduce the functional

$$G_c(u) := c\|u\|_N^N - N\int_{\mathbb{R}^N} F(u)\,dx \quad \forall u \in W^{1,N}(\mathbb{R}^N)$$

and the constrained minimization problem

$$\begin{aligned} A_c &:= \inf\Big\{\, \frac{1}{N}\|\nabla u\|_N^N \;\Big|\; u \in W^{1,N}(\mathbb{R}^N)\backslash\{0\},\ G_c(u) = 0 \,\Big\} \\ &= \inf\big\{\, I_c(u) \;\big|\; u \in W^{1,N}(\mathbb{R}^N)\backslash\{0\},\ G_c(u) = 0 \,\big\} \le m_c. \end{aligned}$$

Some remarks are in order.

Remark 7.1. Let $\mathscr{P}_c$ be the set consisting of all functions in $W^{1,N}(\mathbb{R}^N)\backslash\{0\}$ satisfying the Pohozaev identity for equation (7.1), i.e.

$$\mathscr{P}_c := \big\{u \in W^{1,N}(\mathbb{R}^N) \;\big|\; u \neq 0,\, G_c(u) = 0\big\},$$

so that

$$A_c = \inf_{u \in \mathscr{P}_c} \frac{1}{N} \|\nabla u\|_N^N.$$

We point out that $\mathscr{P}_c$ is not empty. In fact, let $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ be compactly supported and define

$$h(s) := G_c(su) = cs^N \|u\|_N^N - N \int_{\mathbb{R}^N} F(su)\, dx \quad \forall s > 0. \tag{7.2}$$

Then $h(s) > 0$ for $s > 0$ small enough, as a consequence of (f_1) and (f_2), while $h(s) < 0$ for $s > 0$ sufficiently large, as a consequence of (f_2). Therefore we get the existence of $s_0 > 0$ satisfying $h(s_0 u) = 0$, which means that $s_0 u \in \mathscr{P}_c$.

Note also that for any fixed $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$, the function h defined by (7.2) is *strictly positive* for $s > 0$ small enough.

Remark 7.2. Given a minimizing sequence for A_c, that is a sequence $\{u_k\}_k \subset \mathscr{P}_c$ satisfying

$$\frac{1}{N} \|\nabla u_k\|_N^N \to A_c \quad \text{as } k \to +\infty,$$

we may always assume that $\{u_k\}_k \subset W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ and $\|u_k\|_N = 1$. This can be done simply by Schwarz symmetrization and rescaling.

Remark 7.3. If the infimum A_c is attained then the minimizer $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)\backslash\{0\}$ is, under a suitable change of scale, a ground state solution of (7.1). In fact, if u is a minimizer for A_c then there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$-\Delta_N u + c|u|^{N-2}u - f(u) = \theta\big(c|u|^{N-2}u - Nf(u)\big) \quad \text{in } \mathbb{R}^N,$$

namely

$$-\Delta_N u = (N\theta - 1)\big(c|u|^{N-2}u - f(u)\big) \quad \text{in } \mathbb{R}^N.$$

Recalling that $u \in \mathcal{P}_c$,

$$\begin{aligned}\int_{\mathbb{R}^N} \big(c|u|^{N-2}u - f(u)\big)u\,dx &= c\|u\|_N^N - \int_{\mathbb{R}^N} \big(uf(u) \pm NF(u)\big)\,dx \\ &= -\int_{\mathbb{R}^N} \big(uf(u) - NF(u)\big)\,dx < 0\end{aligned}$$

as a consequence of (f_1). Moreover,

$$\int_{\mathbb{R}^N} u\Delta_N u\,dx < 0$$

and hence $N\theta - 1 < 0$. Therefore

$$\tilde{u}(x) := u\bigg(\frac{x}{(1 - N\theta)^{1/N}}\bigg) \quad \text{for a.e. } x \in \mathbb{R}^N$$

is a non-trivial solution of (7.1). Note also that $\tilde{u}$ is a minimizer for A_c and thus $\tilde{u}$ is a ground state solution of (7.1).

Following (18), we begin showing an interesting relation between the attainability of the infimum A_c and the Trudinger-Moser inequality with the exact growth condition (Theorem 1.5). To this aim, as in (18), we introduce the *Trudinger-Moser ratio*

$$\mathrm{C}^{A}_{\mathrm{TM}}(F) := \sup\left\{ \frac{N}{\|u\|_N^N} \int_{\mathbb{R}^N} F(u)\,dx \;\middle|\; u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\},\ \|\nabla u\|_N \leq A \right\},$$

the *Trudinger-Moser threshold*

$$\mathfrak{M}(F) := \sup\{\, A > 0 \mid \mathrm{C}^{A}_{\mathrm{TM}}(F) < +\infty \,\}$$

and we denote by $\mathrm{C}^{\star}_{\mathrm{TM}}(F)$ the ratio at the threshold, i.e.

$$\mathrm{C}^{\star}_{\mathrm{TM}}(F) := C^{\mathfrak{M}(F)}_{\mathrm{TM}}(F).$$

From (f_1) and (f_2), it follows that

$$\lim_{t\to+\infty} \frac{t^{N/(N-1)}F(t)}{e^{\alpha t^{N/(N-1)}}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0. \end{cases} \tag{7.3}$$

Moreover, (f_1) implies

$$\lim_{t\to 0} \frac{F(t)}{t^N} = 0. \tag{7.4}$$

Hence, from Theorem 1.5, we deduce that

$$\mathfrak{M}(F) = \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}.$$

Now, we can state a *sufficient* condition for the attainability of A_c in terms of the Trudinger-Moser threshold $\mathfrak{M}(F)$.

Proposition 7.1. *Let $c > 0$ and assume (f_1), (f_2) and (f_2). If*

$$A_c < \frac{\left(\mathfrak{M}(F)\right)^N}{N}$$

then A_c is attained and $A_c = I_c(u)$ where $u \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\}$ is, under a suitable change of scale, a ground state solution of equation (7.1).

Proof. Let $\{u_k\}_k \subset W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\}$ be a minimizing sequence for A_c, namely $\|u_k\|_N = 1$, $u_k \in \mathcal{P}_c$ for any $k \geq 1$,

$$\frac{1}{N}\|\nabla u_k\|_N^N \to A_c \quad \text{as } k \to +\infty$$

and, we may assume that $u_k \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$ as $k \to +\infty$.

Step 1. First, we prove that $A_c > 0$. Clearly $A_c \geq 0$; by way of contradiction, we assume that $A_c = 0$.

From (7.3) and (7.4), we deduce that F satisfies the growth condition (3) of the compactness theorem (Theorem 1.6) for any $K > 0$ with $N/K^{1/(N-1)} > \alpha_0$ and, since $\|\nabla u_k\|_N \to 0$ as $k \to +\infty$, we get

$$\int_{\mathbb{R}^N} F(u_k)\,dx \to \int_{\mathbb{R}^N} F(u)\,dx \quad \text{as } k \to +\infty.$$

Recalling that $u_k \in \mathscr{P}_c$ and $\|u_k\|_N = 1$, we have also

$$\int_{\mathbb{R}^N} F(u_k)\,dx = c/N \quad \forall k \geq 1.$$

Consequently,

$$\int_{\mathbb{R}^N} F(u)\,dx = \frac{c}{N} > 0. \tag{7.5}$$

On the other hand,

$$0 = \liminf_{k\to+\infty} \|\nabla u_k\|_N \geq \|\nabla u\|_N \geq 0$$

and $u = 0$, which contradicts (7.5).

Step 2. Since

$$0 < NA_c < (\,\mathfrak{M}(F)\,)^N = \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

from (7.3), we deduce that

$$\lim_{|t|\to+\infty} \frac{|t|^{N/(N-1)}F(t)}{e^{\alpha|t|^{N/(N-1)}}} = 0 \quad \forall \alpha \in \left(\alpha_0,\ \frac{\alpha_N}{(NA_c)^{1/(N-1)}}\right].$$

This together with (7.4) and

$$\limsup_{k\to+\infty} \|\nabla u_k\|_N^N = NA_c$$

enable us to apply the compactness theorem (Theorem 1.6), which tells us that

$$\int_{\mathbb{R}^N} F(u_k)\,dx \to \int_{\mathbb{R}^N} F(u)\,dx \quad \text{as } k \to +\infty.$$

Therefore, arguing as in (7.5), we get

$$\int_{\mathbb{R}^N} F(u)\,dx = \frac{c}{N} > 0$$

and $u \neq 0$. Moreover,

$$\frac{1}{N}\|\nabla u\|_N^N \leq \liminf_{k\to+\infty} \frac{1}{N}\|\nabla u_k\|_N^N = A_c$$

and in order to prove that the infimum A_c is attained by u, it remains only to show that $G_c(u) = 0$. Since

$$G_c(u) = c\|u\|_N^N - N\int_{\mathbb{R}^N} F(u)\,dx = c\|u\|_N^N - c \leq \liminf_{k\to+\infty} c\|u_k\|_N^N - c = 0,$$

we argue by contradition assuming $G_c(u) < 0$. If we define

$$h(s) := G_c(su) = cs^N\|u\|_N^N - N\int_{\mathbb{R}^N} F(su)\,dx \quad \forall s > 0,$$

then $h(1) < 0$ and, from Remark 7.1, we deduce that $h(s) > 0$ for $s > 0$ small enough. Consequently, there exists $s_0 \in (0, 1)$ such that $h(s_0u) = 0$, namely $s_0u \in \mathcal{P}_c$, and hence

$$A_c \leq \frac{1}{N}\|\nabla(s_0u)\|_N^N = s_0^N\frac{1}{N}\|\nabla u\|_N^N \leq s_0^N A_c < A_c$$

a contradition. □

Remark 7.4. Note that the case $\alpha_0 = 0$, i.e.

$$\lim_{t\to+\infty} \frac{f(t)}{e^{\alpha t^{N/(N-1)}}} = 0 \quad \forall \alpha > 0, \tag{7.6}$$

corresponds to the subcritical exponential case. If the nonlinear term f satisfies the assumptions of Proposition 7.1 with (f_2) replaced by (7.6), then $\mathfrak{M}(F) = +\infty$ and for any $c > 0$ the infimum A_c is attained by a ground state solution of equation (7.1).

We have also an interesting connection between the attainability of A_c and the Trudinger-Moser ratio at the threshold $C^{\star}_{TM}(F)$.

Proposition 7.2. *Let $c > 0$ and assume (f_1), (f_2) and (f_2). The constrained minimization problem A_c associated to the functional I_c satisfies*

$$A_c < \frac{\left(\mathfrak{M}(F)\right)^N}{N} \tag{7.7}$$

if and only if

$$c < C^{\star}_{TM}(F). \tag{7.8}$$

Proof. First, we prove the sufficiency of (7.8), that is $0 < c < C^{\star}_{TM}(F)$ yields

$$A_c < \frac{\left(\mathfrak{M}(F)\right)^N}{N}.$$

We distinguish between the case $C^{\star}_{TM}(F) < +\infty$ and $C^{\star}_{TM}(F) = +\infty$. In the case $C^{\star}_{TM}(F) < +\infty$, since $0 < c < C^{\star}_{TM}(F)$, we have that $c < C^{\star}_{TM}(F) - \varepsilon_0$ for some $\varepsilon_0 > 0$. From the definition of $C^{\star}_{TM}(F)$, there exists $u_0 \in W^{1,N}(\mathbb{R}^N)\backslash\{0\}$ with $\|\nabla u_0\|_N \leq \mathfrak{M}(F)$ satisfying

$$C^{\star}_{TM}(F) - \varepsilon_0 \leq \frac{N}{\|u_0\|_N^N} \int_{\mathbb{R}^N} F(u_0)\, dx,$$

and hence

$$c < \frac{N}{\|u_0\|_N^N} \int_{\mathbb{R}^N} F(u_0)\, dx, \tag{7.9}$$

namely $G_c(u_0) < 0$. Let $h(s) := G_c(su_0)$ for $s > 0$; since $h(1) < 0$ and $h(s) > 0$ for $s > 0$ small enough (see Remark 7.1), there exists $s_0 \in (0, 1)$ satisfying $h(s_0 u_0) = 0$. Consequently, $s_0 u_0 \in \mathscr{P}_c$ and

$$A_c \leq \frac{1}{N} \|\nabla(s_0 u_0)\|_N^N = s_0^N \frac{1}{N} \|\nabla u_0\|_N^N \leq s_0^N \frac{(\,\mathfrak{M}(F)\,)^N}{N} < \frac{(\,\mathfrak{M}(F)\,)^N}{N}.$$

In the case $\mathrm{C}^{\star}_{\mathrm{TM}}(F) = +\infty$, for any $c > 0$ there exists $u_0 \in W^{1,N}(\mathbb{R}^N)\backslash\{0\}$ with $\|\nabla u_0\|_N \leq \mathfrak{M}(F)$ and satisfying (7.9). Hence we can repeat the same arguments as above to get the conclusion.

Now we prove the necessity of (7.8), that is (7.7) implies (7.8). Let $c > 0$ and assume that (7.7) holds. Obviously, if $\mathrm{C}^{\star}_{\mathrm{TM}}(F) = +\infty$ then $c < \mathrm{C}^{\star}_{\mathrm{TM}}(F)$ and the proof is complete. Therefore, without loss of generality, we may assume that $\mathrm{C}^{\star}_{\mathrm{TM}}(F) < +\infty$. Since the assumptions of Proposition 7.1 are satisfied, we have the existence of a minimizer $u \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\}$ for A_c satisfying $\|\nabla u\|_N < \mathfrak{M}(F)$ and $G_c(u) = 0$, i.e.

$$c = \frac{N}{\|u\|_N^N} \int_{\mathbb{R}^N} F(u)\, dx$$

We introduce the function

$$g(s) := \frac{N}{s^N \|u\|_N^N} \int_{\mathbb{R}^N} F(su)\, dx \quad \forall s > 0,$$

so that $g(1) = c$ and, using (f_1), it is easy to see that g is monotone increasing. If we set

$$\tilde{u} := \frac{\mathfrak{M}(F)}{\|\nabla u\|_N}\, u$$

then $\|\nabla \tilde{u}\|_N = \mathfrak{M}(F)$ and

$$\mathrm{C}^{\star}_{\mathrm{TM}}(F) \geq \frac{N}{\|\tilde{u}\|_N^N} \int_{\mathbb{R}^N} F(\tilde{u})\, dx = g\left(\frac{\mathfrak{M}(F)}{\|\nabla u\|_N}\right) > g(1) = c$$

□

Before proceeding to the proof of Theorem 1.7, we recall that in (19) the authors enlighten an interesting relation between the existence of solutions to problem (7.1) in the semilinear case $N = 2$ and their Trudinger-Moser inequality with the exact growth in $\mathbb{R}^2$.

Theorem 7.3 ((19), Theorem 5.1). *Let $N = 2$ and assume that f satisfies (f_1), (f_2) and (f_2). Then there exists $c_* \in (0, +\infty]$ such that, for each $c \in (0, c_*)$, equation* (7.1) *admits a positive radial solution which has the least energy among all the solutions of* (7.1). *Moreover, $c_* = \mathrm{C}^{\star}_{\mathrm{TM}}(F)$ when $\mathrm{C}^{\star}_{\mathrm{TM}}(F) < +\infty$, while $c_* = +\infty$ is equivalent to*

$$\lim_{t \to +\infty} \frac{t^2 F(t)}{e^{\alpha_0 t^2}} = +\infty.$$

In view of the Trudinger-Moser inequality with the exact growth in $\mathbb{R}^N$ (Theorem 1.5), we can obtain a similar result in the general quasilinear case $N \geq 3$ and thus, in particular, we can prove Theorem 1.7.

Theorem 7.4. *Let* $N \geq 3$ *and assume that* f *satisfies* (f_1), (f_2) *and* (f_2). *Then there exists* $c_* \in (0, +\infty]$ *such that, for each* $c \in (0, c_*)$, *equation* (7.1) *admits a positive radial solution which has the least energy among all the solutions of* (7.1). *Moreover,* $c_* = \mathrm{C}^{\star}_{\mathrm{TM}}(F)$ *when* $\mathrm{C}^{\star}_{\mathrm{TM}}(F) < +\infty$, *while* $c_* = +\infty$ *is equivalent to*

$$\lim_{t\to+\infty} \frac{t^{N/(N-1)}F(t)}{e^{\alpha_0 t^{N/(N-1)}}} = +\infty. \tag{7.10}$$

The proof of Theorem 7.4 follows the same line of (19, Theorem 5.1), but we briefly sketch it for the convenience of the reader.

Proof. If $0 < c < \mathrm{C}^{\star}_{\mathrm{TM}}(F)$ then in view of Proposition 7.2

$$A_c < \frac{\left(\mathfrak{M}(F)\right)^N}{N}$$

Hence the assumptions of Proposition 7.1 are fulfilled and we get the existence of a ground state solution of equation (7.1). Moreover, recalling (7.4) and in light of Theorem 1.5, we can easily see that $\mathrm{C}^{\star}_{\mathrm{TM}}(F) = +\infty$ if and only if (7.10) holds. □

8. OTHER SUFFICIENT CONDITIONS FOR THE EXISTENCE OF GROUND STATES

In Section 7, we showed that for problems of the form (7.1), where $c > 0$ and f satisfies (f_1), (f_2) and (f_2) of Theorem 1.7, the additional growth condition on the nonlinear term f

$$\lim_{t\to+\infty} \frac{t^{N/(N-1)}F(t)}{e^{\alpha_0 t^{N/(N-1)}}} = +\infty \tag{8.1}$$

is a sufficient condition for the existence of a ground state solution. Now, we propose some sufficient conditions that can be considered alternatively to (8.1).

First, we recall the results obtained in (6) and (29) for the semilinear case

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \mathbb{R}^2, \\ u \in W^{1,2}(\mathbb{R}^2),\ u > 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{8.2}$$

where f satisfies the assumptions (f_1), (f_2) and (f_2) of Theorem 1.7 with $N = 2$. Both the papers (6) and (29) concern the existence of ground state solutions for problem (8.2). In (6), the existence is obtained by means of the following additional growth condition on the nonlinearity f:

$$\exists\, q > 2 \text{such that} \quad f(t) \geq \lambda t^{q-1} \quad \forall t \geq 0, \quad \text{where } \lambda > \left(\frac{q-2}{q}\right)^{\frac{q-2}{2}} \left(\frac{\alpha_0}{4\pi}\right)^{\frac{q-2}{2}} C_q^{q/2}. \tag{8.3}$$

Here, the constant C_q is defined as

$$C_q := \inf_{u\in W^{1,2}(\mathbb{R}^2)\setminus\{0\}} \frac{\|\nabla u\|_2^2 + \|u\|_2^2}{\|u\|_q^2}.$$

Note that, since f has exponential growth at infinity (f_2), assumption (8.3) only prescribes the growth of f near the origin. Instead in (29), the existence of a ground state solution for

(8.2) follows from an additional growth condition at infinity and more precisely from the assumption

$$\lim_{t\to+\infty}\frac{tf(t)}{e^{\alpha_0 t^2}}=\beta_0>\frac{2}{\alpha_0}e. \tag{8.4}$$

(Note the flaw there: it is not sufficient that $\beta_0>0$ and the condition $\beta_0>(2/\alpha_0)e$ is needed.)

The arguments of the proofs in (6) and (29) can easily be adapted to study the general quasilinear case (7.1) where $c>0$ and f satisfies (f_1), (f_2) and (f_2) of Theorem 1.7 together with an analogue of the growth condition (8.3) or (8.4) for the N-dimensional case.

In fact, since $c>0$, the quantity

$$\|u\|_c:=\left(\|\nabla u\|_N^N+c\|u\|_N^N\right)^{1/N}\quad\forall u\in W^{1,N}(\mathbb{R}^N)$$

defines a norm on $W^{1,N}(\mathbb{R}^N)$ which is equivalent to the standard one. The functional $I_c\in\mathscr{C}^1(W^{1,N}(\mathbb{R}^N))$ associated to a variational approach to problem (7.1), i.e.

$$I_c(u):=\frac{1}{N}\|u\|_c^N-\int_{\mathbb{R}^N}F(u)\,dx\quad\forall u\in W^{1,N}(\mathbb{R}^N),$$

has a mountain pass geometry, that is

Proposition 8.1. *Let $N\geq 3$, $c>0$ and assume that f satisfies (f_1), (f_2) and (f_2). Then*

- $I_c(0)=0$,
- $\exists\,\rho,\ a>0$ *such that* $I_c(u)\geq a>0\ \forall u\in W^{1,N}(\mathbb{R}^N)$ *with* $\|u\|_c=\rho$,
- $\exists u_0\in W^{1,N}(\mathbb{R}^N)$ *such that* $\|u_0\|_c>\rho$ *and* $I_c(u_0)<0$.

If we denote by $b_c \in \mathbb{R}$ the mountain pass level of the functional I_c, i.e.

$$b_c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_c(\gamma(t))$$

where

$$\Gamma := \left\{\gamma \in \mathscr{C}([0,1], W^{1,N}(\mathbb{R}^N)) \middle| \gamma(0) = 0, I_c(\gamma(t)) < 0\right\},$$

then the following relation between b_c and the constrained minimization problem A_c, introduced in Section 7, holds:

$$A_c \le b_c. \tag{8.5}$$

To prove the above inequality it suffices to argue as in (29, Lemma 7) and apply (15, Lemma 2.5)

In view of (8.5) and Proposition 7.1, in order to obtain the existence of a ground state solution for (7.1), it is enough to get some suitable upper bound for the mountain pass level b_c and more precisely

$$b_c < \frac{(\mathfrak{M}(F))^N}{N} = \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \tag{8.6}$$

Let

$$C_q := \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_c^N}{\|u\|_q^N} = \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_N^N + c\|u\|_N^N}{\|u\|_q^N},$$

which is attained by some non-negative radial function $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ provided $q > N$ (see for instance (32, Proposition 7.2)). If the nonlinear term f satisfies the additional growth condition

$$\exists\, q > N \quad \text{such that} \quad f(t) \geq \lambda t^{q-1} \quad \forall t \geq 0,$$
$$\text{where} \quad \lambda > \left(\frac{q-N}{q}\right)^{\frac{q-N}{N}} \left(\frac{\alpha_0}{\alpha_N}\right)^{\frac{N-1}{N}(q-N)} C_q^{q/N}. \tag{8.7}$$

then, arguing as in (6, Lemma 3.7) and using the fact that C_q is attained when $q > N$, it is easy to get the desired estimate (8.6).

Instead of (8.7), one may also assume the following growth condition at infinity

$$\lim_{t\to+\infty} \frac{tf(t)}{e^{\alpha_0 t^{N/(N-1)}}} = \beta_0 > (N-2)!\frac{N}{\alpha_0^{N-1}} ec. \tag{8.8}$$

In this case the upper bound for the mountain pass level (8.6) can be obtained as in (29) (see also (31, Section 3.2) and (21, Lemma 3.6)) by means of an estimate involving the Moser's sequence of functions.

Therefore, we can finally state the following result

Theorem 8.2. *Let $c > 0$ and consider the quasilinear equation*

$$-\Delta_N u + c|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N,\ N \geq 2. \tag{8.9}$$

We assume that f satisfies the conditions (f_1), (f_2) and (f_2). Then the growth conditions (8.7) *and* (8.8) *are both sufficient to get the existence of a positive radial solution $u \in W^{1,N}(\mathbb{R}^N)$ of* (8.9) *which has the least energy among all the solutions.*

Remark 8.1. Note that both conditions (8.7) and (8.8) entail a relation between the choice of the constant potential $c > 0$ appearing in equation (8.9) and the growth of the nonlinearity f.

Remark 8.2. It seems to be difficult to compare the growth condition (8.7) with (8.8). The point is that (8.8) allows to control the growth of the nonlinear term f at infinity while (8.7) prescribes the growth of f near the origin. However, a comparison between these conditions can be seen in terms of the Trudinger-Moser supremum $\mathrm{C}^{\star}_{\mathrm{TM}}\ (F)$. In fact, both the conditions (8.7) and (8.8) yield the existence of a ground state solution for (8.9) by means of the following property of the constrained minimization problem A_c associated to (8.9)

$$A_c < \frac{\left(\mathfrak{M}(F)\right)^N}{N},$$

(see (8.5) and (8.6)). In view of Proposition 7.2 the above estimate is equivalent to

$$c < \mathrm{C}^{\star}_{\mathrm{TM}}(F),$$

even if we do not know whether or not $\mathrm{C}^{\star}_{\mathrm{TM}}\ (F) = +\infty$. In other words the growth conditions (8.7) and (8.8) implies that the constant potential $c > 0$ appearing in equation (8.9) satisfies $c < \mathrm{C}^{\star}_{\mathrm{TM}}\ (F)$.

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