



Research Article

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Regularity for scalar integrals without structure conditions

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Abstract: Integrals of the Calculus of Variations with p, q -growth may have not smooth minimizers, not even bounded, for general p, q exponents. In this paper we consider the scalar case, which contrary to the vector-valued one allows us not to impose structure conditions on the integrand $f(x, \xi)$ with dependence on the modulus of the gradient, i.e. $f(x, \xi) = g(x, |\xi|)$. Without imposing structure conditions, we prove that if $\frac{q}{p}$ is sufficiently close to 1, then every minimizer is locally Lipschitz-continuous.

Keywords: Elliptic equations, local minimizers, local Lipschitz continuity, p, q -growth, general growth conditions

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1 Introduction

The fundamental classical problem of the Calculus of Variations in the scalar case usually is formulated as finding a function u assuming a given value u_0 at the boundary $\partial\Omega$ of an open bounded set $\Omega \subset \mathbb{R}^n$ which minimizes the integral

$$\int_{\Omega} f(x, Dv) \, dx \quad (1.1)$$

among all functions $v : \Omega \rightarrow \mathbb{R}$, assuming the same boundary value u_0 as u . The precise functional space where to look for solutions depends on the growth conditions of $f = f(x, \xi)$ as $\xi \in \mathbb{R}^n$ grows in modulus to $+\infty$. Usually this growth is stated in terms of an inequality of the type

$$f(x, \xi) \geq M_1 |\xi|^p \quad (1.2)$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$ and for some positive constant M_1 . Here $p = 1$ is associated to the BV(Ω) space of functions with bounded variation, while $p > 1$ is related to the Sobolev space $W^{1,p}(\Omega)$. Usually the condition $p > 1$ and the strict convexity of $f(x, \xi)$ with respect to ξ are sufficient conditions for the existence and uniqueness of minimizers.

A different problem is the regularity of minimizers. A large literature is known about regularity (see for instance [26, 28, 30]) partly based on the nowadays classical well-known Hölder continuity result by De Giorgi [16]. To this aim it seems necessary to impose also a growth condition from above, to be associated

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to the growth condition from below in (1.2), of the type

$$f(x, \xi) \leq M_2(1 + |\xi|^q) \tag{1.3}$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$, for $q \geq p$ and for some positive constant M_2 . The so-called “natural growth conditions” appear if $q = p$, while the more general assumption $q > p$ allows us to consider a much larger class of integrals of the Calculus of Variations, such as for example

$$f(\xi) = |\xi|^p \log(1 + |\xi|) \tag{1.4}$$

or

$$f(x, \xi) = |\xi|^{p(x)} \quad \text{or} \quad f(x, \xi) = (1 + |\xi|^2)^{\frac{p(x)}{2}}. \tag{1.5}$$

We recall also the integrands recently considered in [11, 12, 19, 20], see also [2–4],

$$f(x, \xi) = a(x)|\xi|^p + b(x)|\xi|^q, \tag{1.6}$$

where $a(x), b(x) \geq 0$ and possibly zero on some part of Ω , being at least one of the two coefficients positive at almost every $x \in \Omega$. The above examples (1.4), (1.5), (1.6) enter in the theory presented in this paper. However, here we study the more general case with $f = f(x, \xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ of the type $f(x, \xi) = g(x, |\xi|)$.

We assume that $f : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$ is a convex function with respect to the gradient variable and it is strictly convex only at infinity. More precisely, there exists $M_0 > 0$ such that $f_{\xi\xi}, f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} M_1|\xi|^{p-2}|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x, \xi)\lambda_i\lambda_j, \\ |f_{\xi\xi}(x, \xi)| \leq M_2|\xi|^{q-2}, \\ |f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{\frac{p+q-2}{2}} \end{cases} \tag{1.7}$$

for a.e. $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$, with $|\xi| \geq M_0$ and for positive constants M_1, M_2 . Here $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$.

Model integrands satisfying condition (1.7) are, for instance, the function $f(x, \xi)$ in (1.6) and also

$$f(x, \xi) = |\xi|^p + c(x)|\xi|^s + |\xi_n|^q, \tag{1.8}$$

ξ_n being the last component (or any other component) of the vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, when

$$s \leq \frac{p+q}{2}.$$

For instance, when $s = p$ and $q \geq p$, we are considering energy integrals with integrand of the type (we denote here $a(x) = 1 + c(x)$ a generic positive coefficient)

$$f(x, \xi) = a(x)|\xi|^p + |\xi_n|^q. \tag{1.9}$$

Note that the cases (1.8) and (1.9) can be handled with Theorem 1.1. On the other hand example (1.10) below enters in Theorem 1.2:

$$f(x, \xi) = |\xi|^p + b(x)|\xi|^q. \tag{1.10}$$

The main regularity result that we prove here is the following a-priori estimate.

Theorem 1.1 (A-priori estimate). *Let $u \in W^{1,p}(\Omega)$ be a smooth local minimizer of the integral functional (1.1) with exponents p, q fulfilling*

$$\frac{q}{p} < 1 + 2\left(\frac{1}{n} - \frac{1}{r}\right). \tag{1.11}$$

Under the growth assumption (1.7), there exist positive constants C, β, γ depending on $n, r, p, q, M_0, M_1, M_2$ such that, for every $0 < \rho < R \leq \rho + 1$,

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq C \left(\frac{\|1 + h\|_{L^r(\Omega)}}{R - \rho} \right)^{\beta\gamma} \left(\int_{B_R} \{1 + |Du|^p\} dx \right)^{\frac{\gamma}{p}}. \tag{1.12}$$

Note that to get regularity of solutions it is natural, and also necessary, to assume that the gap $q - p$ is small or that $\frac{q}{p}$ is close to 1, because of the known counterexamples [27, 31, 33].

The L^∞ -bound of the gradient is obtained through several steps. The first step of the a-priori estimate is Lemma 2.3 below, where on the right-hand side of the a-priori estimate there is the norm of the minimizer u in $W^{1,qm}(\Omega)$ ($m = \frac{r}{r-2}$), and the exponents p, q are related by the condition

$$\frac{q}{p} < 1 + \frac{2n}{n-2} \left(\frac{1}{n} - \frac{1}{r} \right) \tag{1.13}$$

with $n \geq 3$. Note that if $r = +\infty$ in (1.11) and (1.13), we recover the bounds in [33, Theorems 2.1 and 3.1]. An interpolation method allows us to obtain (1.12).

The mathematical literature on the regularity under p, q -growth is now very large; we refer to [32–35] and to [36] for a complete survey on the subject. A new impulse to the subject has been given by the recent articles already cited [11, 12, 15, 20] for the case of elliptic equations and by [6–8] for the case of parabolic equations and systems under p, q -growth. We observe that here the ellipticity and growth assumptions hold only for large values of the gradient variable, i.e. we consider functionals which are *uniformly convex only at infinity*. In this context see [10, 14, 25] and recently [13, 19, 20]. The Sobolev dependence on x has recently been considered in [1, 37] and for obstacle problems in [21].

The previous a-priori estimate, more precisely Theorem 2.6, under assumptions (1.7) where the last condition is replaced by

$$|f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{q-1} \tag{1.14}$$

for a.e. $x \in \Omega$, with $|\xi| \geq M_0$, allows us to obtain the following existence and regularity result.

Theorem 1.2 (Existence and regularity). *Assume that f satisfies (1.7) and (1.14) with $1 < p \leq q$ and*

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}.$$

The Dirichlet problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$, with F defined in (1.15) below and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.

Here we emphasize the definition (i.e. the precise meaning) of the integral $F(u)$ to be minimized; in fact, the integral in (1.1) is well defined if $u \in W_{loc}^{1,q}(\Omega)$, due to the growth assumption in (1.3), but a-priori it is not uniquely defined if $u \in W^{1,p}(\Omega) \setminus W_{loc}^{1,q}(\Omega)$. In this context of x -dependence, we cannot a-priori exclude the Lavrentiev phenomenon; however, note that in Section 5 we assume a special form of f to rule out this possibility.

For the gap in the Lavrentiev phenomenon we refer to [9, 39] and recently [22–24] for related results.

For the functional F we adopt the classical definition which refers to the pioneering research by Serrin [38] (see also [29]), which is related to the Γ -convergence theory by De Giorgi [17]. Precisely, for all $u \in W^{1,p}(\Omega)$,

$$F(u) = \inf \left\{ \liminf_k \int_{\Omega} f(x, Du_k) dx : u_k \in W_0^{1,q}(\Omega) + u_0, u_k \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \right\}. \tag{1.15}$$

We discuss more in details in Section 3 the definition of F in (1.15), while in Section 2 we give the proof of the a-priori estimate. Finally, in Section 4 we give the proof of Theorem 1.2.

2 A-priori estimates

Let us start with two technical lemmas.

Lemma 2.1. *The inequality*

$$(1+t)^\beta \leq c_\beta \left(1 + \int_0^t (1+s)^{\beta-2} s ds \right) \tag{2.1}$$

holds for every $t \in [0, +\infty)$ and every $\beta \in (0, +\infty)$, where

$$c_\beta = \frac{\beta}{1 - (1 + \beta(\beta - 1))^{-\frac{1}{\beta}}} \tag{2.2}$$

if $\beta \neq 1$, while (by continuity)

$$c_1 = \lim_{\beta \rightarrow 1} c_\beta = \frac{e}{e-1}. \tag{2.3}$$

Proof. In order to prove inequality (2.1) we first consider the case $\beta = 1$.

Step 1 ($\beta = 1$). We compute the integral on the right-hand side of (2.1):

$$\int_0^t (1+s)^{-1} s \, ds = \int_0^t (1 - (1+s)^{-1}) \, ds = t - \log(1+t)$$

and inequality (2.1) becomes

$$1+t \leq c_1(1+t - \log(1+t)),$$

which is equivalent to

$$\frac{\log(1+t)}{1+t} \leq \frac{c_1 - 1}{c_1}.$$

A computation shows that $g(t) =: \frac{\log(1+t)}{1+t}$ is positive for $t \in (0, +\infty)$ and has a maximum at $t = e - 1$, thus

$$g(t) =: \frac{\log(1+t)}{1+t} \leq g(e-1) = \frac{1}{e};$$

with the position $\frac{c_1-1}{c_1} =: \frac{1}{e}$ we find (2.3).

Step 2 ($\beta \neq 1$). We compute the integral on the right-hand side of (2.1) under the condition $\beta \neq 1$ and with the notation $r =: 1 + s$:

$$\begin{aligned} \int_0^t (1+s)^{\beta-2} s \, ds &= \int_1^{t+1} r^{\beta-2} (r-1) \, dr \\ &= \int_1^{t+1} r^{\beta-1} \, dr - \int_1^{t+1} r^{\beta-2} \, dr \\ &= \left[\frac{r^\beta}{\beta} \right]_{r=1}^{r=t+1} - \left[\frac{r^{\beta-1}}{\beta-1} \right]_{r=1}^{r=t+1} \\ &= \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)}. \end{aligned}$$

Inequality (2.1) takes then the form

$$\frac{1}{c_\beta} (1+t)^\beta \leq 1 + \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)}.$$

We can write it equivalently

$$g(t) \leq 1 + \frac{1}{\beta(\beta-1)}, \tag{2.4}$$

where

$$g(t) =: \frac{(t+1)^{\beta-1}}{\beta-1} - \left(\frac{1}{\beta} - \frac{1}{c_\beta} \right) (t+1)^\beta.$$

We can compute the maximum of $g(t)$ when $t \in [0, +\infty)$. We find that the derivative $g'(t)$ is equal to zero if $t = \frac{\beta}{c_\beta - \beta}$ and, since $c_\beta > \beta$, we obtain

$$\max\{g(t) : t \in [0, +\infty)\} = g\left(\frac{\beta}{c_\beta - \beta}\right) = \left(\frac{c_\beta}{c_\beta - \beta}\right)^{\beta-1} \frac{1}{\beta(\beta-1)}.$$

Therefore inequality (2.4) holds if we choose c_β to satisfy the condition

$$\left(\frac{c_\beta}{c_\beta - \beta}\right)^{\beta-1} \frac{1}{\beta(\beta-1)} = 1 + \frac{1}{\beta(\beta-1)}.$$

A further computation gives for c_β the explicit expression in (2.2). Note that $c_\beta \rightarrow c_1$ as $\beta \rightarrow 1$. □

In the sequel we apply the previous lemma to get the a-priori estimates in particular to deal with the left-hand side of (2.26), with $\beta = \frac{\gamma}{2} + \frac{\beta_0}{2}$, for $\gamma \geq 0$; thus $\beta \geq \frac{\beta_0}{2}$. In the next result in fact we consider $\beta \in [\beta_0, +\infty)$ for some fixed $\beta_0 > 0$.

Lemma 2.2. *Let $\beta_0 > 0$. There exist constants c' and c'' , depending on β_0 but independent of $\beta \geq \beta_0$ and of $t \geq 0$, such that*

$$(1+t)^\beta \leq c' \frac{\beta^2}{\log(1+\beta)} \left(1 + \int_0^t (1+s)^{\beta-2} s \, ds \right), \tag{2.5}$$

$$(1+t)^\beta \leq c'' \beta^2 \left(1 + \int_0^t (1+s)^{\beta-2} s \, ds \right) \tag{2.6}$$

for every $\beta \in [\beta_0, +\infty)$ and every $t \in [0, +\infty)$.

Proof. First we show that the constant c_β is bounded independently of $\beta \leq 1$ if $\beta \in [\beta_0, 1]$ (here we assume that $\beta_0 \in (0, 1)$, otherwise nothing to be proved at this step). Precisely, we show that

$$c_\beta \leq \frac{\beta}{1 - e^{-\beta_0}} \quad \text{for all } \beta \in [\beta_0, 1]. \tag{2.7}$$

In fact, by the inequality $\log t \leq t - 1$, valid for all $t > 0$, by posing $t = 1 + \beta(\beta - 1)$ if $\beta < 1$, we obtain

$$(1 + \beta(\beta - 1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} \leq e^{\frac{\beta(\beta-1)}{1-\beta}} = e^{-\beta}$$

and (2.7) follows if $\beta \in [\beta_0, 1)$, since

$$c_\beta = \frac{\beta}{1 - (1 + \beta(\beta - 1))^{\frac{1}{1-\beta}}} \leq \frac{\beta}{1 - e^{-\beta}} \leq \frac{\beta}{1 - e^{-\beta_0}}.$$

Finally, if $\beta = 1$, then $c_1 = \frac{e}{e-1} < \frac{1}{1 - e^{-\beta_0}}$ holds, since it is equivalent to $1 < e^{1-\beta_0}$.

We now consider the case $\beta > 1$. By Taylor's formula we get

$$(1 + \beta(\beta - 1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} = 1 + \frac{\log(1 + \beta(\beta - 1))}{1 - \beta} + o\left(\frac{\log(1 + \beta(\beta - 1))}{1 - \beta}\right)$$

and thus the quantity

$$\frac{c_\beta \log(1 + \beta)}{\beta^2} = \frac{\log(1 + \beta)}{\beta \left[\frac{\log(1+\beta(\beta-1))}{\beta-1} + o\left(\frac{\log(1+\beta(\beta-1))}{1-\beta}\right) \right]}$$

has a finite limit as $\beta \rightarrow +\infty$ (equal to $\frac{1}{2}$) and it is a bounded function for $\beta \in [1, +\infty)$, let us say bounded by c' . This proves (2.5). The other inequality (2.6) is a direct consequence of (2.5). \square

Let now Ω be an open bounded subset of \mathbb{R}^n for $n \geq 2$ and assume that f satisfies (1.7). We observe that we can transform $f(x, \xi)$ into $f(x, M_0 \xi)$, which satisfies the same assumptions for $|\xi| \geq 1$ (with different constants depending on M_0). Then it is sufficient to obtain the a-priori bound and the regularity results for $v = \frac{1}{M_0} u$. Therefore, for clarity of exposition and without loss of generality, we assume $M_0 = 1$. Throughout the paper we will denote by B_ρ and B_R balls of radii, respectively, ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say, $x_0 \in \Omega$.

In this section we assume the following supplementary assumptions on f . Assume that $f \in \mathcal{C}^2(\Omega \times \mathbb{R}^n)$ and there exist two positive constants k and K such that for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$,

$$\begin{cases} k(1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j, \\ |f_{\xi \xi}(x, \xi)| \leq K(1 + |\xi|^2)^{\frac{q-2}{2}}, \\ |f_{\xi x}(x, \xi)| \leq K(1 + |\xi|^2)^{\frac{q-1}{2}}. \end{cases} \tag{2.8}$$

In the next lemma, we obtain an a-priori estimate for the L^∞ -norm of the gradient of u which is independent of k and K .

Lemma 2.3. *Let u be a local minimizer of the integral functional (1.1) with f satisfying (1.7) and (2.8) with*

$$\frac{q}{p} < 1 + \frac{2\alpha}{n-2} \quad \text{with } \alpha = 1 - \frac{n}{r} \tag{2.9}$$

if $n \geq 3$ and $p < q$ if $n = 2$. Then there exists a positive constants C depending only on n, r, p, q, M_1, M_2 (depending also on $|\Omega|$ when $n = 2$) such that

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq C \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \right]^{\theta \tilde{\beta}} \left(\int_{B_R} \{1 + |Du|^{qm}\} dx \right)^{\frac{\theta}{qm}} \tag{2.10}$$

for every $0 < \rho < R \leq \rho + 1$, where

$$\tilde{\beta} := \frac{2^*}{p \frac{2^*}{2} - qm}, \quad \theta := \frac{qm(\frac{2^*}{2m} - 1)}{p \frac{2^*}{2} - qm}, \quad m := \frac{r}{r-2}. \tag{2.11}$$

Remark 2.4. We observe that

$$1 \leq m := \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}, \quad \text{since } r > n; \tag{2.12}$$

the last inequality holds for $n > 2$, while we set 2^* equal to any fixed real number greater than 2 if $n = 2$. Moreover, we also have

$$\frac{1}{2m} - \frac{1}{2^*} = \frac{r-2}{2r} - \frac{n-2}{2n} = \frac{n(r-2) - r(n-2)}{2nr} = \frac{r-n}{nr} = \frac{1}{n} - \frac{1}{r} = \frac{\alpha}{n}, \tag{2.13}$$

therefore (2.9) can be equivalently expressed as

$$\frac{q}{p} < \frac{2^*}{2m} \tag{2.14}$$

because

$$1 + \frac{2\alpha}{n-2} = 1 + \frac{2^* \alpha}{n} \stackrel{(2.13)}{=} 1 + 2^* \left(\frac{1}{2m} - \frac{1}{2^*} \right) = \frac{2^*}{2m}.$$

Therefore, due to (2.14), in (2.11) we have $\tilde{\beta} > 0$ and $\theta > 1$.

Remark 2.5. The result obtained is sharp in the sense that if $m = 1$ ($r = +\infty$), then the relation between p and q reduces to the analogous one in [33, Theorem 2.1], i.e. $\frac{q}{p} < \frac{n}{n-2}$.

Proof. Let $u \in W^{1,q}(\Omega)$ be a local minimizer of (1.1). Then u satisfies the Euler first variation

$$\int_{\Omega} \sum_{i=1}^n f_{\xi_i}(x, Du) \varphi_{x_i}(x) dx = 0 \quad \text{for all } \varphi \in W_0^{1,q}(\Omega).$$

By (2.8), the technique of the difference quotients (see [18, 30], in particular [28, Chapter 8, Sections 8.1 and 8.2]) gives

$$u \in W_{loc}^{1,\infty}(\Omega) \cap W_{loc}^{2,\min(2,q)}(\Omega) \quad \text{and} \quad (1 + |Du|^2)^{\frac{q-2}{2}} |D^2u|^2 \in L_{loc}^1(\Omega). \tag{2.15}$$

Let $\eta \in C_0^1(\Omega)$ and for any fixed $s \in \{1, \dots, n\}$ define

$$\varphi = \eta^2 u_{x_s} \Phi(|Du| - 1)_+$$

for $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0, +\infty)$, such that $\Phi(0) = \Phi'(0) = 0$ and

$$\Phi'(s)s \leq c_\Phi \Phi(s) \tag{2.16}$$

for a suitable constant $c_\Phi \geq 1$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote

$$\overline{\Phi}(|Du| - 1)_+ = \Phi(|Du| - 1)_+.$$

We have then

$$\varphi_{x_i} = 2\eta\eta_{x_i}u_{x_s}\Phi(|Du| - 1)_+ + \eta^2u_{x_sx_i}\Phi(|Du| - 1)_+ + \eta^2u_{x_s}\Phi'(|Du| - 1)_+[(|Du| - 1)_+]_{x_i}. \tag{2.17}$$

Let $q \geq 2$. By (2.15) we have that $|D^2u|^2 \in L^1_{\text{loc}}(\Omega)$. Otherwise if $1 < q < 2$, we use the fact that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ to infer that there exists $M = M(\text{supp } \varphi)$ such that

$$|Du(x)| \leq M \quad \text{for a.e. } x \in \text{supp } \varphi.$$

Now since $q - 2 < 0$, we have

$$(1 + M^2)^{\frac{q-2}{2}}|D^2u|^2 \leq (1 + |Du|^2)^{\frac{q-2}{2}}|D^2u|^2,$$

and by (2.15) we again get $|D^2u|^2 \in L^1(\text{supp } \varphi)$. Therefore we can insert φ_{x_i} in the following second variation,

$$\int_{\Omega} \left\{ \sum_{i,j=1}^n f_{\xi_i\xi_j}(x, Du)u_{x_jx_s}\varphi_{x_i} + \sum_{i=1}^n f_{\xi_ix_s}(x, Du)\varphi_{x_i} \right\} dx = 0 \quad \text{for all } s = 1, \dots, n,$$

and we obtain

$$\begin{aligned} 0 &= \sum_s \left[\int_{\Omega} 2\eta\Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}u_{x_sx_j} dx \right. \\ &\quad + \int_{\Omega} \eta^2\Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)u_{x_sx_i}u_{x_sx_j} dx \\ &\quad + \int_{\Omega} \eta^2\Phi'(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)u_{x_s}u_{x_sx_j}[(|Du| - 1)_+]_{x_i} dx \\ &\quad + \int_{\Omega} 2\eta\Phi(|Du| - 1)_+ \sum_i f_{\xi_ix_s}(x, Du)\eta_{x_i}u_{x_s} dx \\ &\quad + \int_{\Omega} \eta^2\Phi(|Du| - 1)_+ \sum_i f_{\xi_ix_s}(x, Du)u_{x_sx_i} dx \\ &\quad \left. + \int_{\Omega} \eta^2\Phi'(|Du| - 1)_+ \sum_i f_{\xi_ix_s}(x, Du)u_{x_s}[(|Du| - 1)_+]_{x_i} dx \right] \\ &=: \sum_s (I_1^s + I_2^s + I_3^s + I_4^s + I_5^s + I_6^s). \end{aligned} \tag{2.18}$$

In the following, constants will be denoted by C , regardless of their actual value.

Let us start with the estimate of the first integral in (2.18). By the Cauchy–Schwarz inequality, the Young inequality and (1.7), we have

$$\begin{aligned} \left| \sum_s I_1^s \right| &= \left| \int_{\Omega} 2\eta\Phi(|Du| - 1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}u_{x_sx_j} dx \right| \\ &\leq \int_{\Omega} 2\eta\Phi(|Du| - 1)_+ \left\{ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}\eta_{x_j}u_{x_s} \right\}^{\frac{1}{2}} \left\{ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)u_{x_sx_i}u_{x_sx_j} \right\}^{\frac{1}{2}} dx \\ &\leq C \int_{\Omega} |D\eta|^2\Phi(|Du| - 1)_+|Du|^q dx + \frac{1}{2} \int_{\Omega} \eta^2\Phi(|Du| - 1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)u_{x_sx_i}u_{x_sx_j} dx. \end{aligned}$$

Let us consider the third integral in (2.18). First of all we observe that

$$[(|Du| - 1)_+]_{x_i} \sum_s u_{x_s}u_{x_sx_j} = [(|Du| - 1)_+]_{x_i}|Du|[(|Du| - 1)_+]_{x_j}.$$

This entails using (1.7)

$$\begin{aligned} \sum_s I_3^s &= \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j,s} f_{\xi_i \xi_j}(x, Du) u_{x_s} u_{x_s x_j} [(|Du| - 1)_+]_{x_i} dx \\ &\geq M_1 \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ |Du|^{p-1} |D(|Du| - 1)_+|^2 dx \geq 0. \end{aligned}$$

We now deal with the fourth integral in (2.18). We have

$$\begin{aligned} \left| \sum_s I_4^s \right| &= \left| \int_{\Omega} 2\eta \Phi(|Du| - 1)_+ \sum_{i,s} f_{\xi_i x_s}(x, Du) \eta_{x_i} u_{x_s} dx \right| \\ &\stackrel{(1.7)}{\leq} \int_{\Omega} 2\eta \Phi(|Du| - 1)_+ h(x) |Du|^{\frac{p+q-2}{2}} \sum_{i,s} |\eta_{x_i} u_{x_s}| dx \\ &\leq C \int_{\Omega} (\eta^2 + |D\eta|^2) h(x) \Phi(|Du| - 1)_+ |Du|^q dx. \end{aligned}$$

Consider now the fifth integral in (2.18). We have

$$\begin{aligned} \left| \sum_s I_5^s \right| &= \left| \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ \sum_{i,s} f_{\xi_i x_s}(x, Du) u_{x_s x_j} dx \right| \\ &\stackrel{(1.7)}{\leq} \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ h(x) |Du|^{\frac{p+q-2}{2}} |D^2 u| dx \\ &\leq \int_{\Omega} [\eta^2 \Phi(|Du| - 1)_+ |Du|^{p-2} |D^2 u|^2]^{\frac{1}{2}} [\eta^2 \Phi(|Du| - 1)_+ |h(x)|^2 |Du|^q]^{\frac{1}{2}} dx \\ &\leq \varepsilon \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ |Du|^{p-2} |D^2 u|^2 dx + C_{\varepsilon} \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ |h(x)|^2 |Du|^q dx, \end{aligned}$$

where in the last line we used the Young inequality. Finally, for any $0 < \delta < 1$,

$$\begin{aligned} \left| \sum_s I_6^s \right| &= \left| \int_{\Omega} \eta^2 \sum_{i,s} f_{\xi_i x_s}(x, Du) u_{x_s} \Phi'(|Du| - 1)_+ [(|Du| - 1)_+]_{x_i} dx \right| \\ &\stackrel{(1.7)}{\leq} \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ h(x) |Du|^{\frac{p+q-2}{2}} |Du| |D(|Du| - 1)_+| dx \\ &\leq \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ h(x) |Du|^{\frac{p+q}{2}} |D^2 u| dx \\ &= \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ h(x) [(|Du| - 1)_+ + \delta] [(|Du| - 1)_+ + \delta]^{-1} |Du|^{\frac{p+q}{2}} |D^2 u| dx \\ &\leq \int_{\Omega} \eta^2 \left\{ \frac{1}{c_{\Phi}} \Phi'(|Du| - 1)_+ [(|Du| - 1)_+ + \delta] |Du|^{p-2} |D^2 u|^2 \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ c_{\Phi} \Phi'(|Du| - 1)_+ |h(x)|^2 |Du|^{q+2} [(|Du| - 1)_+ + \delta]^{-1} \right\}^{\frac{1}{2}} dx \\ &\leq C_{\varepsilon} c_{\Phi} \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ |h(x)|^2 |Du|^{q+2} [(|Du| - 1)_+ + \delta]^{-1} dx \\ &\quad + \frac{\varepsilon}{c_{\Phi}} \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ [(|Du| - 1)_+ + \delta] |Du|^{p-2} |D^2 u|^2 dx. \end{aligned}$$

Since $\Omega = \{x : |Du(x)| \geq 2\} \cup \{x : |Du(x)| < 2\}$ and $(|Du| - 1)_+ \geq 1$ in $\{x : |Du(x)| \geq 2\}$, we also have

$$(|Du| - 1)_+ + \delta \leq 2(|Du| - 1)_+ \tag{2.19}$$

as long as we have chosen $\delta < 1$. Therefore, using (2.16), we can estimate the last integral as

$$\begin{aligned}
 & \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ [(|Du| - 1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
 &= \int_{|Du| \geq 2} \eta^2 \Phi'(|Du| - 1)_+ [(|Du| - 1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
 &\quad + \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ [(|Du| - 1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
 &\stackrel{(2.19)}{\leq} 2 \int_{|Du| \geq 2} \eta^2 \Phi'(|Du| - 1)_+ (|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
 &\quad + \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ (|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
 &\quad + \delta \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
 &\stackrel{(2.16)}{\leq} 2c_{\Phi} \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx + \delta \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx.
 \end{aligned}$$

Therefore we finally have

$$\begin{aligned}
 \left| \sum_s I_6^s \right| &\leq C_{\varepsilon} c_{\Phi} \int_{\Omega} \eta^2 \Phi'(|Du| - 1)_+ |h(x)|^2 |Du|^{q+2} [(|Du| - 1)_+ + \delta]^{-1} \, dx \\
 &\quad + 2\varepsilon \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx + \delta\varepsilon \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx.
 \end{aligned}$$

Now, for ε sufficiently small and putting together all the previous estimates, we deduce that there exists a constant C depending on n, r, p, q, M_1 such that

$$\begin{aligned}
 & \int_{\Omega} \eta^2 \Phi(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
 &\leq Cc_{\Phi} \int_{\Omega} (\eta^2 + |D\eta|^2) (1 + h(x))^2 |Du|^q [\Phi(|Du| - 1)_+ + \Phi'(|Du| - 1)_+ |Du|^2 [(|Du| - 1)_+ + \delta]^{-1}] \, dx \\
 &\quad + \delta \int_{1 < |Du| < 2} \eta^2 \Phi'(|Du| - 1)_+ |Du|^{p-2} |D^2u|^2 \, dx. \tag{2.20}
 \end{aligned}$$

Let us now set

$$\Phi(s) := (1 + s)^{\gamma-2} s^2, \quad \gamma \geq 0,$$

with

$$\Phi'(s) = (\gamma s + 2)s(1 + s)^{\gamma-3}.$$

It is easy to check that Φ satisfies (2.16) with $c_{\Phi} = 2(1 + \gamma)$.

We now approximate this function Φ by a sequence of functions Φ_h , each of them being equal to Φ in the interval $[0, h]$, and then extended to $[h, +\infty)$ with the constant value $\Phi(h)$. Since Φ_h and Φ'_h converge monotonically to Φ and Φ' , by inserting Φ_h in (2.20), it is possible to pass to the limit as $h \rightarrow +\infty$ by the Monotone Convergence Theorem.

Therefore, for every $0 < \delta < 1$, since

$$\frac{(|Du| - 1)_+}{(|Du| - 1)_+ + \delta} \leq 1 \quad \text{for all } \delta > 0$$

and $\Phi'(t - 1)_+ \leq C(y)$ when $1 < t < 2$, we obtain

$$\begin{aligned} & \int_{\Omega} \eta^2 (1 + (|Du| - 1)_+)^{y-2} (|Du| - 1)_+^2 |Du|^{p-2} |D^2u|^2 dx \\ & \leq C(1 + y)^2 \int_{\Omega} (\eta^2 + |D\eta|^2) (1 + h(x))^2 (1 + (|Du| - 1)_+)^{y+q} dx + \delta C(y) \int_{1 < |Du| < 2} \eta^2 |Du|^{p-2} |D^2u|^2 dx. \end{aligned} \quad (2.21)$$

Using [20, formula (3.51) of Lemma 3.3], namely the fact that $|Du|^{p-2} \leq C(p)(1 + |Du|^2)^{\frac{p-2}{2}}$ when $|Du| > 1$, we have

$$\int_{1 < |Du| < 2} \eta^2 |Du|^{p-2} |D^2u|^2 dx \leq C \int_{1 < |Du| < 2} \eta^2 (1 + |Du|^2)^{\frac{p-2}{2}} |D^2u|^2 dx < +\infty$$

by (2.15), and for $\delta \rightarrow 0$ and the last term in the previous inequality vanishes.

Since $h \in L^r(\Omega)$, by the Hölder inequality, since $\frac{1}{m} + \frac{2}{r} = 1$, by (2.21) we have

$$\begin{aligned} & \int_{\Omega} \eta^2 (1 + (|Du| - 1)_+)^{y-2} (|Du| - 1)_+^2 |Du|^{p-2} |D((|Du| - 1)_+)|^2 dx \\ & \leq C(1 + y)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_{\Omega} (\eta^2 + |D\eta|^2)^m (1 + (|Du| - 1)_+)^{(y+q)m} dx \right]^{\frac{1}{m}}. \end{aligned} \quad (2.22)$$

Let us introduce

$$G(t) = 1 + \int_0^t (1 + s)^{\frac{y}{2} + \frac{p}{2} - 2} s ds. \quad (2.23)$$

We obtain

$$[G(t)]^2 \leq 4(1 + t)^{y+p} \leq 4(1 + t)^{y+q}, \quad (2.24)$$

where we used the fact that $p \leq q$. On the other hand

$$G'(t) = (1 + t)^{\frac{y}{2} + \frac{p}{2} - 2} t, \quad (2.25)$$

which in turn allows us to give the following estimate for the gradient of the function $w = \eta G((|Du| - 1)_+)$:

$$\begin{aligned} & \int_{\Omega} |D(\eta G((|Du| - 1)_+))|^2 dx \\ & \leq 2 \int_{\Omega} |D\eta|^2 |G((|Du| - 1)_+)|^2 dx + 2 \int_{\Omega} \eta^2 [G_t((|Du| - 1)_+)]^2 [D((|Du| - 1)_+)]^2 dx \\ & \leq C(1 + y)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_{\Omega} (\eta^2 + |D\eta|^2)^m [1 + (|Du| - 1)_+]^{(y+q)m} dx \right]^{\frac{1}{m}}, \end{aligned}$$

the second inequality by (2.22), (2.24), (2.25). By Sobolev's inequality there exists a constant C (depending also on $|\Omega|$ when $n = 2$) such that

$$\left\{ \int_{\Omega} [\eta G((|Du| - 1)_+)]^{2^*} dx \right\}^{\frac{2}{2^*}} \leq C \int_{\Omega} |D(\eta G((|Du| - 1)_+))|^2 dx$$

and by the previous inequality we get (for a different constant)

$$\left\{ \int_{\Omega} [\eta G((|Du| - 1)_+)]^{2^*} dx \right\}^{\frac{2}{2^*}} \leq C(1 + y)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_{\Omega} (\eta^2 + |D\eta|^2)^m [1 + (|Du| - 1)_+]^{(y+q)m} dx \right]^{\frac{1}{m}}. \quad (2.26)$$

We take into account the definition of $G(t)$ in (2.23) and we use Lemma 2.2, and in particular formula (2.6) with $\beta = \frac{y+p}{2}$. Being $\gamma \geq 0$, we have $\beta \geq \beta_0 := \frac{p}{2} > 0$ and

$$(1 + t)^{\frac{y+p}{2}} \leq c'' \left(\frac{y+p}{2} \right)^2 \left(1 + \int_0^t (1 + s)^{\frac{y+p}{2} - 2} s ds \right)$$

for every $\gamma \geq 0$ and every $t \in [0, +\infty)$. In terms of $G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma}{2} + \frac{p}{2} - 2} s \, ds$ equivalently

$$(1+t)^{\frac{\gamma+p}{2}} \leq c'' \left(\frac{\gamma+p}{2} \right)^2 G(t) \quad \text{for all } \gamma \geq 0 \text{ and all } t \geq 0.$$

Therefore, if $t := (|Du| - 1)_+$,

$$(1 + (|Du| - 1)_+)^{\frac{\gamma+p}{2} 2^*} \leq (c'')^{2^*} \left(\frac{\gamma+p}{2} \right)^{2 \cdot 2^*} [G((|Du| - 1)_+)]^{2^*} \quad \text{for all } \gamma \geq 0,$$

and by (2.26) we finally get

$$\begin{aligned} \left\{ \int_{\Omega} \eta^{2^*} [1 + (|Du| - 1)_+]^{\frac{\gamma+p}{2} 2^*} dx \right\}^{\frac{2}{2^*}} &\leq (c'')^2 \left(\frac{\gamma+p}{2} \right)^4 \left\{ \int_{\Omega} [\eta G((|Du| - 1)_+)]^{2^*} dx \right\}^{\frac{2}{2^*}} \\ &\leq C(\gamma+1)^6 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_{\Omega} (\eta^2 + |D\eta|^2)^m [1 + (|Du| - 1)_+]^{(\gamma+q)m} dx \right]^{\frac{1}{m}} \end{aligned}$$

with a new constant C and for every $\gamma \geq 0$.

As usual we consider a test function η equal to 1 in a ball B_ρ , with $\text{supp } \eta \subset B_R$ and such that $|D\eta| \leq \frac{2}{(R-\rho)}$. We get

$$\left[\int_{B_\rho} [1 + (|Du| - 1)_+]^{[(\gamma+p)m] \frac{2^*}{2m}} dx \right]^{\frac{2m}{2^*}} \leq C_0 \|1 + h\|_{L^r(\Omega)}^{2m} \frac{(\gamma+q)^{6m}}{(R-\rho)^{2m}} \int_{B_R} [1 + (|Du| - 1)_+]^{(\gamma+q)m} dx, \quad (2.27)$$

where the constant C_0 only depends on n, r, p, q, M_1, M_2 but is independent of γ .

Fixed $0 < \rho_0 < R_0 \leq \rho_0 + 1$, we define the following decreasing sequence of radii $\{\rho_k\}_{k \geq 1}$:

$$\rho_k = \rho_0 + \frac{R_0 - \rho_0}{2^k} \quad \text{for all } k \geq 1.$$

We define recursively a sequence α_k in the following way:

$$\alpha_1 := 0, \quad \alpha_{k+1} := (\alpha_k + pm) \frac{2^*}{2m} - qm. \quad (2.28)$$

Then we have the following representation formula for α_k which can easily be proved by induction:

$$\alpha_k = \left(p \frac{2^*}{2} - qm \right) \frac{[(\frac{2^*}{2m})^{k-1} - 1]}{\frac{2^*}{2m} - 1}. \quad (2.29)$$

We rewrite (2.27) with $R = \rho_k, \rho = \rho_{k+1}, \gamma = \frac{\alpha_k}{m}$ and observe that

$$R - \rho := \rho_k - \rho_{k+1} = \frac{R_0 - \rho_0}{2^{k+1}}.$$

Set, for all $k \geq 1$,

$$\begin{aligned} A_k &:= \left(\int_{B_{\rho_k}} [1 + (|Du| - 1)_+]^{\alpha_k + qm} dx \right)^{\frac{1}{\alpha_k + qm}}, \\ C_k &:= C_0 \|1 + h\|_{L^r(\Omega)}^{2m} \left(\frac{(\alpha_k + qm)^3 2^{k+1}}{R_0 - \rho_0} \right)^{2m}; \end{aligned}$$

we obtain for every $k \geq 1$,

$$A_{k+1} \leq C_k^{\frac{1}{\alpha_k + pm}} A_k^{\frac{\alpha_k + qm}{\alpha_k + pm}}. \quad (2.30)$$

Let θ be defined by

$$\theta := \prod_{i=1}^{\infty} \frac{\alpha_i + qm}{\alpha_i + pm}. \quad (2.31)$$

We show that θ is finite and is given by

$$\theta = \frac{qm(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm}.$$

Indeed by (2.31), the recursive definition of α_k , i.e. (2.28) and the representation formula for α_k , namely (2.29), we have

$$\begin{aligned} \theta_k &:= \prod_{i=1}^k \frac{\alpha_i + qm}{\alpha_i + pm} \stackrel{(2.28)}{=} \frac{qm}{\alpha_k + pm} \left(\frac{2^*}{2m}\right)^{k-1} \stackrel{(2.29)}{=} \frac{qm(\frac{2^*}{2m})^{k-1}}{\frac{(p\frac{2^*}{2} - qm)[(\frac{2^*}{2m})^{k-1} - 1]}{\frac{2^*}{2m} - 1} + pm} \\ &= \frac{qm(\frac{2^*}{2m})^{k-1}(\frac{2^*}{2m} - 1)}{pm(\frac{2^*}{2m} - 1) + (p\frac{2^*}{2} - qm)[(\frac{2^*}{2m})^{k-1} - 1]}, \end{aligned}$$

which yields (2.31) once we pass to the limit as $k \rightarrow \infty$ as long as $\frac{2^*}{2m} > 1$ in view of (2.12). Note that θ makes sense due to (2.12) and the bound (2.14) in Remark 2.4.

Iterating (2.30), we deduce

$$A_{k+1} \leq \tilde{C} \left(\left[\frac{\|1 + h\|_{L^r(\Omega)}}{R_0 - \rho_0} \right]^{\tilde{\beta}} A_1 \right)^{\theta_k}, \quad k \geq 1, \tag{2.32}$$

where

$$\tilde{C} := C_0^{\tilde{\beta}\theta} \exp \left[\theta \sum_{i=1}^{\infty} \frac{\log[(\alpha_i + qm)^{6m} 2^{2m(i+1)}]}{\alpha_i + pm} \right] < +\infty,$$

which is finite because the series is convergent (α_i from the representation formula (2.29) grows exponentially) and

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{2m}{\alpha_i + pm} &\stackrel{(2.29)}{=} \sum_{i=1}^{\infty} \frac{2m}{(p\frac{2^*}{2} - qm)\frac{[(\frac{2^*}{2m})^{i-1} - 1]}{\frac{2^*}{2m} - 1} + pm} \leq \sum_{i=1}^{\infty} \frac{2m}{(p\frac{2^*}{2} - qm)\frac{(\frac{2^*}{2m})^{i-1}}{\frac{2^*}{2m} - 1}} \\ &\leq \frac{2m(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm} \sum_{i=0}^{\infty} \left(\frac{2m}{2^*}\right)^i = \frac{2m(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm} \frac{1}{(1 - \frac{2m}{2^*})} = \frac{2^*}{p\frac{2^*}{2} - qm} =: \tilde{\beta}, \end{aligned}$$

where in the first inequality we used the fact that

$$\left(p\frac{2^*}{2} - qm\right)\frac{[(\frac{2^*}{2m})^{i-1} - 1]}{\frac{2^*}{2m} - 1} + pm \geq \left(p\frac{2^*}{2} - qm\right)\frac{(\frac{2^*}{2m})^{i-1}}{\frac{2^*}{2m} - 1} \iff -\frac{p\frac{2^*}{2} - qm}{\frac{2^*}{2m} - 1} + pm \geq 0 \iff q \geq p.$$

By letting $k \rightarrow +\infty$ in (2.32), we have (2.10). Therefore, the proof of Lemma 2.3 is complete. □

The a-priori estimate (1.12) in Theorem 1.1 follows by the classical interpolation inequality

$$\|v\|_{L^s(B_\rho)} \leq \|v\|_{L^p(B_\rho)}^{\frac{p}{s}} \|v\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}} \tag{2.33}$$

for any $s \geq p$, which permits to estimate the essential supremum of the gradient of the local minimizer in terms of its L^p -norm.

Proof of Theorem 1.1. Let us set

$$V(x) := 1 + (|Du|(x) - 1)_+$$

then estimate (2.10) becomes

$$\sup_{x \in B_\rho} |V(x)| \leq C \left(\left[\frac{\|1 + h\|_{L^r(\Omega)}}{R - \rho} \right]^{\tilde{\beta}} \|V\|_{L^{qm}(B_R)} \right)^\theta \tag{2.34}$$

for every ρ, R such that $0 < \rho < R \leq \rho + 1$ and where $C = C(n, r, p, q, M_1, M_2)$.

In the following we denote

$$s := qm.$$

At this point, (2.34) and (2.33) give

$$\|V\|_{L^s(B_\rho)} \leq C^{1-\frac{p}{s}} \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}} \|V\|_{L^s(B_R)} \right)^{\theta(1-\frac{p}{qm})}. \tag{2.35}$$

We observe that

$$\tau := \theta \left(1 - \frac{p}{qm} \right) < 1, \tag{2.36}$$

because

$$\theta \left(1 - \frac{p}{qm} \right) < 1 \iff \frac{q\frac{2^*}{2} - qm - \frac{p2^*}{2m} + p}{p\frac{2^*}{2} - qm} < 1 \iff q < p \left(1 - \frac{2}{2^*} + \frac{1}{m} \right) \stackrel{(2.13)}{=} p \left(1 + \frac{2\alpha}{n} \right).$$

For $0 < \rho < R$ and for every $k \geq 0$, let us define $\rho_k := R - (R - \rho)2^{-k}$. By inserting in (2.35) $\rho = \rho_k$ and $R = \rho_{k+1}$, (so that $R - \rho = (R - \rho)2^{-(k+1)}$) we have, for every $k \geq 0$,

$$\|V\|_{L^s(B_{\rho_k})} \leq C^{1-\frac{p}{s}} \|V\|_{L^p(B_{\rho_k})}^{\frac{p}{s}} \left(2^{\tilde{\beta}(k+1)} \left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}} \|V\|_{L^s(B_{\rho_{k+1}})} \right)^\tau. \tag{2.37}$$

By iteration of (2.37), we deduce for $k \geq 0$,

$$\|V\|_{L^s(B_{\rho_0})} \leq \left(C^{1-\frac{p}{s}} \left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}\tau} \|V\|_{L^p(B_{\rho_k})}^{\frac{p}{s}} \right)^{\sum_{i=0}^k \tau^i} 2^{\tilde{\beta} \sum_{i=0}^{k+1} i\tau^i} (\|V\|_{L^s(B_{\rho_{k+1}})})^{\tau^{k+1}}. \tag{2.38}$$

By (2.36), the series appearing in (2.38) are convergent. Since

$$\|V\|_{L^s(B_{\rho_k})} \leq \|V\|_{L^s(B_R)},$$

we can pass to the limit as $k \rightarrow +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, q, M_1, M_2)$ independent of k ,

$$\|V\|_{L^s(B_\rho)} \leq C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}\tau} \|V\|_{L^p(B_R)}^{\frac{p}{s}} \right)^{\frac{1}{1-\tau}}. \tag{2.39}$$

Combining (2.34) and (2.39), by setting $\rho' = \frac{R+\rho}{2}$ we have

$$\begin{aligned} \|V\|_{L^\infty(B_\rho)} &\leq C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \right]^{\tilde{\beta}} \|V\|_{L^s(B_{\rho'})} \right)^\theta \\ &\leq C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \right]^{\tilde{\beta}(1-\tau)} \left[\frac{1+\|h\|_{L^r(\Omega)}}{R-\rho'} \right]^{\tilde{\beta}\tau} \|V\|_{L^p(B_R)}^{\frac{p}{s}} \right)^{\frac{\theta}{1-\tau}}; \end{aligned}$$

now, since

$$(\rho' - \rho) = (R - \rho') = \frac{R - \rho}{2},$$

we get

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^\beta \left(\int_{B_R} \{1 + |Du|^p\} dx \right)^{\frac{1}{p}} \right)^\gamma,$$

where

$$\beta := \tilde{\beta} \frac{qm}{p} = \frac{\frac{2^*q}{p}m}{p\frac{2^*}{2} - qm}, \quad \gamma := \frac{\theta p}{qm(1 - \theta(1 - \frac{p}{qm}))}, \quad \theta := \frac{qm(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm},$$

so (1.12) follows. □

Let now f satisfy (1.7) and (1.14). Under these assumptions on f , we obtain the following result.

Theorem 2.6. Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the integral functional (1.1). Assume that $f = f(x, \xi)$ in (1.1) satisfies (1.7), (1.14) and (2.8), with

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{2.40}$$

Then there exist positive constants $C, \hat{\beta}, \hat{\gamma}$ depending on $n, r, p, q, M_1, M_2, \rho, R$ such that

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq C[\|1 + h\|_{L^r(\Omega)}]^{\hat{\beta}} \hat{\gamma} \left(\int_{B_R} \{1 + f(x, Du)\} dx \right)^{\frac{\hat{\gamma}}{p}} \tag{2.41}$$

for every $0 < \rho < R \leq \rho + 1$.

Proof. Let $t := 2q - p$. Then (1.14) can be written explicitly in the form

$$|f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{\frac{t+p-2}{2}}, \quad |\xi| \geq 1.$$

Moreover, (2.40) in terms of p and t is equivalent to

$$\frac{t}{p} < 1 + \frac{2\alpha}{n} \quad \text{with } \alpha = 1 - \frac{n}{r}.$$

Thus all the assumptions of Theorem 1.1 are satisfied with q replaced by t . In particular, the second inequality in (1.7) holds with q replaced by t since $t \geq q$. Then the conclusion of Theorem 1.1 holds with $q = t$ which corresponds to (2.41) with

$$\hat{\beta} := \frac{\frac{2^*}{p}(2q-p)m}{p\frac{2^*}{2} - (2q-p)m}, \quad \hat{\gamma} := \frac{\theta p}{(2q-p)m(1-\theta) + p\theta},$$

since $f(x, \xi) \geq C|\xi|^p$ for every $|\xi| \geq 1$. □

3 Extension of the integral energy

Let $f : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$ be a continuous function, convex in ξ such that

$$|\xi|^p \leq f(x, \xi) \leq C(1 + |\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n. \tag{3.1}$$

For $u_0 \in W^{1,q}(\Omega)$, we define the extension to $W^{1,p}(\Omega)$ of the integral functional $\int_\Omega f(x, Du) dx$, i.e.

$$F(u) = \inf \left\{ \liminf_k \int_\Omega f(x, Du_k) dx : u_k \in W_0^{1,q}(\Omega) + u_0, u_k \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \right\} \tag{3.2}$$

with

$$\int_\Omega f(x, Du_0) dx < +\infty.$$

It is easy to check that

$$F(u) = \int_\Omega f(x, Du) dx \quad \text{for } u \in W_0^{1,q}(\Omega) + u_0.$$

In fact, for $u_k = u$ for all k ,

$$F(u) \leq \int_\Omega f(x, Du) dx.$$

On the other hand, by the semicontinuity of $\int_\Omega f(x, Du) dx$ with respect to the weak topology of $W^{1,p}$, the inverse inequality also holds.

Lemma 3.1. For each $v \in W_0^{1,p}(\Omega) + u_0$, there exists a sequence $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and

$$F(v) = \lim_{k \rightarrow +\infty} \int_\Omega f(x, Dv_k) dx.$$

Proof. The proof follows similarly as in [5]. We give the sketch of the proof.

Let $v \in W_0^{1,p}(\Omega) + u_0$ such that $F(v) < \infty$. Then, for all k , there exists $v_h^{(k)} \in W_0^{1,q}(\Omega) + u_0$ such that $v_h^{(k)} \xrightarrow{w} v$, as $h \rightarrow +\infty$, weakly in $W^{1,p}(\Omega)$ and

$$F(v) \leq \lim_{h \rightarrow +\infty} \int_{\Omega} f(x, Dv_h^{(k)}) \, dx \leq F(v) + \frac{1}{k}.$$

Moreover, by the weak convergence of $v_h^{(k)}$ in $W^{1,p}(\Omega)$ we get

$$\lim_{h \rightarrow +\infty} \|v_h^{(k)} - v\|_{L^p(\Omega)} = 0$$

and for h sufficiently large,

$$\int_{\Omega} |Dv_h^{(k)}|^p \, dx \leq \int_{\Omega} f(x, Dv_h^{(k)}) \, dx \leq F(v) + 1.$$

Then for all k there exists h_k such that for all $h \geq h_k$,

$$\|v_h^{(k)} - v\|_{L^p(\Omega)} < \frac{1}{k}$$

and for $h = h_k$, by denoting $w_k = v_{h_k}^{(k)}$, we have

$$\|w_k - v\|_{L^p(\Omega)} < \frac{1}{k} \quad \text{and} \quad \int_{\Omega} |Dw_k|^p \, dx \leq C;$$

then $w_k \xrightarrow{w} v$ as $k \rightarrow +\infty$ in the weak topology of $W^{1,p}(\Omega)$ and

$$F(v) \leq \int_{\Omega} f(x, Dw_k) \, dx \leq F(v) + \frac{1}{k},$$

i.e.

$$\lim_{k \rightarrow +\infty} \int_{\Omega} f(x, Dw_k) \, dx = F(v). \quad \square$$

4 Existence and regularity

First of all we prove an approximation theorem for f through a suitable sequence of regular functions.

Proposition 4.1. *Let f be satisfying the growth conditions (3.1), $f_{\xi\xi}$ and $f_{\xi x}$ Carathéodory functions, satisfying (1.7) and (1.14) with $M_0 = 1$ and f strictly convex at infinity. Then there exists a sequence of C^2 -functions $f^{\ell k} : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$, $f^{\ell k}$ convex in the last variable and strictly convex at infinity, such that $f^{\ell k}$ converges to f as $\ell \rightarrow \infty$ and $k \rightarrow \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$, where $\Omega_0 \subset\subset \Omega$ and K being a compact set of \mathbb{R}^n . Moreover:*

- there exists \tilde{C} , independently of k, ℓ , such that

$$|\xi|^p \leq f^{\ell k}(x, \xi) \leq \tilde{C}(1 + |\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n, \tag{4.1}$$

- there exists $\tilde{M}_1 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$,

$$\tilde{M}_1 |\xi|^{p-2} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}^{\ell k}(x, \xi) \lambda_i \lambda_j, \quad \lambda \in \mathbb{R}^n, \tag{4.2}$$

- there exists $c(k) > 0$ such that for all $(x, \xi) \in \Omega \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$,

$$c(k)(1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}^{\ell k}(x, \xi) \lambda_i \lambda_j, \tag{4.3}$$

- there exists $\tilde{M}_2 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$,

$$|f_{\xi\xi}^{\ell k}(x, \xi)| \leq \tilde{M}|\xi|^{q-2}, \tag{4.4}$$

- there exists $C(k)$ such that for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$|f_{\xi\xi}^{\ell k}(x, \xi)| \leq C(k)(1 + |\xi|^2)^{\frac{q-2}{2}}, \tag{4.5}$$

- there exists a constant $C > 0$ such that for a.e. $x \in \Omega$ and $|\xi| > 2$,

$$|f_{\xi x}^{\ell k}(x, \xi)| \leq Ch_\ell(x)|\xi|^{q-1}, \tag{4.6}$$

where $h_\ell \in C^\infty(\Omega)$ is the regularized function of h which converges to h in $L^r(\Omega)$,

- for $\Omega_0 \subset\subset \Omega$, there exists a constant C such that for $x \in \Omega_0$ and $\xi \in \mathbb{R}^n$,

$$|f_{\xi x}^{\ell k}(x, \xi)| \leq C(k, \ell, \Omega_0)(1 + |\xi|^2)^{\frac{q-1}{2}}. \tag{4.7}$$

Proof. We argue as in the proof of [25, Theorem 2.7 (Step 3)] and [20, Lemma 4.3]. For the sake of completeness, we give a sketch of the arguments of the proof.

Let B be the unit ball of \mathbb{R}^n centered in the origin and consider a positive decreasing sequence $\varepsilon_\ell \rightarrow 0$. We introduce

$$f^\ell(x, \xi) = \int_{B \times B} \rho(y)\rho(\eta)f(x + \varepsilon_\ell y, \xi + \varepsilon_\ell \eta) d\eta dy,$$

where ρ is a positive symmetric mollifier, and

$$f^{\ell k}(x, \xi) = f^\ell(x, \xi) + \frac{1}{k}(1 + |\xi|^2)^{\frac{q}{2}}. \tag{4.8}$$

It is easy to check that the sequence $f^{\ell k}$ satisfies conditions (4.1), (4.2), (4.3), (4.4), (4.5), (4.7). Let us verify (4.6). For $|\xi| > 2$ we have

$$|f_{\xi x}^{\ell k}(x, \xi)| \leq \int_{B \times B} \rho(y)\rho(\eta)|\xi + \varepsilon_\ell \eta|^{q-1}h(x + \varepsilon_\ell y) dy d\eta \leq Ch_\ell(x)|\xi|^{q-1},$$

where

$$h_\ell(x) = \int_B \rho(y)h(x + \varepsilon_\ell y) dy,$$

h_ℓ is a smooth function and it converges to h in $L^r(\Omega)$. Moreover,

$$|f_{\xi x}^{\ell k}(x, \xi)| \leq C(k, \Omega_0)[\|1 + h_\ell\|_{L^\infty(\Omega_0)}](1 + |\xi|^2)^{\frac{q-1}{2}}.$$

This concludes the proof. □

Proof of Theorem 1.2. For $u_0 \in W^{1,q}(\Omega)$, let us consider the variational problems

$$\inf \left\{ \int_{\Omega} f^{\ell k}(x, Dv) dx : v \in W_0^{1,q}(\Omega) + u_0 \right\}, \tag{4.9}$$

where $f^{\ell k}$ are defined in (4.8). By semicontinuity arguments, there exists $v^{\ell k} \in u_0 + W_0^{1,q}(\Omega)$, a solution to (4.9). By the growth conditions and the minimality of $v^{\ell k}$, we get

$$\begin{aligned} \int_{\Omega} |Dv^{\ell k}|^p dx &\leq \int_{\Omega} f^{\ell k}(x, Dv^{\ell k}) dx \\ &\leq \int_{\Omega} f^{\ell k}(x, Du_0) dx \\ &= \int_{\Omega} f^\ell(x, Du_0) dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} dx. \end{aligned}$$

Moreover, the properties of the convolutions imply that

$$f^\ell(x, Du_0) \xrightarrow{\ell \rightarrow \infty} f(x, Du_0) \quad \text{a.e. in } \Omega,$$

and since

$$\int_{\Omega} f^\ell(x, Du_0) \, dx \leq C \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx,$$

by the Lebesgue Dominated Convergence Theorem we deduce therefore

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \int_{\Omega} |Dv^{\ell k}|^p \, dx &\leq \lim_{\ell \rightarrow \infty} \int_{\Omega} f^\ell(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx \\ &= \int_{\Omega} f(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx. \end{aligned}$$

By Proposition 4.1, the functions $f^{\ell k}$ satisfy (1.7), (1.14) and (2.8), so we can apply the a-priori estimate (2.41) to $v^{\ell k}$ and obtain, by standard covering arguments for all $\Omega' \subset\subset \Omega$,

$$\|Dv^{\ell k}\|_{L^\infty(\Omega'; \mathbb{R}^n)} \leq C(\Omega') [\|1 + h_\ell\|_{L^r(\Omega)}]^{\hat{\beta}\hat{\gamma}} \left[\int_{\Omega} (1 + f^{\ell k}(x, Dv^{\ell k})) \, dx \right]^{\frac{\hat{\gamma}}{p}}.$$

Since $\|1 + h_\ell\|_{L^r(\Omega)} = \|(1 + h)_\ell\|_{L^r(\Omega)} \leq \|1 + h\|_{L^r(\Omega)}$, we obtain

$$\begin{aligned} \|Dv^{\ell k}\|_{L^\infty(\Omega'; \mathbb{R}^n)} &\leq C(\Omega') [\|1 + h\|_{L^r(\Omega)}]^{\hat{\beta}\hat{\gamma}} \left[\int_{\Omega} (1 + f^{\ell k}(x, Dv^{\ell k})) \, dx \right]^{\frac{\hat{\gamma}}{p}} \\ &\leq C(\Omega') [\|1 + h\|_{L^r(\Omega)}]^{\hat{\beta}\hat{\gamma}} \left[\int_{\Omega} 1 + f^\ell(x, Du_0) + \frac{1}{k} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx \right]^{\frac{\hat{\gamma}}{p}}, \end{aligned}$$

where $C, \hat{\gamma}, \hat{\beta}$ depend on $n, r, p, q, M_1, M_2, \rho, R$ but are independent of ℓ, k . Therefore we conclude that

$$\begin{aligned} v^{\ell k} &\xrightarrow{\ell \rightarrow \infty} v^k \quad \text{weakly in } W_0^{1,p}(\Omega) + u_0, \\ v^{\ell k} &\xrightarrow{\ell \rightarrow \infty} v^k \quad \text{weakly star in } W_{loc}^{1,\infty}(\Omega), \end{aligned}$$

and by the previous estimates

$$\begin{aligned} \|Dv^k\|_{L^p(\Omega; \mathbb{R}^n)} &\leq \liminf_{\ell \rightarrow \infty} \|Dv^{\ell k}\|_{L^p(\Omega; \mathbb{R}^n)} \\ &\leq \int_{\Omega} f(x, Du_0) \, dx + \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx \end{aligned}$$

and

$$\begin{aligned} \|Dv^k\|_{L^\infty(\Omega'; \mathbb{R}^n)} &\leq \liminf_{\ell \rightarrow \infty} \|Dv^{\ell k}\|_{L^\infty(\Omega'; \mathbb{R}^n)} \\ &\leq C(\Omega') [\|1 + h\|_{L^r(\Omega)}]^{\hat{\beta}\hat{\gamma}} \left[\int_{\Omega} 1 + f(x, Du_0) \, dx + \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx \right]^{\frac{\hat{\gamma}}{p}}. \end{aligned}$$

Thus we can deduce that there exists, up to subsequences, $\bar{u} \in u_0 + W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} v^k &\rightarrow \bar{u} \quad \text{weakly in } W_0^{1,p}(\Omega) + u_0, \\ v^k &\rightarrow \bar{u} \quad \text{weakly star in } W_{loc}^{1,\infty}(\Omega). \end{aligned}$$

Now, for any fixed $k \in \mathbb{N}$, using the uniform convergence of f^ℓ to f in $\Omega_0 \times K$ (for any K compact subset of \mathbb{R}^n)

and the minimality of $v^{\ell k}$, we get for all $v \in W_0^{1,q}(\Omega) + u_0$,

$$\begin{aligned} \int_{\Omega_0} f(x, Dv^k) dx &\leq \liminf_{\ell \rightarrow \infty} \int_{\Omega_0} f(x, Dv^{\ell k}) dx \\ &= \liminf_{\ell \rightarrow \infty} \int_{\Omega_0} f^\ell(x, Dv^{\ell k}) dx \\ &\leq \liminf_{\ell \rightarrow \infty} \int_{\Omega_0} f^\ell(x, Dv^{\ell k}) dx + \frac{1}{k} \int_{\Omega} (1 + |Dv^{\ell k}|^2)^{\frac{q}{2}} dx \\ &\leq \liminf_{\ell \rightarrow \infty} \int_{\Omega} f^\ell(x, Dv^{\ell k}) dx + \frac{1}{k} \int_{\Omega} (1 + |Dv^{\ell k}|^2)^{\frac{q}{2}} dx \\ &\leq \liminf_{\ell \rightarrow \infty} \int_{\Omega} f^\ell(x, Dv) dx + \frac{1}{k} \int_{\Omega} (1 + |Dv|^2)^{\frac{q}{2}} dx. \end{aligned}$$

Then, for $\Omega_0 \rightarrow \Omega$,

$$\int_{\Omega} f(x, Dv^k) dx \leq \int_{\Omega} f(x, Dv) dx + \frac{1}{k} \int_{\Omega} (1 + |Dv|^2)^{\frac{q}{2}} dx.$$

By definition (3.2), we have

$$F(\bar{u}) \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Dv^k) dx \leq \int_{\Omega} f(x, Dv) dx \quad \text{for all } v \in W_0^{1,q}(\Omega) + u_0. \quad (4.10)$$

Let $w \in W_0^{1,p}(\Omega) + u_0$. By Lemma 3.1, there exists $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$ and

$$\lim_{k \rightarrow \infty} \int_{\Omega} f(x, Dv_k) dx = F(w).$$

By (4.10),

$$F(\bar{u}) \leq \int_{\Omega} f(x, Dv_k) dx,$$

and we can conclude that

$$F(\bar{u}) \leq \lim_{k \rightarrow \infty} \int_{\Omega} f(x, Dv_k) dx = F(w) \quad \text{for all } w \in W_0^{1,p}(\Omega) + u_0.$$

Then $\bar{u} \in W_{\text{loc}}^{1,\infty}(\Omega)$ is a solution to the problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$. \square

5 Regularity of local minimizers in a special case

Let us consider now the case of a special form of integrand

$$f(x, \xi) = \sum_{i=1}^N a_i(x) g_i(\xi) \quad (5.1)$$

with $a_i(x) > 0$ a.e. in Ω , $a_i \in W^{1,r}(\Omega)$, $r > n$, $g_i : \mathbb{R}^n \rightarrow [0, +\infty)$ convex in ξ and strictly convex for ξ such that $|\xi| \geq M_0$. The following regularity result holds.

Theorem 5.1. *Assume that $f = f(x, \xi)$ as in (5.1) satisfies the assumptions of Theorem 2.6. Then every local minimizer $u \in W^{1,p}(\Omega)$ of the integral functional*

$$\int_{\Omega} f(x, Dv) dx = \int_{\Omega} \sum_{i=1}^N a_i(x) g_i(Dv) dx \quad (5.2)$$

is locally Lipschitz continuous in Ω .

Proof. Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the integral functional (5.2). For a suitable φ_σ mollifier, consider $u_\sigma = u * \varphi_\sigma \in W^{1,q}_{loc}(\Omega)$. Consider the following sequence of problems in $B_R \subset\subset \Omega$:

$$\inf \left\{ \int_{B_R} f^{\ell k}(x, Dv) \, dx : v \in W^{1,q}_0(B_R) + u_\sigma \right\}, \tag{5.3}$$

where $f^{\ell k}$ are defined in Proposition 4.1.

For fixed σ, ℓ, k , problem (5.3) has a unique solution $v^{\ell k}_\sigma \in W^{1,q}_0(B_R) + u_\sigma$. By proceeding as in the previous theorem, we have that for each fixed σ , by the minimality of $v^{\ell k}_\sigma$,

$$\begin{aligned} v^{\ell k}_\sigma &\xrightarrow{\ell \rightarrow \infty} v^k_\sigma \quad \text{weakly in } W^{1,p}_0(B_R) + u_\sigma, \\ v^{\ell k}_\sigma &\xrightarrow{\ell \rightarrow \infty} v^k_\sigma \quad \text{weakly star in } W^{1,\infty}_{loc}(B_R). \end{aligned}$$

We also have

$$\begin{aligned} v^k_\sigma &\xrightarrow{k \rightarrow \infty} v_\sigma \quad \text{weakly in } W^{1,p}_0(B_R) + u_\sigma, \\ v^k_\sigma &\xrightarrow{k \rightarrow \infty} v_\sigma \quad \text{weakly star in } W^{1,\infty}_{loc}(B_R) \end{aligned}$$

and

$$\begin{aligned} \|Dv_\sigma\|_{L^\infty(B_\rho; \mathbb{R}^n)} &\leq C \liminf_{k \rightarrow \infty} \left[1 + \int_{B_R} f(x, Du_\sigma) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}} \, dx \right]^{\frac{1}{p}} \\ &= C \liminf_{k \rightarrow \infty} \left[1 + \int_{B_R} f(x, Du_\sigma) \, dx \right]^{\frac{1}{p}} \end{aligned} \tag{5.4}$$

for any $0 < \rho < R$ and where C is independent of k, σ . For fixed k , by proceeding as in the previous theorem we have

$$\int_{B_R} f(x, Dv^k_\sigma) \, dx \leq \int_{B_R} f(x, Du_\sigma) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}} \, dx.$$

Then, by semicontinuity,

$$\begin{aligned} \int_{B_R} f(x, Dv_\sigma) \, dx &\leq \liminf_{k \rightarrow \infty} \int_{B_R} f(x, Du_\sigma) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}} \, dx \\ &\leq \int_{B_R} f(x, Du_\sigma) \, dx. \end{aligned} \tag{5.5}$$

Now we claim that, by the particular form of f , we may deduce

$$\liminf_{\sigma \rightarrow 0} \int_{B_R} f(x, Du_\sigma) \, dx \leq \int_{B_R} f(x, Du) \, dx. \tag{5.6}$$

Since g_i is convex, for $i = 1, \dots, N$, Jensen's inequality (applied to each g_i) yields

$$\begin{aligned} \int_{B_R} a_i(x) g_i(Du_\sigma) \, dx &= \int_{B_R} a_i(x) g_i \left(\int_{B_\sigma} Du(y) \varphi_\sigma(x-y) \, dy \right) \, dx \\ &\leq \int_{B_R} a_i(x) \int_{B_\sigma} g_i(Du(y)) \varphi_\sigma(x-y) \, dy \, dx \\ &= \int_{B_R} \int_{B_\sigma} a_i(x) \varphi_\sigma(x-y) \, dy \, g_i(Du(y)) \, dx \\ &\leq \int_{B_{R+\sigma}} (a_i)_\sigma(y) g_i(Du(y)) \, dy. \end{aligned}$$

Then

$$\sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du_\sigma) \, dx \leq \sum_{i=1}^N \int_{B_{R+\sigma}} (a_i)_\sigma(x) g_i(Du) \, dx$$

so that passing to the limit as $\sigma \rightarrow 0$,

$$\liminf_{\sigma \rightarrow 0} \sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du_\sigma) \, dx \leq \sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du) \, dx$$

because $(a_i)_\sigma \rightarrow a_i$ in $L^\infty(B_R)$, $g_i(Du) \in L^1(B_R)$, the Dominated Convergence Theorem may be applied, and (5.6) holds.

By collecting (5.5) and (5.6),

$$\liminf_{\sigma \rightarrow 0} \int_{B_R} f(x, Dv_\sigma) \, dx \leq \int_{B_R} f(x, Du) \, dx. \tag{5.7}$$

On the other hand, the growth assumption on f yields, since u is a local minimizer of (5.2),

$$\liminf_{\sigma \rightarrow 0} \int_{B_R} |Dv_\sigma|^p \, dx \leq \liminf_{\sigma \rightarrow 0} \int_{B_R} f(x, Dv_\sigma) \, dx \stackrel{(5.7)}{\leq} \int_{B_R} f(x, Du) \, dx < +\infty.$$

Thus there exists $\bar{v} \in u + W_0^{1,p}(B_R)$ such that, up to a subsequence,

$$v_\sigma \rightharpoonup \bar{v} \text{ weakly in } W^{1,p}(B_R).$$

By the semicontinuity of the functional, using (5.5) and (5.7),

$$\int_{B_R} f(x, D\bar{v}) \, dx \leq \liminf_{\sigma \rightarrow 0} \int_{B_R} f(x, Dv_\sigma) \, dx \leq \int_{B_R} f(x, Du) \, dx. \tag{5.8}$$

Moreover, since (5.4) holds, Dv_σ converges to $D\bar{v}$ as $\sigma \rightarrow 0$ in the weak star topology of L^∞ and there exists a constant C such that, for any $0 < \rho < R$,

$$\|D\bar{v}\|_{L^\infty(B_\rho; \mathbb{R}^n)} \leq C \left[1 + \int_{B_R} f(x, Du) \, dx \right]^{\frac{\rho}{p}}.$$

Consider the following problem in $B_R \subset\subset \Omega$:

$$\inf \left\{ \int_{B_R} f(x, Dv) \, dx : v \in W_0^{1,p}(B_R) + u \right\}. \tag{5.9}$$

Then (5.8) implies that \bar{v} and u are solutions to (5.9) and $\bar{v} \in W_{\text{loc}}^{1,\infty}(B_R)$.

In the present case the functional is not strictly convex; we proceed as in [20, Theorem 2.1] (see also [25]) and we have that $u \in W_{\text{loc}}^{1,\infty}(B_R)$. Indeed, set

$$E_0 := \left\{ x \in B_R : \left| \frac{Du(x) + D\bar{v}(x)}{2} \right| > M_0, Du(x) \neq D\bar{v}(x) \right\} \text{ and } w := \frac{u + \bar{v}}{2}.$$

If E_0 has positive measure, then from the convexity of $f(x, \cdot)$ we have

$$\int_{B_R \setminus E_0} f(x, Dw) \, dx \leq \frac{1}{2} \int_{B_R \setminus E_0} f(x, Du) \, dx + \frac{1}{2} \int_{B_R \setminus E_0} f(x, D\bar{v}) \, dx. \tag{5.10}$$

Now, by the strict convexity of $f(x, \xi)$ for ξ such that $|\xi| \geq M_0$ and applying two times the inequality

$$f(x, \eta) > f(x, \xi) + \langle f_\xi(x, \xi), \eta - \xi \rangle \text{ for } \xi \text{ such that } |\xi| \geq M_0$$

first with $\xi = Dw$ and $\eta = Du$, then for $\xi = Dw$ and $\eta = D\bar{v}$, finally by adding up the two inequalities obtained, we have

$$\int_{B_R \cap E_0} f(x, Dw) dx < \frac{1}{2} \int_{B_R \cap E_0} f(x, Du) dx + \frac{1}{2} \int_{B_R \cap E_0} f(x, Dv) dx. \quad (5.11)$$

Adding (5.10) and (5.11), we get a contradiction with the minimality of u and \bar{v} . Therefore the set E_0 has zero measure, which implies that

$$\sup_{B_\rho} |Du(x)| \leq \sup_{B_\rho} |Du(x) + D\bar{v}(x)| + \sup_{B_\rho} |D\bar{v}(x)| \leq 2M_0 + \sup_{B_\rho} |D\bar{v}(x)|$$

and this yields the thesis. \square

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