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# NONLINEAR STARK-WANNIER EQUATION

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**ABSTRACT.** In this paper we consider stationary solutions to the nonlinear one-dimensional Schrödinger equation with a periodic potential and a Stark-type perturbation. In the limit of large periodic potential the Stark-Wannier ladders of the linear equation become a dense energy spectrum because a cascade of bifurcations of stationary solutions occurs when the ratio between the effective nonlinearity strength and the tilt of the external field increases.

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## 1. INTRODUCTION

The dynamics of a quantum particle in a periodic potential under an homogeneous external field is one of the most important problems in solid-state physics and, more recently, in the theory of Bose Einstein Condensates (BECs). Because of the periodicity of the potential, it is expected the existence of families of stationary (metastable) states with associated energies displaced on regular ladders, the so-called Stark-Wannier ladders [15, 17, 27], and the wavefunction would perform Bloch oscillations.

Quantum dynamics becomes more interesting when we take into account the interaction among particles. In fact, in the framework of BECs accelerated ultracold atoms moving in an optical lattice [4, 5, 22, 26, 29] has opened the field to multiple applications, as well as the measurement of the value of the gravity acceleration  $g$  using ultracold Strontium atoms confined in a vertical optical lattice [11, 21], direct measurement of the universal Newton gravitation constant  $G$  [24] and of the gravity-field curvature [25].

Motivated by such physical applications we study, as a model for a confined accelerated BECs in a periodic optical lattice under the effect of the gravitational force, the nonlinear one-dimensional time-dependent Schrödinger equation with a cubic nonlinearity, a periodic potential  $V$  and an accelerating Stark-type potential  $W$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\epsilon} V\psi + \alpha_1 W\psi + \alpha_2 |\psi|^2 \psi, \quad (1)$$

in the limit of large periodic potential, i.e.  $0 < \epsilon \ll 1$ ; that is equation (1) is the so called Gross-Pitaevskii equation. Here,  $\hbar$  is the Planck's constant,  $m$  is the mass of the atom and  $\alpha_2$  is the strength of the nonlinearity term; the real valued parameters  $m$ ,  $\hbar$ ,  $\alpha_1$  and  $\alpha_2$  are assumed to be fixed. In particular  $W(x)$  is a Stark-type potential with strength  $\alpha_1$ , that is it is locally a linear function:  $W(x) = x$  for any  $x$  belonging to a fixed interval large enough.

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We name equation (1) *nonlinear Wannier-Stark equation*. The well-known *Wannier-Stark equation*, where  $\alpha_2 = 0$ , has been extensively studied since the papers by Bloch [3] and Zener [37]. Assuming that the periodic potential  $V$  is regular enough, then the spectrum of the associated operator covers the whole real axis. On the other side, if we neglect the coupling term between different bands, then it turns out that the spectrum of such a decoupled band approximation consists of a sequence on infinite ladders of real eigenvalues [30, 31]. The crucial point is to understand what happens to these eigenvalues when we restore the interband coupling term [32, 35, 36]. This question has been largely debated and it has been proved that these ladders of real eigenvalues will turn into ladders of quantum resonances, the so-called Wannier-Stark resonances (see [27] and the references therein). Analysis of the nonlinear Wannier-Stark equation, where  $\alpha_2 \neq 0$ , is a completely open problem and it is motivated by recent experiments of BECs in accelerating optical lattices.

By means of a simple recasting we swap the limit of large potential  $\epsilon \ll 1$  to a semiclassical equation (see eq. (3) below) where the strength of the Stark-type potential and the nonlinearity strength will depend on a semiclassical parameter  $\hbar$ . In the semiclassical limit of  $\hbar \rightarrow 0$  we will show that the time-independent nonlinear Schrödinger equation may be approximated by means of a discrete time-independent nonlinear Schrödinger equation whose stationary solutions may be explicitly calculated. In particular, a cascade of bifurcations occurs when the ratio between the nonlinearity strength and the strength of the Stark-type potential increases; in the opposite situation, that is when this ratio goes to zero, we recover a local Wannier-Stark ladders picture.

Existence and computation of stationary solutions to equation (1) has been already considered by [13, 19, 20] when  $\alpha_1 = 0$ ; in these papers the authors reduce the problem of the existence and calculation of stationary solutions to the one related to a discrete nonlinear Schrödinger equation. In this latter problem has been observed by [2] that stationary solutions may bifurcate when some parameters of the model assume critical values. Here, we extend such analysis to the case where an external Stark-type potential is present, that is when  $\alpha_1 \neq 0$ . To this end we must introduce some technical assumptions on  $W$ , that is  $W$  must be a locally linear bounded function with compact support; in fact in the case of a *true* Stark potential where  $W(x) = x$  some basic estimates useful in our analysis don't work because  $W$  is not a bounded operator. Some results, like the occurrence of a cascade of bifurcations for the discrete nonlinear Schrödinger equation in the anticontinuous limit has been already announced in a physics-oriented paper [28] without mathematical details. We should also mention a recent paper [14] where bifurcations are observed in rotating Bose-Einstein condensates.

The paper is organized as follows: in §2 we introduce the model and we state our assumptions; in §3 we recall some technical results obtained by [13]; in §4 we derive the discrete nonlinear Schrödinger Wannier-Stark equation; in §5 we compute the finite-mode stationary solutions of the discrete nonlinear Schrödinger Wannier-Stark equation in the anticontinuous limit, it turns out that a bifurcation tree picture occurs; in §6 we prove the stability of these stationary solutions when we recover the discrete nonlinear Schrödinger Wannier-Stark equation; finally, in §7-8 we prove that stationary solutions to the complete equation (6) can be approximated by means of the finite-mode solutions derived in §5.

**Notation.** By  $\ell_{\mathbb{R}}^p$  we denote the space of vectors  $\mathbf{c} = \{c_n\}_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$  such that  $c_n \in \mathbb{R}$  are real valued. Similarly,

$$L_{\mathbb{R}}^p = \{\psi \in L^p : \psi \text{ is a real valued function}\}.$$

Let  $f$  and  $g$  two vectors belonging to a normed space with norm  $\|\cdot\|$ , and depending on the semiclassical parameter  $h$ . By the notation  $f = g + \tilde{\mathcal{O}}(e^{-S_0/h})$ , as  $h \rightarrow 0$ , we mean that for any  $\rho \in (0, S_0)$  there exist a positive constant  $C := C_\rho > 0$  (independent of  $h$ ) such that

$$\|f - g\| \leq C e^{-(S_0 - \rho)/h}, \quad \forall h \in (0, h^*),$$

for some  $h^* > 0$ . By the notation  $f \sim g$ , as  $h \rightarrow 0$ , we mean that  $\lim_{h \rightarrow 0^+} \frac{f}{g} = C$  for some  $C \in (0, +\infty)$ . By the notation  $f = \mathcal{O}(h^q)$ , as  $h \rightarrow 0$ , we mean that there exists  $h^* > 0$  and a positive constant  $C$  independent of  $h$  such that  $|f| \leq Ch^q$  for any  $h \in (0, h^*)$ .

By  $C$  we denote a generic positive constant independent of  $h$  whose value may change from line to line.

## 2. DESCRIPTION OF THE MODEL AND ASSUMPTIONS

Here we consider the nonlinear Schrödinger equation (1) where the following assumptions hold true.

**Hyp.1**  $V(x)$  is a smooth, real-valued, periodic and non negative function with period  $a$ , i.e.

$$V(x) = V(x + a), \quad \forall x \in \mathbb{R},$$

and with minimum point  $x_0 \in [-\frac{1}{2}a, +\frac{1}{2}a]$  such that

$$V(x) > V(x_0), \quad \forall x \in \left[-\frac{1}{2}a, +\frac{1}{2}a\right] \setminus \{x_0\}.$$

For argument's sake we assume that  $V(x_0) = 0$  and  $x_0 = 0$ .

In the following let us denote by  $x_n = x_0 + na$ .

**Remark 1.** In physical experiments [11, 21] on accelerated BECs in optical lattices the periodic potential has the form  $V(x) = V_0 \sin^2(k_L x)$  for some  $V_0, k_L > 0$ ; hence  $V(x)$  has a unique minimum point  $x_0 = 0$  in the interval  $\left[-\frac{\pi}{2k_L}, +\frac{\pi}{2k_L}\right)$ . However, we could, in principle, adapt our treatment to a more general case where  $V(x)$  has more than one absolute minimum point in such interval.

**Hyp.2**  $W(x)$  is a smooth real-valued function such that

$$W(x) = x \quad \text{if } |x| \leq Na,$$

for some  $N \in \mathbb{N}$ . Furthermore  $W$  has compact support  $\Omega \supset [-Na, Na]$ .

**Remark 2.** We require that  $W(x)$  is a bounded function with compact support for technical reasons; indeed, this assumption will play a crucial role in order to prove the results given in §6, 7, 8. However, in practical experiments [11, 21] on accelerated BECs in optical lattices it is expected that BECs perform Bloch oscillations in a finite region; hence, a model where the external field  $W$  has a compact support and it is locally linear in the finite region where Bloch oscillations occur would fit the physical device.

By recasting

$$F = \epsilon\alpha_1, \quad h = \hbar\sqrt{\epsilon/2m}, \quad \tau = t/\sqrt{\epsilon/2m} \quad \text{and} \quad \eta = \epsilon\alpha_2 \quad (2)$$

then the above equation takes the form

$$ih \frac{\partial \psi}{\partial \tau} = -h^2 \frac{\partial^2 \psi}{\partial x^2} + V\psi + FW\psi + \eta|\psi|^2\psi \quad (3)$$

and the limit of large periodic potential  $\epsilon \rightarrow 0^+$  is equivalent to the semiclassical limit  $h \rightarrow 0^+$  where

$$\eta \sim F \sim h^2 \quad \text{as } h \text{ goes to zero.} \quad (4)$$

We recall here some results by [7, 8, 9] concerning the solution to the time-dependent nonlinear Schrödinger equation (3). Let  $H_B$  be the Bloch operator formally defined on  $L^2(\mathbb{R}, dx)$  as

$$H_B := -h^2 \frac{d^2}{dx^2} + V. \quad (5)$$

For any  $N \in \mathbb{N}$ ,  $N > 0$ , the linear operator  $H$ , formally defined as

$$H = H_B + FW$$

on the Hilbert space  $L^2(\mathbb{R}, dx)$ , admits a self-adjoint extension, still denoted by  $H$ . The following estimate hold true (see Proposition 2.1 by [9]): let  $(q, r)$  be an admissible pair  $\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$  with  $2 \leq q, r \leq +\infty$ . Let  $T > 0$ , then there exists  $C := C(q, T, h)$  such that

$$\left\| e^{-i\tau H/h} \psi \right\|_{L^q([-T, T]; L^r(\mathbb{R}))} \leq C \|\psi\|_{L^2(\mathbb{R})}, \quad \forall \psi \in L^2(\mathbb{R}).$$

In order to discuss the local and global existence of solutions to (3) [9] introduced the following set in a more general situation where the potential is not bounded

$$\Sigma = \left\{ \psi \in H^1(\mathbb{R}) : \|\psi\|_\Sigma := \|\psi\|_{H^1(\mathbb{R})} + \|(V + FW)\psi\|_{L^2(\mathbb{R})} < \infty \right\}.$$

Then (see Theorem 4.2 by [9]), if  $\psi_0 \in \Sigma$  there exists a unique solution  $\psi \in C([-T, T]; \Sigma)$  to (3) with initial datum  $\psi_0$ , such that

$$\psi, \psi \partial_x (V + FW), \partial_x \psi \in L^8([-T, T]; L^4(\mathbb{R})),$$

for some  $T > 0$  depending on  $\|\psi_0\|_\Sigma$ . We must underline that in our case  $\Sigma \equiv H^1(\mathbb{R})$  because  $V$  and  $W$  are bounded functions.

In fact, this solution is global in time for any  $\eta \in \mathbb{R}$  because  $1 < 2/d$ , where  $d = 1$  is the spatial dimension, and (3) enjoys the conservation of the mass

$$\|\psi(\cdot, \tau)\|_{L^2(\mathbb{R})} = \|\psi_0(\cdot)\|_{L^2(\mathbb{R})}$$

and of the energy

$$\mathcal{E}(\psi) = \mathcal{E}(\psi_0)$$

where

$$\begin{aligned} \mathcal{E}(\psi) &:= \langle H\psi, \psi \rangle + \frac{\eta}{2} \|\psi\|_{L^4}^4 \\ &= h^2 \|\partial_x \psi\|_{L^2(\mathbb{R})}^2 + \langle V\psi, \psi \rangle + F \langle W\psi, \psi \rangle + \frac{\eta}{2} \|\psi\|_{L^4}^4 \end{aligned}$$

We may remark that such results hold true even when the Stark-type potential is replaced by an actual Stark potential, i.e.  $W(x) \equiv x$ . In such a case  $\Sigma \subset H^1(\mathbb{R})$ .

Here, we look for stationary solutions to equation (3) of the form

$$\psi(x, \tau) = e^{-i\lambda\tau/\hbar}\psi(x)$$

for some energy  $\lambda \in \mathbb{R}$  and wave function  $\psi(x)$ . Hence, equation (3) takes the form

$$H\psi + \eta|\psi|^2\psi = \lambda\psi. \quad (6)$$

**Remark 3.** *We must underline that when a stationary solution  $\psi$  to equation (6) is regular enough then  $\psi$  is, up to a phase factor, a real-valued function (see Lemma 3.7 by [18] adapted to (6)). Hence, equation (6) can be replaced by the following equation*

$$H\psi + \eta\psi^3 = \lambda\psi. \quad (7)$$

where  $\psi$  is real-valued.

Our aim is to look for real-valued stationary solutions  $\psi \in H^1$  to (7) with associated energy  $\lambda \in \mathbb{R}$ .

**Remark 4.** *Let  $(T_a\psi)(x) = \psi(x-a)$  be the translation operator. Since  $[H_B, T_a] = 0$  and  $[Fx, T_a] = Fa$  then the stationary solutions to (7) when  $W$  is a Stark potential, i.e.  $W(x) \equiv x$ , have associated energies  $\lambda$  displaced on regular ladders; that is, if  $\psi(x)$  is a solution to (7) associated with  $\lambda$ , then  $\psi(x-a)$  is a solution to the same equation associated with  $\lambda + Fa$ . From this fact we expect that, under some circumstances, the dominant term of the energies  $\lambda$  associated to stationary solutions to (7) are displaced on ladders for some range of values of  $\lambda$ , even when  $W(x)$  is a Stark-type potential satisfying Hyp.2.*

### 3. PRELIMINARY RESULTS. BLOCH FUNCTIONS IN THE SEMICLASSICAL LIMIT

**3.1. Bloch Decomposition and Wannier functions.** Here, we briefly resume some known results by [6, 23] concerning the spectral properties of the self-adjoint realization, still denoted by  $H_B$ , of the Bloch operator formally defined on  $L^2(\mathbb{R}, dx)$  as (5). Its spectrum is given by bands. Let  $\mathcal{B} = \mathbb{R}/b\mathbb{Z} = (-\frac{1}{2}b, +\frac{1}{2}b]$ , where  $b = \frac{2\pi}{a}$  and  $a$  is the period of the periodic potential  $V$ , be the Brillouin zone, the elements of the Brillouin zone are denoted by  $k$  and they are usually named quasi-momentum (or crystal momentum) variable.

Let  $\varphi_l(x, k)$  denote the Bloch functions associated to the band functions  $E_l(k)$ ,  $l \in \mathbb{N}$ . Here, we collect some basic properties about the Bloch and band functions. The band and Bloch functions satisfy to the following eigenvalues problem

$$H_B\varphi = E\varphi \quad (8)$$

with quasi-periodic boundary conditions

$$\varphi(a, k) = e^{ika}\varphi(0, k) \quad \text{and} \quad \frac{\partial\varphi(a, k)}{\partial x} = e^{ika}\frac{\partial\varphi(0, k)}{\partial x}.$$

The Bloch functions  $\varphi_l$  may be written as

$$\varphi_l(x, k) = e^{ikx}\Theta_l(x, k)$$

where  $\Theta_l(x, k)$  is a periodic function with respect to  $x$ :  $\Theta_l(x+a, k) = \Theta_l(x, k)$ . For any fixed  $k \in \mathcal{B}$  the spectral problem (8) has a sequence of real eigenvalues

$$E_1(k) \leq E_2(k) \leq \dots \leq E_l(k) \leq \dots,$$

such that  $\lim_{l \rightarrow \infty} E_l(k) = +\infty$ . As functions on  $k$ , both Bloch and band functions are periodic with respect to  $k$ :

$$E_l(k) = E_l(k + b) \quad \text{and} \quad \varphi_l(x, k) = \varphi_l(x, k + b),$$

and they satisfy to the following properties for any real-valued  $k$ :

$$\varphi_l(x, -k) = \overline{\varphi_l(x, k)} \quad \text{and} \quad E_l(-k) = E_l(k).$$

Furthermore, if  $V(x)$  is an even potential, i.e.  $V(-x) = V(x)$ , then  $\varphi_l(-x, k) = \overline{\varphi_l(x, k)}$ ,  $\varphi_l(x, 0)$  are even functions while  $\varphi_l(x, b/2)$  are odd functions. The band functions  $E_l(k)$  are monotone increasing (resp. decreasing) functions for any  $k \in [0, \frac{1}{2}b]$  if the index  $l$  is an odd (resp. even) natural number. The spectrum of  $H_B$  is purely absolutely continuous and it is given by bands:

$$\sigma(H_B) = \cup_{l \in \mathbb{N}} [E_l^b, E_l^t] \quad \text{where} \quad [E_l^b, E_l^t] = \{E_l(k), k \in \mathcal{B}\}.$$

In particular we have that

$$E_l^b = \begin{cases} E_l(0) & \text{for odd } l \\ E_l(b/2) & \text{for even } l \end{cases} \quad \text{and} \quad E_l^t = \begin{cases} E_l(b/2) & \text{for odd } l \\ E_l(0) & \text{for even } l \end{cases}.$$

The intervals  $(E_l^t, E_{l+1}^b)$  are named gaps; a gap  $(E_l^t, E_{l+1}^b)$  may be empty, that is  $E_{n+1}^b = E_l^t$ , or not. It is well known that, in the case of one-dimensional crystals, all the gaps are empty if, and only if, the periodic potential is a constant function. Because we assume that the periodic potential is not a constant function then one gap, at least, is not empty. In particular when  $h$  is small enough then we have that the following asymptotic behavior [16, 33, 34]

$$\frac{1}{C}h \leq E_2^b - E_1^t \leq Ch \tag{9}$$

holds true for some  $C > 0$ ; hence, the first gap between  $E_1^t$  and  $E_2^b$  is not empty in the semiclassical limit. Furthermore, the first band turns out to be exponentially small, i.e.

$$E_1^t - E_1^b = \tilde{\mathcal{O}}(e^{-C/h}) \quad \text{for some } C > 0; \tag{10}$$

in (13) we will give an expression for such a constant  $C$ .

The Bloch functions are assumed to be normalized to 1 on the interval  $[0, a]$ :

$$\frac{2\pi}{a} \int_0^a \overline{\varphi_j(x, k)} \varphi_l(x, k) dx = \delta_j^l,$$

where  $\delta_j^l = 1$  when  $j = l$  and  $\delta_j^l = 0$  when  $j \neq l$  (see Eq. (4.1.8) by [6]). Furthermore, the Bloch functions are such that (see Eq. (4.1.6a) by [6])

$$\int_{\mathbb{R}} \overline{\varphi_j(x, k)} \varphi_l(x, q) dx = \delta_j^l \delta(k - q)$$

and (see Eq. (4.1.10) by [6])

$$\sum_{l \in \mathbb{N}} \int_{\mathcal{B}} \overline{\varphi_m(x, k)} \varphi_l(x', k) dk = \delta(x - x'),$$

where  $\delta(\cdot)$  denotes the Dirac's  $\delta$ . From the Bloch decomposition formula it follows that any vector  $\psi \in L^2$  can be written as (see Eq. (5.1.5) by [6] or Theorem XIII.98

by [23])

$$\psi(x) = \sum_{l \in \mathbb{N}} \int_{\mathcal{B}} \varphi_l(x, k) \phi_l(k) dk.$$

The family of functions  $\{\phi_l(k)\}_{l \in \mathbb{N}}$  is called the crystal momentum representation of the wave function  $\psi$  and it is defined as

$$\phi_l(k) = \int_{\mathbb{R}} \overline{\varphi_l(x, k)} \psi(x) dx.$$

By construction any function  $\phi_l(k)$  is a periodic function and the transformation

$$\psi \in L^2(\mathbb{R}, dx) \rightarrow \mathcal{U}\psi := \{\phi_l\}_{l \in \mathbb{N}} \in \mathcal{H} := \otimes_{l \in \mathbb{N}} L^2(\mathcal{B}, dk) \quad (11)$$

is unitary:

$$\|\psi\|_{L^2(\mathbb{R}, dx)}^2 = \sum_{l \in \mathbb{N}} \|\phi_l\|_{L^2(\mathcal{B}, dk)}^2.$$

Let  $W_l(x)$  be the *basic* Wannier function associated to the  $l$ -th band, that is

$$W_l(x) = \sqrt{\frac{a}{2\pi}} \int_{\mathcal{B}} \varphi_l(x, k) dk.$$

We define a family of Wannier functions  $\{W_{l,n}(x)\}_{l \in \mathbb{N}, n \in \mathbb{Z}}$  as

$$W_{l,n}(x) = W_l(x - na) = \sqrt{\frac{a}{2\pi}} \int_{\mathcal{B}} \varphi_l(x, k) e^{-ina k} dk.$$

Basically, in the semiclassical limit of  $h$  small, the Wannier function  $W_{l,n}$  is localized on the  $n$ -th well, that is in a neighborhood of  $x_n$ . The following properties hold true

$$\int_{\mathbb{R}} \overline{W_{l,n}(x)} W_{m,n}(x) dx = \delta_l^m, \quad \sum_{l \in \mathbb{N}, n \in \mathbb{Z}} \overline{W_{l,n}(x)} W_{l,n}(x') = \delta(x - x')$$

and we have the following relation between the Wannier and the Bloch functions:

$$\varphi_l(x, k) = \sqrt{\frac{a}{2\pi}} \sum_{n \in \mathbb{Z}} e^{ina x} W_{l,n}(x).$$

If we set

$$c_l^n = \int_{\mathbb{R}} \overline{W_{l,n}(x)} \psi(x) dx$$

then we may represent a wave function  $\psi$  as

$$\psi \in L^2 \rightarrow \mathcal{W}\psi = \{c_l^n\}_{l \in \mathbb{N}, n \in \mathbb{Z}} \in \ell^2(\mathbb{N} \times \mathbb{Z})$$

Such a transformation  $\mathcal{W}$  is unitary

$$\|\psi\|_{L^2}^2 = \sum_{l \in \mathbb{N}, n \in \mathbb{Z}} |c_l^n|^2$$

with inverse

$$\psi(x) = \sum_{l \in \mathbb{N}, n \in \mathbb{Z}} c_l^n W_{l,n}(x). \quad (12)$$



**Remark 5.** *The standard “tight binding” model is obtained by substituting (12) in (3), and it reduces (3) to a discrete nonlinear Schrödinger equation. In fact, in order to improve the estimate of the remainder terms of the discrete nonlinear Schrödinger equation we decompose the wave function  $\psi(x)$  on a different base where the vectors of such a base are obtained by means of the single well semiclassical approximation described in §3.2.*

**3.2. Semiclassical construction.** Here we restrict our attention to just one band, say the first one  $[E_1^b, E_1^t]$ . By assuming  $h$  small enough then the gap between the first band and the remainder of the spectrum is open, see equation (9). Let  $\Pi$  be the spectral projection of  $H_B$  on the first band; by [10] we can find a “good” orthonormal basis  $\{u_n\}_{n \in \mathbb{Z}}$  of  $\Pi[L^2(\mathbb{R})]$ .

In one dimension let

$$d_A(x, y) = \int_x^y \sqrt{V(q)} dq$$

be the Agmon distance between  $x$  and  $y$  (associated to the energy level corresponding to the minimum value  $V(x_0) = 0$  of the potential  $V(x)$ ) and let

$$S_0 = d_A(x_n, x_{n+1}) \quad (13)$$

be the Agmon distance between two adjacent wells; by periodicity of the potential  $V(x)$  then  $S_0$  is independent of the index  $n$ .

Here we summarize some important properties of  $\{u_n\}_{n \in \mathbb{Z}}$  (see [10] and Appendix A by [13]). Let  $\tilde{V}$  be the “single well potential” obtained by filling all the well, but one; that is  $\tilde{V}(x) = V(x) + \theta(x)$  where  $\theta(x)$  is a smooth and non negative function such that  $\theta(x) = 0$  in a small neighborhood  $(x_0 - \delta, x_0 + \delta)$  of  $x_0$  and  $\theta(x) > \varepsilon$  for any  $x \notin (x_0 - 2\delta, x_0 + 2\delta)$  for some  $\varepsilon > 0$  and  $0 < \delta < \frac{1}{4}a$  is fixed. Then the operator  $\tilde{H} = -h^2 \frac{d^2}{dx^2} + \tilde{V}$  has discrete spectrum in the interval  $[0, \varepsilon]$  and we call such eigenvalues single well states. We denote by  $\Lambda_1$  the first one, the so called “single well ground state”, and by  $w_0(x)$  the associated eigenvector.

**Remark 6.** *By means of semiclassical arguments it follows that [16, 33, 34]*

$$\text{dist}(\Lambda_1, [E_1^b, E_1^t]) = \tilde{O}\left(e^{-S_0/h}\right).$$

Furthermore,

$$E_2^b - \Lambda_1 \geq Ch$$

for some  $C > 0$ .

If we denote  $w_n(x) = w_0(x - na)$  then the family  $\{w_n\}_{n \in \mathbb{Z}}$  is a family of linearly independent vectors localized on the  $n$ -th well. Then, taking their projection  $\Pi w_n$  on  $\Pi[L^2(\mathbb{R})]$  and orthonormalizing the obtained family we finally get the base  $\{u_n\}_{n \in \mathbb{Z}}$  of  $\Pi[L^2(\mathbb{R})]$ .

**Lemma 1.** *The vectors  $u_n$  of the orthonormal base of  $\Pi[L^2(\mathbb{R})]$  are such that:*

- i. *The matrix with real-valued elements  $\langle u_m, H_B u_n \rangle$  can be written as*

$$(\langle u_m, H_B u_n \rangle) = \Lambda_1 \mathbb{1} - \beta \mathcal{T} + \tilde{D},$$

where  $\mathcal{T}$  is the tridiagonal Toeplitz matrix, i.e.,

$$(\mathcal{T})_{m,n} = \begin{cases} 0 & \text{if } |m - n| \neq 1 \\ 1 & \text{if } |m - n| = 1 \end{cases},$$

$\beta > 0$  is such that for any  $\rho > 0$  then

$$\frac{1}{C}e^{-(S_0+\rho)/h} \leq \beta \leq Ce^{-(S_0-\rho)/h} \quad (14)$$

for some positive constant  $C := C_\rho > 0$ , and the remainder term  $\tilde{D}$  is a bounded linear operator from  $\ell^p(\mathbb{Z})$  to  $\ell^p(\mathbb{Z})$  with bound

$$\|\tilde{D}\|_{\mathcal{L}(\ell^p \rightarrow \ell^p)} \leq Ce^{-(S_0+\zeta)/h}, \quad p \in [1, +\infty], \quad (15)$$

for some positive constant  $\zeta > 0$  independent of  $h$  and  $p$ , and for some positive constant  $C$  which depends only on  $p$ .

- ii. Let  $T_a$  be the translation operator  $(T_a\psi)(x) = \psi(x-a)$ , where  $a$  is the period of  $V$ . Then,  $u_n = T_a^n u_0$ .
- iii. All the functions  $u_n$  can be chosen to be real-valued by means of a suitable gauge choice.
- iv. For any  $\rho', \rho'' > 0$  and for some positive constant  $C > 0$  independent on the indexes  $n$  and  $m$ , we have that

$$\|u_m u_n\|_{L^1} \leq Ce^{-[(S_0-\rho')|m-n|-\rho'']/h}, \quad m \neq n.$$

- v. There exists a constant  $C > 0$  independent of  $h$  such that

$$\left\| \sum_{n \in \mathbb{Z}} |u_n| \right\|_{L^\infty} \leq Ch^{-1/2}.$$

- vi. For any  $p \in [2, \infty]$ ,  $\|u_n\|_{L^p} \leq Ch^{-(p-2)/4p}$ , and  $\left\| \frac{du_n}{dx} \right\|_{L^2} \leq Ch^{-1/2}$ , where the constants  $C > 0$  are independent of  $h$  and  $n$ .

#### 4. CONSTRUCTION OF THE DISCRETE NONLINEAR STARK-WANNIER EQUATION

Let  $\Pi$  the projection operator associated to the first band  $[E_1^b, E_1^t]$  of  $H_B$  (see §3.1) and let  $\Pi_\perp = \mathbf{1} - \Pi$ . Let

$$\psi = \psi_1 + \psi_\perp \quad \text{where } \psi_1 = \Pi\psi \quad \text{and} \quad \psi_\perp = \Pi_\perp\psi. \quad (16)$$

By the Carlsson's construction resumed in §3.2 we may write  $\psi_1$  by means of a linear combination of a suitable orthonormal base  $\{u_n\}_{n \in \mathbb{Z}}$  of the space  $\Pi[L^2(\mathbb{R})]$ , that is

$$\psi_1(x) = \sum_{n \in \mathbb{Z}} c_n u_n(x). \quad (17)$$

where  $u_n \in H^1(\mathbb{R})$  and

$$\mathbf{c} = \{c_n\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^2(\mathbb{Z})$$

because  $\psi$ , and then  $\psi_1$ , is a real-valued function by Remark 3 and  $u_n$  are real valued too since Lemma 1.iii. In fact, when we make use of the fixed point argument in §7 and when we prove the existence result of stationary solutions in §8 we work with vectors  $\mathbf{c} \in \ell^1(\mathbb{R})$ ; then in the sequence we may assume that  $\mathbf{c} \in \ell^p(\mathbb{R})$  for any  $p \in [1, +\infty]$ .

**Remark 7.** *By construction*

$$\begin{aligned} \|\psi_1\|_{L^p} &= \left\| \sum_{n \in \mathbb{Z}} c_n u_n \right\|_{L^p} \leq \sum_{n \in \mathbb{Z}} |c_n| \max_n \|u_n\|_{L^p} \leq \|\mathbf{c}\|_{\ell^1} \|u_0\|_{L^p} \\ &\leq Ch^{-(p-2)/4p} \|\mathbf{c}\|_{\ell^1} \end{aligned}$$

by Lemma 1.ii and Lemma 1.vi.

**Remark 8.** We must underline that the standard tight-binding model is constructed by making use of the Wannier functions (see (12)) instead of (16) and (17). In fact, the decomposition (12) turns out to be more natural and it has the advantage to work for any range of  $h$ ; decompositions (16) and (17) are more powerful than (12) in the semiclassical regime of  $h \ll 1$  and they have the great advantage that the vectors  $u_n$  are explicitly constructed by means of the semiclassical approximation (see Lemma 1).

By inserting (16) and (17) in equation (7) then it takes the form

$$\begin{cases} \lambda c_n &= \langle u_n, H_B \psi \rangle + F \langle u_n, W \psi \rangle + \eta \langle u_n, \psi^3 \rangle, \quad n \in \mathbb{Z} \\ \lambda \psi_\perp &= \Pi_\perp H_B \psi + F \Pi_\perp W \psi + \eta \Pi_\perp \psi^3 \end{cases}, \quad (18)$$

where  $\mathbf{c} \in \ell^2_{\mathbb{R}}$  and  $\psi_\perp$  are such that

$$\|\psi\|_{L^2}^2 = \|\mathbf{c}\|_{\ell^2}^2 + \|\psi_\perp\|_{L^2}^2.$$

The following result immediately follows by Lemma 1.

**Lemma 2.** We have that

$$\langle u_n, H_B \psi \rangle = \Lambda_1 c_n - \beta(c_{n+1} + c_{n-1}) + r_1^n,$$

where  $\beta$  satisfies (14) and

$$r_1^n := \sum_{m \in \mathbb{Z}} \tilde{D}_{n,m} c_m$$

where  $\tilde{D}_{n,m}$  is defined by Lemma 1.i and it satisfies to the following estimate for some  $\zeta > 0$ : let  $\mathbf{r}_1 = \{r_1^n\}_{n \in \mathbb{Z}}$  and  $\mathbf{c} = \{c_n\}_{n \in \mathbb{Z}} \in \ell^p_{\mathbb{R}}$ , then

$$\|\mathbf{r}_1\|_{\ell^p} \leq C e^{-(S_0 + \zeta)/h} \|\mathbf{c}\|_{\ell^p}, \quad \forall p \in [1, +\infty],$$

for some positive constant  $C := C_p > 0$ .

Let  $d_A(x, y)$  be the Agmon distance between two points  $x, y \in \mathbb{R}$  and let  $S_0 := d_A(x_n, x_{n+1})$ ,  $n \in \mathbb{Z}$ , be the Agmon distance between the bottoms  $x_n$  and  $x_{n+1}$  of two adjacent wells of the periodic potential  $V$  (for further details see §3.2); by periodicity  $S_0$  does not depend on the index  $n$ .

**Lemma 3.** We have that

$$\langle u_n, W \psi \rangle = \tilde{a} \tilde{\xi}(n) c_n + r_2^n + r_3^n,$$

where for any  $\rho > 0$  there exists  $C := C_\rho$  such that

$$\|\mathbf{r}_2\|_{\ell^1} \leq C e^{-(S_0 - \rho)/h} \|\mathbf{c}\|_{\ell^1}$$

and there exists  $C > 0$  such that

$$\|\mathbf{r}_3\|_{\ell^1} \leq C \|\psi_\perp\|_{L^2}.$$

Furthermore,  $|\tilde{\xi}(n)| \leq C$  for any  $n$  because  $W$  is bounded, and

$$\tilde{\xi}(n) = \frac{C_0}{a} + \xi(n) + \tilde{\mathcal{O}}\left(e^{-S_0/h}\right) \quad (19)$$

where  $\xi(n)$  is a bounded function such that

$$\xi(n) = n \text{ if } |n| \leq N, \text{ and } C_0 = \int_{a-}^{a+} x |u_0(x)|^2 dx$$

where  $a_{\pm}$  are such that  $a_- < x_0 = 0 < a_+$  and  $d_A(a_-, x_0) = d(x_0, a_+) = \frac{1}{2}S_0$ ; by construction  $a_+ - a_- = a$ .

*Proof.* By inserting (16) and (17) in  $\langle u_n, W\psi \rangle$  one gets

$$\begin{aligned} \langle u_n, W\psi \rangle &= \langle u_n, W\psi_1 \rangle + \langle u_n, W\psi_{\perp} \rangle \\ &= \langle u_n, Wu_n \rangle c_n + \sum_{m \neq n} \langle u_n, Wu_m \rangle c_m + \langle u_n, W\psi_{\perp} \rangle. \end{aligned}$$

Now, we set

$$r_2^n = \sum_{m \neq n} \langle u_n, Wu_m \rangle c_m, \quad r_3^n = \langle u_n, W\psi_{\perp} \rangle \quad \text{and} \quad \tilde{\xi}(n) = \frac{1}{a} \langle u_n, Wu_n \rangle.$$

Estimates of  $\mathbf{r}_2 = \{r_2^n\}_{n \in \mathbb{Z}}$  and  $\mathbf{r}_3 = \{r_3^n\}_{n \in \mathbb{Z}}$  directly come from the properties collected in Lemma 1. Indeed

$$|\langle u_n, Wu_m \rangle| \leq \|W\|_{\infty} \|u_n u_m\|_{L^1} \leq C e^{-[(S_0 - \rho')|m-n| - \rho'']/h}$$

for any  $\rho', \rho'' > 0$  and some  $C > 0$ , because  $W$  is bounded; hence the estimate

$$\|\mathbf{r}_2\|_{\ell^1} \leq \sum_n \sum_{m \neq n} |\langle u_n, Wu_m \rangle| |c_m| \leq C e^{-(S_0 - \rho)/h} \|\mathbf{c}\|_{\ell^1}$$

follows. Similarly,

$$\begin{aligned} \|\mathbf{r}_3\|_{\ell^1} &= \sum_n |\langle u_n, W\psi_{\perp} \rangle| \leq \left\langle \sum_n |u_n| \chi_{\Omega}, |W\psi_{\perp}| \right\rangle \\ &\leq C \|\psi_{\perp}\|_{L^2} \end{aligned}$$

where  $\chi_{\Omega}$  is the characteristic function and where  $\Omega$  is the compact support of  $W$ . Concerning estimate (19) we consider the term  $\langle u_n, Wu_n \rangle$  when  $|n| \leq N$ ; let

$$\begin{aligned} \langle u_n, Wu_n \rangle &= \int_{na+a_-}^{na+a_+} W(x) |u_n(x)|^2 dx + \\ &+ \int_{-\infty}^{na+a_-} W(x) |u_n(x)|^2 dx + \int_{na+a_+}^{+\infty} W(x) |u_n(x)|^2 dx \end{aligned}$$

where

$$\begin{aligned} \int_{na+a_-}^{na+a_+} W(x) |u_n(x)|^2 dx &= \int_{na+a_-}^{na+a_+} x |u_n(x)|^2 dx \\ &= C_0 + na \int_{a_-}^{a_+} |u_0(y)|^2 dy = C_0 + na \left[ 1 + \tilde{\mathcal{O}}(e^{-S_0/h}) \right] \end{aligned}$$

because  $u_n(y + na) = u_0(y)$ , Lemma 1.ii and Lemmata 4.iii and 7 by [13]. More precisely, let  $\Omega_0 = \mathbb{R} \setminus [a_-, a_+]$  then

$$\int_{a_-}^{a_+} |u_0(y)|^2 dy = 1 - \int_{\mathbb{R}} |\chi_{\Omega_0}(y)|^2 |u_0(y)|^2 dy$$

where  $\chi_{\Omega_0}$  is the characteristic function on  $\Omega_0$ . Then (the properties below concerning  $w_0$  are given in Lemma 4.iii by [13], where  $w_0(x)$  is the single well ground state defined in §3.2)

$$\begin{aligned} \|\chi_{\Omega_0} u_0\|_{L^2} &\leq \|\chi_{\Omega_0} w_0\|_{L^2} + \|\chi_{\Omega_0} (u_0 - w_0)\|_{L^2} \leq \tilde{\mathcal{O}}(e^{-S_0/2h}) + \tilde{\mathcal{O}}(e^{-S_0/h}) \\ &= \tilde{\mathcal{O}}(e^{-S_0/2h}). \end{aligned}$$

Hence,

$$\int_{a_-}^{a_+} |u_0(y)|^2 dy = 1 - \tilde{\mathcal{O}}(e^{-S_0/h}).$$

Concerning the estimate of the remainder terms we have that

$$\left| \int_{-\infty}^{na+a_-} W(x) |u_n(x)|^2 dx \right| \leq C \int_{\mathbb{R}} \chi_{(-\infty, na+a_-)}^2(x) |u_n(x)|^2 dx = \tilde{\mathcal{O}}(e^{-S_0/h})$$

because  $W$  is bounded and by making use of the same arguments as before. Similarly we get the same estimate for  $\int_{na+a_+}^{+\infty} W(x) |u_n(x)|^2 dx$ .  $\square$

**Remark 9.** By construction and since  $u_0$  is normalized to one it follows that  $|C_0| \leq C$  for some positive constant  $C > 0$  independent of  $h$ .

Finally, concerning the nonlinear term we recall the following result which follows by [13] (where we choose  $\sigma = 1$ , for the purpose of completeness the detailed proof is given in a separate appendix).

**Lemma 4.** We have that

$$\langle u_n, \psi^3 \rangle = C_1 c_n^3 + r_4^n,$$

where

$$C_1 = \|u_n\|_4^4 \equiv \|u_0\|_4^4$$

and

$$r_4^n = (\langle u_n, \psi^3 \rangle - C_1 c_n^3)$$

satisfies to the following estimate: let  $\mathbf{r}_4 = \{r_4^n\}_{n \in \mathbb{Z}}$ , then for any  $\rho > 0$  there exists  $C := C_\rho$  such that

$$\|\mathbf{r}_4\|_{\ell^1} \leq C \left[ h^{-1/2} \|\psi_\perp\|_{H^1}^3 + \|\mathbf{c}\|_{\ell^1} \|\psi_\perp\|_{H^1}^2 + \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} \|\psi_\perp\|_{H^1} + \|\mathbf{c}\|_{\ell^1}^3 e^{-(S_0-\rho)/h} \right]$$

**Remark 10.** By Lemma 1.vi it follows that  $C_1 \sim h^{-1/2}$  as  $h$  goes to zero.

Therefore, equation (18) takes the form

$$\begin{cases} \lambda c_n &= (\Lambda_1 + FC_0)c_n - \beta(c_{n+1} + c_{n-1}) + F\xi(n)ac_n + \eta C_1 c_n^3 + r^n, \\ \lambda \psi_\perp &= H_B \psi_\perp + F\Pi_\perp W\psi + \eta \Pi_\perp \psi^3 \end{cases} \quad (20)$$

where

$$r^n = r_1^n + F(r_2^n + r_3^n) + \eta r_4^n + F r_5^n, \quad r_5^n := a \left[ \tilde{\xi}(n) - \xi(n) - \frac{C_0}{a} \right] c_n. \quad (21)$$

**Definition 1.** We define the discrete nonlinear Stark-Wannier equation (hereafter DNLSWE) as

$$\lambda g_n = (\Lambda_1 + FC_0)g_n - \beta(g_{n+1} + g_{n-1}) + F\xi(n)ag_n + \eta C_1 g_n^3 \quad (22)$$

where  $\mathbf{g} = \{g_n\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^2(\mathbb{Z})$ .

As already explained in Remark 2 we expect that the solutions to equation (22) are displaced, when  $\xi(n) \equiv n$  (corresponding to the case  $W(x) \equiv x$ ), on regular ladders, that is the solutions  $\lambda$  are of the form  $\lambda_j = \lambda_0 + jFa$  for some  $\lambda_0 \in \mathbb{R}$  and any  $j \in \mathbb{Z}$ . We will call, hereafter, the value  $\lambda_j$  as the  $j$ -th rung of the ladder connected to  $\lambda_0$ . In the case that  $W(x)$  is a linear function on an interval  $[-Na, Na]$ , according with Hyp. 2, then we will see that the structure of the ladder

locally occurs, so even in such a case we may speak of *rungs* of such a kind of ladders of stationary solutions.

## 5. ANTICONTINUOUS LIMIT OF THE DNLSWE

Let us set

$$\tilde{\lambda} := \lambda - (\Lambda_1 + FC_0), \quad \nu := \eta C_1, \quad f := Fa \quad (23)$$

where

$$f \sim h^2, \quad \nu \sim h^{3/2} \quad \text{and} \quad \beta = \tilde{O}\left(e^{-S_0/h}\right) \quad (24)$$

since Remark 10 and Lemma 1.i. For argument's sake, we assume that  $f, \nu \geq 0$ . Hence (22) takes the form

$$\tilde{\lambda} g_n = -\beta (g_{n+1} + g_{n-1}) + f\xi(n)g_n + \nu g_n^3. \quad (25)$$

and in the *anticontinuous limit*  $\beta \rightarrow 0$  then (25) becomes

$$(\tilde{\lambda} - \nu d_n^2) d_n = f\xi(n)d_n, \quad \mathbf{d} = \{d_n\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^2(\mathbb{Z}). \quad (26)$$

**5.1. Finite-mode solutions to the anticontinuous limit equation (26).** Here, we look for stationary solutions  $\mathbf{d} \in \ell_{\mathbb{R}}^2$  to (26) under the normalization condition

$$\|\mathbf{d}\|_{\ell^2}^2 = \sum_{n \in \mathbb{Z}} d_n^2 = 1.$$

**Definition 2.** We say that the anticontinuous limit equation (26), under the normalization condition  $\|\mathbf{d}\|_{\ell^2} = 1$ , has a one-mode solution if there exists a set  $S \subset \mathbb{Z}$ , hereafter called *solution-set*, with finite cardinality, a real value  $\mu^S$  and a normalized vector  $\mathbf{d}^S = \{d_n^S\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^2(\mathbb{Z})$  where  $\mu^S$  and  $\mathbf{d}^S$  solve

$$(\mu^S - \nu d_n^2) d_n = f\xi(n)d_n, \quad \text{with } d_n \neq 0 \text{ if } n \in S \quad (27)$$

and where  $d_n^S = 0$  if  $n \notin S$ . The real value  $\mu^S$  is hereafter called the “energy” associated to the stationary solution  $\mathbf{d}^S$ .

When  $\nu = 0$  then we simply recover a (kind of) Stark-Wannier ladder, that is the solution-sets are given by simple sets of the form  $S = \{j\}$  for any  $j \in \mathbb{Z}$  and we have a family of admitted “energies”  $\mu^S = f\xi(j)$  with associated stationary solutions  $\mathbf{d} = \{\delta_n^j\}_{n \in \mathbb{Z}}$ . In fact, it is an exact Stark-Wannier ladder when  $\xi(n) \equiv n$ .

Assume now that the effective nonlinearity strength is not zero, that is  $\nu > 0$  for argument's sake. In such a case, equation (27) has finite mode solutions  $\mathbf{d}^S = \{d_n^S\}_{n \in \mathbb{Z}}$ , associated to sets  $S \subset \mathbb{Z}$  with finite cardinality  $\mathcal{N} = \#S < \infty$ , given by

$$d_n^S = \begin{cases} 0 & \text{if } n \notin S \\ \pm \left[ \frac{\mu^S - f\xi(n)}{\nu} \right]^{1/2} & \text{if } n \in S \end{cases}, \quad (28)$$

with the condition

$$\mu^S - f\xi(n) > 0, \quad n \in S, \quad (29)$$

because we have assumed that  $d_n^S \in \mathbb{R}$  and  $\nu > 0$ . The normalization condition reads

$$1 = \|\mathbf{d}^S\|_{\ell^2}^2 = \sum_{n \in S} (d_n^S)^2 = \sum_{n \in S} \frac{\mu^S - f\xi(n)}{\nu}. \quad (30)$$

In the case  $\mathcal{N} = 1$  then  $S = \{j\}$  again for any  $j \in \mathbb{Z}$  and (30) reduces to

$$\mu^S = \nu + f\xi(j)$$

where condition (29) holds true because we have assumed that  $\nu > 0$ ; the associated stationary solution  $\mathbf{d}^S$  takes the form:

$$d_n^S = \begin{cases} 0 & \text{if } n \neq j \\ \pm 1 & \text{if } n = j \end{cases}.$$

That is we recover a kind of (perturbed) Stark-Wannier ladder.

**Remark 11.** *From this fact we can conclude that the anticontinuous limit (26) always admits a ladder-type family of normalized one-mode solutions.*

**5.2. Finite-mode solutions to equation (27) associated to solution-sets  $S$  with finite cardinality bigger than 1.** In order to look for finite-mode solutions with  $\mathcal{N} > 1$  the normalization condition (30) implies that

$$\mu^S = \frac{\nu}{\mathcal{N}} + \frac{f}{\mathcal{N}} \sum_{n \in S} \xi(n) \quad \text{with} \quad \max_{n \in S} \xi(n) < \frac{\mu^S}{f}. \quad (31)$$

**5.2.1. Existence of finite-mode solutions.** Stationary solutions  $\mathbf{d}^S$  associated to the energy (31) are given by

$$d_n^S = \begin{cases} 0 & \text{if } n \notin S \\ \pm \left[ \frac{1}{\mathcal{N}} + \frac{f}{\nu \mathcal{N}} \sum_{\ell \in S} \xi(\ell) - \frac{f}{\nu} \xi(n) \right]^{1/2} & \text{if } n \in S \end{cases}. \quad (32)$$

In the case  $\mathcal{N} = 2$  let  $S = \{j, j + \ell_1\}$  with  $\ell_1 > 0$ . The eigenvalue equation (31) becomes

$$\mu^S = \frac{\nu}{2} + \frac{f}{2} [\xi(j) + \xi(j + \ell_1)] \quad (33)$$

where condition  $\max_{n \in S} \xi(n) < \frac{\mu^S}{f}$  becomes

$$f\xi(j + \ell_1) < \frac{1}{2}\nu + \frac{1}{2}f [\xi(j) + \xi(j + \ell_1)],$$

that is

$$0 \leq [\xi(j + \ell_1) - \xi(j)] < \frac{\nu}{f}. \quad (34)$$

In conclusion, if

- $\frac{\nu}{f} \leq [\xi(j + \ell_1) - \xi(j)]$  then (34) is not satisfied and there are no stationary solutions associated to solution-sets of the form  $S = \{j + \ell_1, j\}$  with cardinality 2;
- $0 \leq [\xi(j + \ell_1) - \xi(j)] < \frac{\nu}{f}$  we have a family of two-mode solutions associated to solution-sets  $S = \{j, j + \ell_1\}$  with  $\mu^S$  given by (33) and where

$$d_n^S = \begin{cases} 0 & \text{if } n \neq j, j + \ell_1 \\ \pm \left[ \frac{1}{2} + \frac{1}{2} \frac{f}{\nu} (\xi(j + \ell_1) - \xi(j)) \right]^{1/2} & \text{if } n = j \\ \pm \left[ \frac{1}{2} - \frac{1}{2} \frac{f}{\nu} (\xi(j + \ell_1) - \xi(j)) \right]^{1/2} & \text{if } n = j + \ell_1 \end{cases}$$

Finally, we can extend such an argument to any integer number  $\mathcal{N} > 1$  obtained the following result.

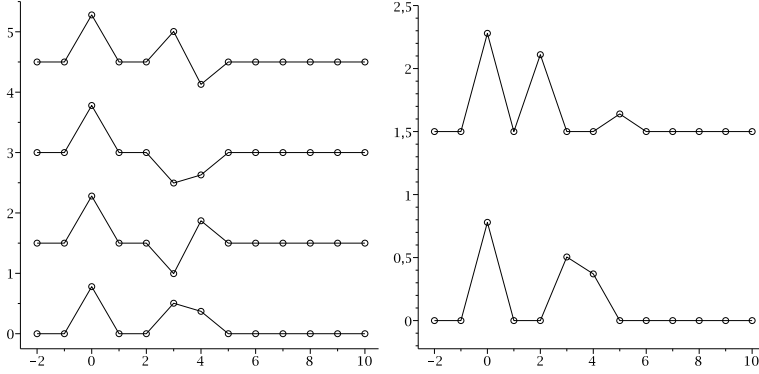


FIGURE 1. In the left panel we plot the 4 solutions  $\mathbf{d}^{S_1}$  corresponding to the solution-set  $S_1 = \{0, 3, 4\}$ . In the right panel we plot the solutions  $\mathbf{d}^{S_1}$  and  $\mathbf{d}^{S_2}$  given by (32) with sign +, corresponding to the solution-sets  $S_1$  and  $S_2 = \{0, 2, 5\}$ ; both solutions are associated to the same value of the energy  $\mu$ .

**Theorem 1.** Let  $S = \{j + \ell_0, j + \ell_1, \dots, j + \ell_{N-1}\}$ , with  $j \in \mathbb{Z}$  and  $0 = \ell_0 < \ell_1 < \ell_2 < \dots < \ell_{N-1}$  positive integer numbers such that

$$\xi(j + \ell_{N-1}) < \frac{\nu}{fN} + \frac{1}{N} \sum_{k=0}^{N-1} \xi(j + \ell_k) \quad (35)$$

Then  $S$  is a solution-set connected to the  $j$ -th rung of a (kind of) Stark-Wannier ladder and equation (27) has a  $N$ -mode solution with

$$\mu^S = \frac{\nu}{N} + \frac{f}{N} \sum_{k=0}^{N-1} \xi(j + \ell_k) \quad (36)$$

and associated normalized stationary solution given by (32).

**Remark 12.** We should underline that some of such a solution may be associated to the same “energy”  $\mu^S$ . For instance let  $N > 5$  and let us consider the sets  $S_1 = \{0, 3, 4\}$  and  $S_2 = \{0, 2, 5\}$ . Recalling that  $\xi$  is a linear function in both sets  $S_1$  and  $S_2$  then they are associated to the same value (where we assume, for argument sake, that  $a = 1$  and  $C_0 = 0$ ) of energy

$$\mu = \frac{1}{3}\nu + \frac{7}{3}f.$$

In Figure 1 - left panel - we plot the 4 solutions (32) corresponding to the set  $S_1$ . In Figure 1 - right panel - we plot the solutions (32) with sign +, corresponding to the sets  $S_1$  and  $S_2$ .

**Remark 13.** If the solution-set  $S = \{0, \ell_1, \dots, \ell_{N-1}\} \subset [-N, +N]$  then  $\xi_n(\ell)$ ,  $\ell \in S$ , is linear and thus we locally recover a Stark-Wannier ladder structure. That is  $S' = \{j, j + \ell_1, \dots, j + \ell_{N-1}\}$  is a solution-set too, provided that  $|j|, |j + \ell_{N-1}| \leq N$ , and  $\mu^{S'} - \mu^S = jf$ . We will say that  $\mu^S$  is connected with the 0-th rung of the ladder, and that  $\mu^{S'}$  is connected with the  $j$ -th rung of the ladder.



**Remark 14.** *In the limit of  $h$  small enough then  $\frac{\nu}{f} = \frac{\eta C_1}{F_a} \sim C_1 \sim h^{-1/2}$  since Remark 10 and (24); therefore the stationary solutions  $\mathbf{d}^S$  takes the value  $d_n^S = 0$  if  $n \notin S$  and  $d_n^S \sim \pm 1$  if  $n \in S$ , and the energy  $\mu^S$  belongs to an interval with center  $\nu \sim Ch^{3/2}$ , for some  $C > 0$ , and with amplitude of order  $h^2$ .*

**5.2.2. Bifurcation of stationary solutions.** We consider solution-sets  $S$  associated to a given rung of the (kind of) Stark-Wannier ladder satisfying the condition  $S \subseteq [-N, +N]$  where  $\xi(n)$  is a linear function. That is we consider energies  $\mu^S$  in the interval  $[\nu - fN, \nu + fN]$ . We can see that stationary solutions to equation (27) associated to such solution-sets  $S$  may bifurcate when the ratio  $\nu/f$  is a positive integer number.

In order to count how many stationary solutions we have let us introduce the following function (see Abramowitz and Stegun [1], p. 825).

**Definition 3.** *Let  $Q(n)$ ,  $n \in \mathbb{N}$ , be the number of ways of writing the integer number  $n$  as a sum of positive integers without regard to order, with the constraint that all integers in a given partition are distinct.*

E.g.:  $Q(1) = 1$ ,  $Q(2) = 1$ ,  $Q(3) = 2$  and  $Q(4) = 2$ .

**Theorem 2.** *When  $\nu/f$  takes the value of a positive integer number then stationary solutions to (27), associated to solution-set  $S \subset [-N, N]$ , bifurcate. Furthermore, the total number of solutions-sets  $S$  associated to a given rung of the (kind of) Wannier-Stark ladder, assuming that all these sets  $S$  are contained in the interval  $[-N, +N]$ , is given by*

$$M(\nu/f) = \sum_{0 < n < \nu/f} Q(n). \quad (37)$$

*Proof.* First of all, because the stationary problem (27) is translation invariant  $n \rightarrow n + \ell$  and  $\mu^S \rightarrow \mu^S - f\ell$ , provided that the solution-sets are contained in the interval  $[-N, N]$ , then we can always restrict ourselves to the 0-th rung of the ladder such that  $\min S = 0$ , that is the solution-set has the form  $S = \{0, \ell_1, \dots, \ell_{N-1}\}$  with  $0 < \ell_1 < \ell_2 < \dots < \ell_{N-1} < N$  positive and integer numbers. Hence, (31) becomes

$$\mu^S = \frac{\nu}{N} + \frac{f}{N} \sum_{\ell \in S} \ell.$$

and condition (29) implies the following condition on the solution-set  $S$

$$\frac{\nu}{f} > N \max S - \sum_{\ell \in S} \ell = \sum_{\ell \in S} [\max S - \ell] > \sum_{\ell^* \in S^*} \ell^* \quad (38)$$

where

$$S^* = \{\ell^* := \max S - \ell : \ell \in S\}.$$

Let  $S^*(\nu/f)$  be the collection of sets  $S^*$  satisfying (38), and let  $\mathcal{Q}^*(n)$  be the collection of sets of all non negative integer numbers, including the number 0, which sum is equal to  $n$ , without regard to order with the constraint that all integers in a given partition are distinct; e.g.  $\mathcal{Q}^*(1) = \{\{0, 1\}\}$ ,  $\mathcal{Q}^*(2) = \{\{0, 2\}\}$  and  $\mathcal{Q}^*(3) = \{\{0, 3\}, \{0, 1, 2\}\}$ . Hence, by construction

$$S^*(n+1) = S^*(n) \cup \mathcal{Q}^*(n).$$

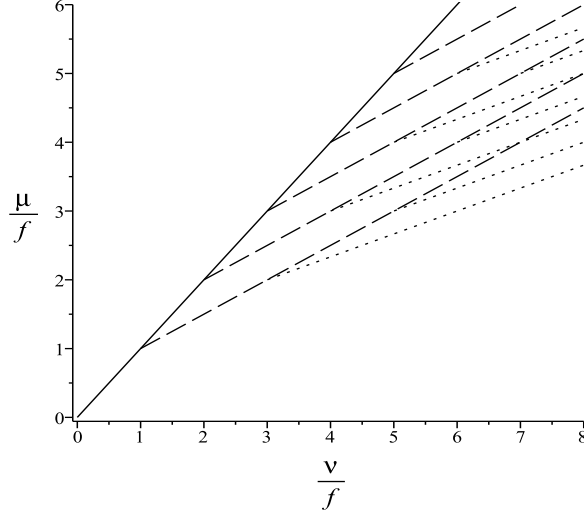


FIGURE 2. Here we plot the values of  $\mu/f$  associated to stationary solution-sets  $S$  such that  $\min S = 0$  and where  $\mathcal{N} = 1, 2, 3$ ; we can see a cascade of bifurcations when  $\nu/f$  increases. Full line represents the solution corresponding to the 0-th rung of the Stark-Wannier ladder localized on the 0-th cell ( $\mathcal{N} = 1$ ), broken lines represent the solutions of the same rung of the Stark-Wannier ladder localized on two cells ( $\mathcal{N} = 2$ ), and finally point lines represent the solutions of the same rung of the Stark-Wannier ladder localized on three cells ( $\mathcal{N} = 3$ ).

In conclusion, we have shown that the counting function  $M(\nu/f)$  defined as the number of solution-sets  $S$  of integer numbers satisfying the conditions (38) and such that  $\min S = 0$ , is given by

$$M(\nu/f) = \sum_{0 < n < \nu/f} Q(n).$$

Theorem 2 is so proved.  $\square$

**Remark 15.** A cascade of bifurcation points, when  $\nu/f$  takes the value of any positive integer, occurs; indeed, when the ratio  $\nu/f$  becomes larger than a positive integer  $n$  then  $Q(n)$  new stationary solutions appear. This fact can be seen in Figure 2, where we plot the values of  $\mu/f$ , when  $\nu/f$  belongs to the interval  $[0, 10]$ , associated to solution-sets  $S$  such that  $\min S = 0$ , that is we plot the value of energies associated to the 0-th rung of the (kind of) Wannier-Stark ladder. By translation  $\mu \rightarrow \mu + jf$ ,  $j \in \mathbb{Z}$ , and thus this picture occurs for each rung of the ladder and then the collection of values of  $\mu$  associated to stationary solutions is going to densely cover intervals of the real axis.

**Remark 16.** One can see that  $M(\nu/f)$  grows quite fast, indeed the following asymptotic behavior holds true [1]:

$$Q(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4 \cdot 3^{1/4} n^{3/4}} \quad \text{as } n \rightarrow \infty.$$

Hence

$$M(n) \sim \frac{1}{2} \operatorname{erfi} \left[ \sqrt{\pi} (n/3)^{1/4} \right] \sim \frac{\exp \left[ \pi (n/3)^{1/2} \right]}{2\pi (n/3)^{1/4}}$$

as  $n$  goes to infinity, where  $\operatorname{erfi}(x) = -i \operatorname{erf}(ix)$  is the imaginary error function. In particular, because (see Remark 14)  $\frac{\nu}{f} \sim C_1 \sim h^{-1/2}$ , then we have that the energy  $\mu$  lies in an interval  $[\nu - fN, \nu + fN]$  with center at  $\nu \sim h^{3/2}$  and amplitude of order  $h^2$ , and the number of stationary solutions is of order

$$M \left( \frac{\nu}{f} \right) \sim h^{1/8} e^{Ch^{-1/4}} \quad \text{as } h \text{ goes to zero,}$$

for some positive constant  $C$ . That is the energy spectrum densely fill the interval  $[\nu - fN, \nu + fN]$  when  $h$  goes to zero.

**Remark 17.** Since we assumed that the parameters  $\hbar$ ,  $m$ ,  $\alpha_1$  and  $\alpha_2$  are fixed and that  $\epsilon \ll 1$  then the rescaling (2) immediately implies that the effective nonlinear coupling strength parameter  $\eta$  is of order  $h^2$ , where  $h$  plays the role of a semiclassical parameter. Hence, the bifurcation parameter is such that

$$\frac{\nu}{f} = \frac{\eta C_1}{Fa} \sim h^{-1/2} \gg 1$$

and we have a dense energy spectrum. One can also consider the case where the parameters depend on  $\epsilon$ ; a similar approach has been used, e.g., by [12]. If the nonlinear coupling strength parameter  $\alpha_2$  depends on some power by  $\epsilon$  then  $\eta \sim h^u$  for some power  $u$  and

$$\frac{\nu}{f} = \frac{\eta C_1}{Fa} \sim h^{u-5/2}.$$

If  $u < \frac{5}{2}$  then we have a dense energy spectrum as in the case above; if  $u > \frac{5}{2}$  then we have a single Stark-Wannier ladder; if  $u = \frac{5}{2}$  then we observe the bifurcation phenomenon.

5.2.3. When do  $\mathcal{N}$ -mode stationary solutions arise from  $(\mathcal{N} - 1)$ -mode stationary solutions? If one looks with more detail the bifurcation cascade one can see that we have  $\mathcal{N}$ -mode solutions for any value of  $\mathcal{N}$ , provided that  $S \subset [-N, N]$  for some  $N$  large enough. Let us restrict our analysis, for sake of simplicity, to solution-sets  $S$  contained in the interval  $[-N, N]$  where  $\xi(n)$  is a linear function.

As said above,  $\mathcal{N}$ -mode stationary solutions are associated to solution-sets of the form

$$S = \{j, j + \ell_1, \dots, j + \ell_{\mathcal{N}-1}\} \quad (39)$$

under condition (29). Now, let us consider, as a particular family of  $\mathcal{N}$ -mode solutions, solution-sets of the form (39) for any  $j \in \mathbb{Z}$  and  $\ell_{r+1} - \ell_r = 1$ , assuming that  $|j|, |j + \mathcal{N} - 1| \leq N$ . They are associated to

$$\mu^S = \frac{\nu}{\mathcal{N}} + fj + \frac{1}{2}f(\mathcal{N} - 1)$$

and then condition (29) implies that

$$\frac{\mathcal{N}(\mathcal{N}-1)}{2} < \frac{\nu}{f}$$

Hence, we can observe a second bifurcation phenomenon: stationary solutions associated to solution-sets with  $\mathcal{N}$  elements arise from stationary solutions associated to solution-sets with  $\mathcal{N}-1$  elements when  $\nu/f$  becomes bigger than the critical value  $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ .

We can summarize such a result as follows

**Theorem 3.** *If  $\nu/f < \mathcal{N}(\mathcal{N}-1)/2$  then stationary solutions to (26), associated to solution-sets  $S \subset [-N, +N]$ , are localized on a number of sites less than  $\mathcal{N}$ , at  $\nu/f = \mathcal{N}(\mathcal{N}-1)/2$  a stationary solution localized on  $\mathcal{N}-1$  sites bifurcates and a new stationary solution localized on  $\mathcal{N}$  sites arises.*

## 6. EXISTENCE OF SOLUTIONS TO THE DNLSWE

Let  $\mu^S$  and  $\mathbf{d}$  be a finite-mode solution to (26) associated to a solution-set  $S$  given by Theorem 1. Now, we will prove, by a stability argument, that this solution becomes a solution to (25) when  $\beta$  is small enough. To this end we have to remind that  $\beta$  goes to zero when  $h$  goes to zero according with Lemma 1.i.

**Theorem 4.** *Let  $\frac{\nu}{f} \notin \mathbb{N}$ . Let  $S$  be a solution-set to (26) with associated energy  $\mu^S$  and normalized stationary solution  $\mathbf{d}^S$  given by (28). We assume that  $S \subset [-N, N]$ . Then, if  $\beta$  is small enough there exists a stationary solution  $\mathbf{g}^S \in \ell_{\mathbb{R}}^1$  to the DNLSWE (25) associated to  $\tilde{\lambda} = \mu^S$  and such that*

$$\|\mathbf{g}^S - \mathbf{d}^S\|_{\ell^1} = \tilde{\mathcal{O}}\left(e^{-S_0/h}\right) \quad \text{and} \quad \|\mathbf{g}^S\|_{\ell^2} = 1 + \tilde{\mathcal{O}}\left(e^{-S_0/h}\right).$$

*Proof.* First of all let us recall that from Remark 14 then  $\mu^S \sim Ch^{3/2} \neq 0$ . In the following let us omit the upper letters  $S$  in  $\mathbf{d}^S$  and  $\mathbf{g}^S$  for sake of simplicity. If we rescale  $g_n = \left[\frac{\tilde{\lambda}}{\nu}\right]^{1/2} g'_n$  and  $d_n = \left[\frac{\tilde{\lambda}}{\nu}\right]^{1/2} d'_n$ , and if we set  $\beta' = \beta/\tilde{\lambda}$  and  $f' = f/\tilde{\lambda}$  then equations (25) and (26) take the form

$$\left(1 - g_n'^2\right) g'_n = \beta'(g'_{n+1} + g'_{n-1}) + f'\xi(n)g'_n. \quad (40)$$

with anticontinuous limit

$$\left(1 - d_n'^2\right) d'_n = f'\xi(n)d'_n. \quad (41)$$

Therefore, any solution  $\tilde{\lambda}$  and  $\mathbf{g}$  to (25) is associated to a solution  $\mathbf{g}'$  to (40), and all the solutions  $\mathbf{d}'$  to (41) associated to the values  $\tilde{\lambda} = \mu^S$  given by Theorem 1 are isolated in  $\ell^1$  by construction when there are no bifurcations, that is for  $\frac{\nu}{f} \notin \mathbb{N}$ .

Let  $\mathcal{F}_1 : (-\delta, +\delta) \times \ell_{\mathbb{R}}^1(\mathbb{Z}) \rightarrow \ell_{\mathbb{R}}^1(\mathbb{Z})$  be the map defined as

$$(\mathcal{F}_1(\beta', \mathbf{g}'))_n := -\left(1 - g_n'^2\right) g'_n - \beta'(g'_{n+1} + g'_{n-1}) - f'\xi(n)g'_n.$$

We are going to look for solutions  $\mathbf{g}'(\beta')$  to equation  $\mathcal{F}_1(\beta', \mathbf{g}') = 0$ ; where we already know that equation  $\mathcal{F}_1(0, \mathbf{d}') = 0$  has solutions  $\mathbf{d}'$  associated to the ones

given by Theorem 1. We may extend the solutions to (41), obtained in the anti-continuous limit  $\beta' \rightarrow 0$ , to the solutions to equation (40) for  $\beta'$  small enough if the tridiagonal matrix

$$T(\beta') := (D_{\mathbf{g}'} \mathcal{F}_1)(\beta', \mathbf{g}') = \text{tridiag}(\beta', f'\xi(n) - 1 + 3g_n'^2, \beta'),$$

obtained deriving the previous equation by  $g'_n$ , is not singular at  $\beta' = 0$ , where  $g'_n$  takes the value of a solution  $d'_n$  to (32) obtained for  $\beta' = 0$ . The linearized map  $(D_{\mathbf{g}'} \mathcal{F}_1)(0, \mathbf{d}')$  can be written as  $\text{diag}(T_n)$ , where

$$T_n = \frac{f\xi(n)}{\mu^S} - 1 + 3d_n'^2 \quad \text{and} \quad d_n'^2 = \begin{cases} 0 & \text{if } n \notin S \\ \frac{\mu^S - f\xi(n)}{\mu^S} & \text{if } n \in S \end{cases}. \quad (42)$$

Hence

$$T_n = \begin{cases} \frac{f\xi(n) - \mu^S}{\mu^S} & \text{if } n \notin S \\ -2\frac{f\xi(n) - \mu^S}{\mu^S} & \text{if } n \in S \end{cases}.$$

**Lemma 5.** *Let  $h$  be small enough, then it follows that*

$$\inf_{n \in \mathbb{Z}} |T_n| > \frac{1}{2}.$$

*Proof.* Assume at first that  $\mathcal{N} = 1$ . In this case  $S = \{j\}$  for any  $j \in \mathbb{Z}$  such that  $|j| \leq N$  and

$$\mu^S = \nu + f\xi(j) = \nu + fj + f\frac{C_0}{a} + \tilde{\mathcal{O}}\left(e^{-S_0/h}\right)$$

as  $h$  goes to zero, because of Lemma 3. Hence

$$T_n = \frac{f[\xi(n) - \xi(j)] - \nu}{\mu^S} + \tilde{\mathcal{O}}\left(e^{-S_0/h}\right).$$

Recalling now that  $\nu = \eta C_1 \sim h^{3/2}$  and  $f = Fa \sim h^2$  then  $T_n \sim -1$ . Similarly, we can easily extend such arguments to any integer number  $\mathcal{N} > 1$ . In this case  $S = \{j, j + \ell_1, \dots, j + \ell_{\mathcal{N}-1}\}$ , for  $j \in \mathbb{Z}$  and  $\ell_k \in \mathbb{N}$  such that  $|j|, |j + \ell_k| < M$ ,  $k = 1, \dots, \mathcal{N} - 1$ . Then

$$\mu^S = \frac{1}{\mathcal{N}}\nu + \frac{f}{\mathcal{N}} \sum_{\ell \in S} \ell + f\frac{C_0}{a} \sim h^{3/2};$$

hence

$$T_n = \frac{f(\mathcal{N}\xi(n) - \sum_{\ell \in S} \ell - \frac{\mathcal{N}C_0}{a}) - \nu}{\mathcal{N}\mu^S} \sim -1, \forall n.$$

The proof of the Lemma is so completed.  $\square$

Now we are ready to conclude the proof of Theorem 4. Indeed, by Lemma 5, the linearized map  $(D_{\mathbf{g}'} \mathcal{F}_1)(0, \mathbf{d}')$  is invertible with inverse uniformly bounded. Therefore, by the Implicit Function Theorem, there exists a neighborhood  $\mathcal{U}$  of 0 such that if  $\beta'$  belongs to such a neighborhood  $\mathcal{U}$  then there exists a unique solution  $\mathbf{g}' := \mathbf{g}'(\beta')$  to equation  $\mathcal{F}_1(\beta', \mathbf{g}') = 0$  in a  $\ell^1$ -neighborhood of  $\mathbf{d}'$ , where  $\mathbf{d}'$  is an isolated solution to  $\mathcal{F}_1(0, \mathbf{d}') = 0$  because bifurcations occur at  $\frac{\nu}{f} \in \mathbb{N}$ . Since  $\beta' = \tilde{\mathcal{O}}(e^{-S_0/h})$ , for any  $h$  small enough, then  $\beta' \in \mathcal{U}$  for any  $h \in (0, h^*)$  for some  $h^* > 0$ . From this fact and because the map  $\beta' \rightarrow \mathbf{g}'(\beta')$  is  $C^1$  then we

can conclude that when  $h$  is small enough then there exists a solution  $\mathbf{g}' \in \ell_{\mathbb{R}}^1$  to equation  $\mathcal{F}_1(\beta', \mathbf{g}') = 0$  such that

$$\|\mathbf{g}' - \mathbf{d}'\|_{\ell^1} = \tilde{\mathcal{O}}\left(e^{-S_0/h}\right). \quad (43)$$

By construction it follows also that when  $\tilde{\lambda} = \mu^S$  then

$$\|\mathbf{d}\|_{\ell^2} = 1, \quad \|\mathbf{d}'\|_{\ell^2} = \left(\frac{\nu}{\mu^S}\right)^{1/2} \quad \text{and} \quad \|\mathbf{g}'\|_{\ell^2} = \left(\frac{\nu}{\mu^S}\right)^{1/2} \|\mathbf{g}\|_{\ell^2}$$

hence

$$|\|\mathbf{g}\|_{\ell^2} - 1| \leq \|\mathbf{g} - \mathbf{d}\|_{\ell^2} \leq \|\mathbf{g} - \mathbf{d}\|_{\ell^1} = \tilde{\mathcal{O}}\left(e^{-S_0/h}\right)$$

from (43). Then the proof of the Theorem is given.  $\square$

**Remark 18.** *Since we can always normalize  $\mathbf{g}^S$  to 1 by means of suitable rescaling of the nonlinearity parameter  $\nu$  we can conclude that  $\mathbf{g}^S$  is a normalized solution to (25) associated to  $\mu^S$  for some  $\tilde{\nu} = \nu + \tilde{\mathcal{O}}(e^{-S_0/h})$ . Furthermore, by construction (see §3.2), the map  $\nu \rightarrow \mu^S(\nu)$  is  $C^1$  when we are far from the bifurcation points  $\frac{\nu}{f} \in \mathbb{N}$ ; then we can conclude that for any  $\nu$  fixed and such that  $\frac{\nu}{f} \notin \mathbb{N}$  then equation (25) has a solution  $\tilde{\lambda}$  and  $\mathbf{g}^S$  where  $\mathbf{g}^S$  is normalized and it satisfies (43) and  $\tilde{\lambda} = \mu^S + \tilde{\mathcal{O}}(e^{-S_0/h})$ .*

## 7. FIXED POINT ARGUMENT

Here, we go back to equation (18) and, at first, we justify the existence of  $\psi_{\perp}$  by means of a fixed point argument. Recalling that  $\lambda = \tilde{\lambda} + (\Lambda_1 + FC_0)$  and  $\tilde{\lambda} = \mu^S + \tilde{\mathcal{O}}(e^{-S_0/h})$  where  $\mu^S \in [\nu - fN, \nu + fN]$  and where  $|C_0| \leq C$ , then the value of  $\lambda$  corresponding to  $\mu^S$  is such that  $\lambda = \Lambda_1 + \nu + \mathcal{O}(h^2)$ . Hence, we consider the second equation of (18) for  $\lambda$  in a neighborhood of  $\Lambda_1$  with width of order  $h^{3/2}$ .

**Theorem 5.** *Let  $\frac{\nu}{f} \notin \mathbb{N}$ ,  $\psi = \psi_1 + \psi_{\perp} \in L^2$ , where  $\psi_1 = \Pi\psi = \sum_{n \in \mathbb{Z}} c_n u_n$  and  $\psi_{\perp} = \Pi_{\perp}\psi$ , let  $h > 0$  small enough. Let  $\delta_0 > 0$  be any fixed real and positive number, then for any  $\mathbf{c} = (c_n)_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^1(\mathbb{Z})$ , with  $\|\mathbf{c}\|_{\ell^1(\mathbb{Z})} \leq \delta_0$ , there exists a unique smooth map*

$$\hat{\psi}_{\perp} : \ell_{\mathbb{R}}^1(\mathbb{Z}) \rightarrow H^1(\mathbb{R})$$

*such that  $\psi_{\perp} = \hat{\psi}_{\perp}(\mathbf{c})$  is a solution to the second equation of (18) for small  $h > 0$ . Moreover,  $\hat{\psi}_{\perp}(\mathbf{c})$  is small as  $h \rightarrow 0$  in the sense that there exists a positive constant  $C > 0$ , dependent on  $\delta_0$  and independent of  $h$ , such that*

$$\|\hat{\psi}_{\perp}(\mathbf{c})\|_{H^1} \leq Ch^{1/2}. \quad (44)$$

*Proof.* We make use here of same ideas already developed by [13] adapted to the case of a tilted periodic potential. Let  $\Lambda_1$  be defined as in §3.2 and let  $E \in \mathbb{R}$  be fixed. Note that the operator  $H_B + FW - (\Lambda_1 - \Gamma E)$  on  $\Pi_{\perp}L^2$  has inverse operator for  $h$  sufficiently small provided that

$$\Gamma = \mathcal{O}(h^{3/2})$$

and thanks to the fact that the  $\text{dist}[\Lambda_1, \sigma(\Pi_{\perp}H_B)] = \mathcal{O}(h)$ , that  $W$  is a bounded operator, and that  $F = \mathcal{O}(h^2)$  from (4). Precisely, there exists a constant  $C > 0$  independent of  $h$  such that

$$\|[H_B + FW - (\Lambda_1 - \Gamma E)]^{-1}\Pi_{\perp}\|_{\mathcal{L}(L^2 \rightarrow H^1)} \leq Ch^{-1}.$$

Then the second equation of (18) may be written as

$$\psi_{\perp} = \mathcal{F}_2(\psi_{\perp}), \quad (45)$$

where  $\mathbf{c} \in \ell_{\mathbb{R}}^1$  is fixed and where we set  $\Lambda_1 = \lambda + \Gamma E$ ,  $\psi = \psi_1 + \psi_{\perp}$  and

$$\mathcal{F}_2(\psi_{\perp}) = \Pi_{\perp} [H_B + FW - (\Lambda_1 - \Gamma E)]^{-1} \Pi_{\perp} \{-\eta\psi^3 - FW\psi_1\}. \quad (46)$$

We are going to show that  $\mathcal{F}_2$  is a contraction map in

$$K_{\gamma} = \{\psi_{\perp} \in H^1(\mathbb{R}) \cap \Pi_{\perp} L^2(\mathbb{R}) : \|\psi_{\perp}\|_{H^1} \leq \gamma\}$$

for some  $\gamma > 0$ . Indeed, let  $\psi_{\perp}, \phi_{\perp} \in K_{\gamma}$  and let  $\psi = \psi_1 + \psi_{\perp}$  and  $\phi = \psi_1 + \phi_{\perp}$ , we have

$$\begin{aligned} \|\mathcal{F}_2(\psi_{\perp})\|_{H^1} &\leq C \left[ \frac{|\eta|}{h} \|\psi^3\|_{L^2} + \frac{F}{h} \|W\psi_1\|_{L^2} \right] \\ &\leq \frac{|\eta|C}{h} (\|\psi_1\|_{L^6}^3 + \|\psi_{\perp}\|_{L^6}^3) + Ch\|\mathbf{c}\|_{\ell^1} \end{aligned}$$

since Remark 7 and (4). Then, by the Gagliardo-Nirenberg inequality, it follows that

$$\|\psi_{\perp}\|_{L^6}^3 \leq C \|\partial_x \psi_{\perp}\|_{L^2} \|\psi_{\perp}\|_{L^2}^2 \leq C\gamma^3$$

since  $\psi_{\perp} \in K_{\gamma}$ , and because by Remark 7 and Lemma 1.vi

$$\|\psi_1\|_{L^6}^3 \leq Ch^{-1/2} \|\mathbf{c}\|_{\ell^1}^3 \leq Ch^{-1/2} \delta_0^3$$

Hence

$$\|\mathcal{F}_2(\psi_{\perp})\|_{H^1} \leq C_2 := C_2(h) = C \left[ |\eta|h^{-3/2} + |\eta|h^{-1}\gamma^3 + h \right] = Ch^{1/2} < \gamma$$

for some  $h$  small enough. Furthermore

$$\mathcal{F}_2(\psi_{\perp}) - \mathcal{F}_2(\phi_{\perp}) = -\eta \Pi_{\perp} [H_B + FW - (\Lambda_1 - \Gamma E)]^{-1} \Pi_{\perp} (\psi^3 - \phi^3),$$

hence

$$\begin{aligned} \|\mathcal{F}_2(\psi_{\perp}) - \mathcal{F}_2(\phi_{\perp})\|_{H^1} &\leq \frac{|\eta|C}{h} (\|\psi\|_{L^4}^2 + \|\phi\|_{L^4}^2) \|\psi_{\perp} - \phi_{\perp}\|_{H^1} \\ &\leq C|\eta|h^{-1} (\|\psi_{\perp}\|_{L^4}^2 + \|\phi_{\perp}\|_{L^4}^2 + \|\psi_1\|_{L^4}^2) \|\psi_{\perp} - \phi_{\perp}\|_{H^1} \\ &\leq C|\eta|h^{-1} \left( \|\partial_x \psi_{\perp}\|_{L^2}^{1/2} \|\psi_{\perp}\|_{L^2}^{3/2} + \|\partial_x \phi_{\perp}\|_{L^2}^{1/2} \|\phi_{\perp}\|_{L^2}^{3/2} + h^{-1/4} \delta_0^2 \right) \|\psi_{\perp} - \phi_{\perp}\|_{H^1} \\ &\leq C|\eta|h^{-1} \left( \gamma^{1/2} + h^{-1/4} \delta_0^2 \right) \|\psi_{\perp} - \phi_{\perp}\|_{H^1} \\ &\leq C_3 \|\psi_{\perp} - \phi_{\perp}\|_{H^1} \end{aligned}$$

where

$$C_3 := C_3(h) = C|\eta|h^{-5/4} < 1$$

since

$$\eta = \mathcal{O}(h^2).$$

Then there exists a unique solution  $\hat{\psi}_{\perp} = \hat{\psi}_{\perp}(\mathbf{c}) \in K_{\gamma}$  to equation (45) for small  $h > 0$ . Moreover, by construction the solution  $\hat{\psi}_{\perp}$  is given by

$$\hat{\psi}_{\perp} = \sum_{j=1}^{\infty} (\psi_{\perp,j} - \psi_{\perp,j-1}) + \psi_{\perp,0}$$

where  $\psi_{\perp,j} = \mathcal{F}_2(\psi_{\perp,j-1})$  and  $\psi_{\perp,0} = 0$ . Hence

$$\begin{aligned} \|\psi_{\perp,j} - \psi_{\perp,j-1}\|_{H^1} &= \|\mathcal{F}_2(\psi_{\perp,j-1}) - \mathcal{F}_2(\psi_{\perp,j-2})\|_{H^1} \\ &\leq C_3 \|\psi_{\perp,j-1} - \psi_{\perp,j-2}\|_{H^1} \\ &\leq C_3^{j-1} \|\psi_{\perp,1} - \psi_{\perp,0}\|_{H^1} \leq C_2 C_3^{j-1} \end{aligned}$$

and thus

$$\|\hat{\psi}_{\perp}\|_{H^1} \leq \sum_{j=1}^{\infty} C_2 C_3^{j-1} = \frac{C_2}{1 - C_3} \leq Ch^{1/2}$$

for some positive constant  $C$ . This fact completes the proof.  $\square$

We must underline that  $\psi_{\perp}$  linearly depends on  $\mathbf{c}$  and thus the map  $\mathbf{c} \rightarrow \hat{\psi}_{\perp}(\mathbf{c})$  is a smooth map. In particular the following result holds true.

**Lemma 6.** *Let  $\mathbf{c} \in \ell_{\mathbb{R}}^1$  be such that  $\|\mathbf{c}\|_{\ell^1} \leq \delta_0$ , where  $\delta_0$  is any fixed and positive number. Then for any  $\mathbf{q}$  such that  $\|\mathbf{c} + \mathbf{q}\|_{\ell^1} \leq \delta_0$  then  $\hat{\psi}_{\perp}(\mathbf{c} + \mathbf{q})$  there exists and it is such that*

$$\|\hat{\psi}_{\perp}(\mathbf{c} + \mathbf{q}) - \hat{\psi}_{\perp}(\mathbf{c})\|_{H^1} \leq Ch \|\mathbf{q}\|_{\ell^1}.$$

*Proof.* Indeed, equation (45) becomes

$$\psi_{\perp} + \psi_q = \mathcal{F}_2(\psi_{\perp} + \psi_q) \quad (47)$$

where we set

$$\psi_{\perp} := \hat{\psi}_{\perp}(\mathbf{c}) \quad \text{and} \quad \psi_q := \hat{\psi}_{\perp}(\mathbf{c} + \mathbf{q}) - \hat{\psi}_{\perp}(\mathbf{c}).$$

A straightforward computation gives that

$$\mathcal{F}_2(\psi_{\perp} + \psi_q) = \mathcal{F}_2(\psi_{\perp}) + \mathcal{R}$$

where

$$\begin{aligned} \mathcal{R} &= \Pi_{\perp} [H_B + FW - (\Lambda_1 - \Gamma E)]^{-1} \Pi_{\perp} \times \\ &\quad \times \left\{ -\eta (3\psi_{\perp}^2 \psi_q + 3\psi_{\perp} \psi_q^2 + \psi_q^3) - FW \left( \sum_{n \in \mathbb{Z}} q_n u_n \right) \right\} \end{aligned}$$

and (47) reduces to

$$\psi_q = \mathcal{R}. \quad (48)$$

The same arguments used in the proof of Theorem 5 yields to the following estimate

$$\begin{aligned} \|\psi_q\|_{H^1} &= \|\mathcal{R}\|_{H^1} \\ &\leq C \left[ \frac{|\eta|}{h} (\|\psi_{\perp}\|_{L^{\infty}}^2 + \|\psi_{\perp}\|_{L^{\infty}} \|\psi_q\|_{L^{\infty}} + \|\psi_q\|_{L^{\infty}}^2) \|\psi_q\|_{L^2} + \frac{|F|}{h} \|\mathbf{q}\|_{\ell^1} \right] \\ &\leq Ch^2 \|\psi_q\|_{H^1} + Ch \|\mathbf{q}\|_{\ell^1} \end{aligned}$$

because  $\|\psi_{\perp}\|_{H^1}, \|\psi_q\|_{H^1} \leq Ch^{1/2}$  and (4). Then  $\|\psi_q\|_{H^1} \leq Ch \|\mathbf{q}\|_{\ell^1}$  immediately follows.  $\square$

**Remark 19.** *From Lemma 6 it follows that the linear map  $D_{\mathbf{c}}(\hat{\psi}_{\perp})$  satisfies the estimate*

$$\left\| D_{\mathbf{c}}(\hat{\psi}_{\perp}) \right\|_{\mathcal{L}(\ell^1 \rightarrow H^1)} \leq Ch.$$



## 8. EXISTENCE OF STATIONARY SOLUTIONS

**Theorem 6.** *Let  $\frac{\nu}{f} \notin \mathbb{N}$  and let  $h > 0$  small enough. Let  $\mathbf{d}^S$  be a finite-mode normalized solution associated to a solution-set  $S$  satisfying the assumption of Theorem 4. Then there exists a stationary solution  $\psi^S$  to equation (20) such that*

$$\left\| \psi^S - \sum_{n \in S} d_n^S u_n \right\|_{H^1} \leq Ch^{1/4}.$$

*Proof.* Let us omit, for the sake of simplicity, the upper letter  $S$ . We have to consider the first equation of (20) where  $\tilde{\lambda}$ ,  $f$  and  $\nu$  are defined by (23):

$$\tilde{\lambda}c_n = -\beta(c_{n+1} + c_{n-1}) + f\xi(n)c_n + \nu c_n^3 + r^n \quad (49)$$

where  $r_n$  is defined by (21) and where the map

$$(\mathbf{c}, \psi_\perp) \in \ell_{\mathbb{R}}^1 \times H^1 \rightarrow \mathbf{r} = \{r^n\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^1(\mathbb{Z})$$

is norm bounded by (see Lemma 2, Lemma 3, equation (24), Lemma 4 and Theorem 5)

$$\begin{aligned} \|\mathbf{r}\|_{\ell^1} &\leq \|\mathbf{r}_1\|_{\ell^1} + |F| [\|\mathbf{r}_2\|_{\ell^1} + \|\mathbf{r}_3\|_{\ell^1}] + |\eta| \|\mathbf{r}_4\|_{\ell^1} + |F| \|\mathbf{r}_5\|_{\ell^1} \\ &\leq C e^{-(S_0+\zeta)/h} \|\mathbf{c}\|_{\ell^1} + C_\rho e^{-(S_0-\rho)/h} \|\mathbf{c}\|_{\ell^1} + C|F|h^{1/2} + \\ &\quad + C_\rho \|\mathbf{c}\|_{\ell^1}^3 e^{-(S_0-\rho)/h} + C|\eta|h^{1/4} \|\mathbf{c}\|_{\ell^1}^2 + C_\rho e^{-(S_0-\rho)/h} \|\mathbf{c}\|_{\ell^1} \end{aligned}$$

for some  $\zeta > 0$  and for any  $\rho \in (0, S_0)$ , where  $C$  is a positive constant and  $C_\rho$  is a positive constant depending on  $\rho$ .

Now, let us consider the following mapping

$$(\mathbf{c}, \kappa) \in \ell_{\mathbb{R}}^1(\mathbb{Z}) \times \mathbb{R} \rightarrow \mathcal{G}(\mathbf{c}, \kappa) = \{\mathcal{G}_n(\mathbf{c}, \kappa)\}_{n \in \mathbb{Z}} \in \ell_{\mathbb{R}}^1(\mathbb{Z})$$

defined as

$$\mathcal{G}_n(\mathbf{c}, \kappa) = \tilde{\lambda}c_n + \beta(c_{n+1} + c_{n-1}) - f\xi(n)c_n - \nu c_n^3 - \kappa h^{-2} r^n \quad (50)$$

where  $\mathbf{r} := \mathbf{r}(\mathbf{c}) = \mathbf{r}(\mathbf{c}, \psi_\perp)$  and where  $\psi_\perp = \hat{\psi}_\perp(\mathbf{c})$  is the solution to the second equation of (20) for small  $h > 0$  given by Theorem 5.

By construction,  $\mathcal{G}_n(\mathbf{c}, 0) = 0$  coincides with the discrete nonlinear Schrödinger equation DNLSWE (22), while  $\mathcal{G}_n(\mathbf{c}, h^2) = 0$  coincides with equation (49).

**Lemma 7.**  *$\mathcal{G}$  is a  $C^1$  map in  $(\mathbf{c}, \kappa)$ . In particular:*

- i. *for any fixed  $\rho \in (0, S_0)$  there exists a positive constant  $C := C_\rho > 0$  such that: the map  $\mathbf{r}_1 : \ell_{\mathbb{R}}^1 \rightarrow \ell_{\mathbb{R}}^1$  satisfies*

$$\|(D_{\mathbf{c}}\mathbf{r}_1)(\mathbf{c})\|_{\mathcal{L}(\ell^1 \rightarrow \ell^1)} \leq C(1 + \|\mathbf{c}\|_{\ell^1}^2) e^{-(S_0-\rho)/h}. \quad (51)$$

- ii. *the maps  $\mathbf{r}_2 : \ell_{\mathbb{R}}^1 \rightarrow \ell_{\mathbb{R}}^1$  and  $\mathbf{r}_5 : \ell_{\mathbb{R}}^1 \rightarrow \ell_{\mathbb{R}}^1$  are linear maps such that*

$$\|(D_{\mathbf{c}}\mathbf{r}_j)(\mathbf{c})\|_{\mathcal{L}(\ell^1 \rightarrow \ell^1)} = \tilde{\mathcal{O}}\left(e^{-S_0/h}\right), \quad j = 2, 5. \quad (52)$$

- iii. *the map  $\mathbf{r}_3 : H_1 \rightarrow \ell_{\mathbb{R}}^1$  does not directly depend on  $\mathbf{c}$  and it is such that*

$$\|(D_{\mathbf{c}}\mathbf{r}_3)(\hat{\psi}_\perp(\mathbf{c}))\|_{\ell^1} \leq Ch \quad (53)$$

*for any  $\mathbf{c} \in \ell_{\mathbb{R}}^1$  such that  $\|\mathbf{c}\|_{\ell^1} \leq \delta_0$ .*

- iv. *the map  $\mathbf{r}_4 : \ell_{\mathbb{R}}^1 \times H^1 \rightarrow \ell_{\mathbb{R}}^1$  satisfies*

$$\|(D_{\mathbf{c}}\mathbf{r}_4)(\mathbf{c}, (\hat{\psi}_\perp(\mathbf{c})))\|_{\mathcal{L}(\ell^1 \rightarrow \ell^1)} \leq Ch^{1/4} \|\mathbf{c}\|_{\ell^1}. \quad (54)$$

*In conclusion*

$$\|D_{\mathbf{c}}\mathbf{r}(\mathbf{c})\|_{\mathcal{L}(\ell^1 \rightarrow \ell^1)} \leq C \left[ h^{9/4} \|\mathbf{c}\|_{\ell^1} + h^3 + (1 + \|\mathbf{c}\|_{\ell^1}^2) e^{-(S_0 - \rho)/h} \right].$$

*Proof.* Estimate (51) has been already proved (see estimate (37) by [13]). Concerning  $\mathbf{r}_2$  we recall that it is the linear map defined in Lemma 3, hence  $D_{\mathbf{c}}\mathbf{r}_2$  is independent of  $\mathbf{c}$  and such that (see Lemma 1.iv):  $\|D_{\mathbf{c}}\mathbf{r}_2\|_{\ell^1} = \tilde{\mathcal{O}}(e^{-S_0/h})$ . Similarly for  $\mathbf{r}_5$  as defined by (21). Concerning  $\mathbf{r}_3$  we recall that it is defined in Lemma 3 and it does not directly depend on  $\mathbf{c}$ , furthermore the estimate (52) on the  $\ell^1$ -norm comes from the fact that  $\mathbf{r}_3$  linearly depends of  $\psi_{\perp}$  and from Lemma 6. Concerning the term  $\mathbf{r}_4$  it is defined as  $r_4^n = \langle u_n, \psi^3 \rangle - c_n^3 C_1$ ; then immediately follows that the map  $\mathbf{c} \rightarrow \mathbf{r}_4(\mathbf{c}, \hat{\psi}_{\perp}(\mathbf{c}))$  is smooth. Furthermore, a straightforward calculation yields to the following expression

$$\begin{aligned} r_4^n &:= r_4^n(\mathbf{c}, \tilde{\psi}_1, \psi_{\perp}) = \langle u_n, \psi_{\perp}^3 \rangle + \langle u_n, \tilde{\psi}_1^3 \rangle + 3\langle u_n, \psi_{\perp}(c_n u_n + \tilde{\psi}_1)^2 \rangle + \\ &\quad + 3\langle u_n, (c_n u_n + \tilde{\psi}_1)\psi_{\perp}^2 \rangle + 3\langle u_n, c_n^2 u_n^2 \tilde{\psi}_1 \rangle + 3\langle u_n, c_n u_n \tilde{\psi}_1^2 \rangle \end{aligned}$$

where we set  $\tilde{\psi}_1 = \psi_1 - c_n u_n = \sum_{m \neq n} c_m u_m$ . Since  $u_n \tilde{\psi}_1 = \tilde{\mathcal{O}}(e^{-S_0/h})$  by Lemma 1.iv then the leading term in  $r_4^n$  is given by

$$r_4^n(\mathbf{c}, 0, \psi_{\perp}) = \langle u_n, \psi_{\perp}^3 \rangle + 3c_n^2 \langle u_n, \psi_{\perp} u_n^2 \rangle + 3c_n \langle u_n, u_n \psi_{\perp}^2 \rangle.$$

From this fact and because  $\|\psi_{\perp}\|_{H^1} \leq Ch^{1/2}$  (Theorem 5),  $\|D_{\mathbf{c}}\hat{\psi}_{\perp}\|_{\mathcal{L}(\ell^1 \rightarrow H^1)} \leq Ch$  (Remark 19),  $\|u_n\|_{L^{\infty}} \leq Ch^{-1/4}$  (Lemma 1.vi) and Lemma 1.v then it follows that the leading term in  $D_{\mathbf{c}}\mathbf{r}_4$  is estimated by

$$6\|\mathbf{c}\|_{\ell^1} \max_n \|u_n\|_{L^2}^2 \|\psi_{\perp} u_n\|_{L^{\infty}} \leq Ch^{1/4} \|\mathbf{c}\|_{\ell^1}.$$

By collecting all these facts and since (4) the the proof follows.  $\square$

Now, we fix  $\delta_0 \geq 1$ , then

$$\sup_{\|\mathbf{c}\|_{\ell^1} \leq \delta_0} \|\mathbf{r}\|_{\ell^1} \leq Ch^{9/4} \quad \text{and} \quad \sup_{\|\mathbf{c}\|_{\ell^1} \leq \delta_0} \|D_{\mathbf{c}}\mathbf{r}\|_{\ell^1} \leq Ch^{9/4}. \quad (55)$$

**Lemma 8.** *Let  $\mathbf{g}^S$  and  $\tilde{\lambda}$  be a solution to equation  $\mathcal{G}(\mathbf{g}^S, 0) = 0$ , as given by Theorem 4; the linear map  $D_{\mathbf{c}}\mathcal{G}(\mathbf{g}^S, 0)$  is one-to-one and onto.*

*Proof.* Again, let us omit the upper letter  $S$  when this does not cause misunderstanding. By construction, the linear map

$$D_{\mathbf{c}}\mathcal{G}(\mathbf{g}^S, 0) : \ell_{\mathbb{R}}^1 \rightarrow \ell_{\mathbb{R}}^1$$

is associated to a tridiagonal matrix defined as

$$\text{tridiag} \left( \beta, \tilde{\lambda} - f\xi(n) - 3\nu g_n^2, \beta \right)$$

Here, we make use of the result given in Appendix A by [2]; in particular, because  $\beta$  is exponentially small as  $h$  goes to zero we only have to check that

$$\left| \tilde{\lambda} - f\xi(n) - 3g_n^2 \nu \right| \geq Ch^{3/2} > 0 \quad (56)$$

uniformly holds true with respect to  $n$ , where  $\mathbf{g}^S$  is close to  $\mathbf{d}^S$  and  $\tilde{\lambda}$  is close to  $\mu^S$ . Indeed, the left hand side of (56) turns out to be close to  $|\tilde{\lambda} T_n|$ , where  $T_n$  is given by (42) and, by Lemma 5, it is such that  $|T_n| > \frac{1}{2}$  for any  $n$ ; furthermore  $\tilde{\lambda} \sim h^{3/2}$ . From this fact and from the argument given in Appendix A by [2] then the linear map  $D_{\mathbf{c}}\mathcal{G}(\mathbf{g}^S, 0)$  is one-to-one and onto.  $\square$

Now, we are ready to conclude the proof of Theorem 6. Let  $\mathbf{g}^S$  be the solution to (22) associated to the finite-mode solution  $\mathbf{d}^S$  satisfying the assumptions of Theorem 6. By the Implicit Function Theorem, there exist an  $h$ -independent  $\delta > 0$  such that if  $|\kappa| \leq \delta$  then there exists a unique solution  $\mathbf{c}(\kappa)$  in a  $\ell^1$ -neighborhood of  $\mathbf{g}^S$  satisfying  $\mathcal{G}(\mathbf{c}, \kappa) = 0$ . Then we can conclude that there exists  $h^* > 0$  such that for any  $h < h^*$  there is a unique solution  $\mathbf{c}^S \in \ell^1_{\mathbb{R}}$  to  $\mathcal{G}(\mathbf{c}, h^2) = 0$ , and it is such that

$$\|\mathbf{c}^S - \mathbf{g}^S\|_{\ell^1} = \mathcal{O}(h^{3/4}) \quad (57)$$

since  $|T_n| > Ch^{3/2}$  and (55). Then the Theorem follows where  $\psi^S = \psi_1^S + \psi_{\perp}^S$ ,  $\psi_1^S = \sum_{n \in \mathbb{Z}} c_n^S u_n$  and  $\psi_{\perp}^S = \hat{\psi}_{\perp}(\mathbf{c}^S)$ , furthermore

$$\begin{aligned} \left\| \psi^S - \sum_n d_n^S u_n \right\|_{H^1} &\leq \|\psi_{\perp}^S\|_{H^1} + \left\| \psi_1^S - \sum_n d_n^S u_n \right\|_{H^1} \\ &\leq \|\psi_{\perp}^S\|_{H^1} + \left\| \sum_n (c_n^S - d_n^S) u_n \right\|_{H^1} \\ &\leq \|\psi_{\perp}^S\|_{H^1} + [\|\mathbf{c}^S - \mathbf{g}^S\|_{\ell^1} + \|\mathbf{g}^S - \mathbf{d}^S\|_{\ell^1}] \|u_0\|_{H^1} \\ &\leq Ch^{1/2} + C \left[ h^{3/4} + \tilde{\mathcal{O}}(e^{-S_0/h}) \right] h^{-1/2} \leq Ch^{1/4} \end{aligned}$$

because of Theorems 4, 5, equation (57) and Lemma 1.vi. Theorem 6 is so proved.  $\square$

#### APPENDIX A. PROOF OF LEMMA 4

Let us recall that  $\psi = \psi_1 + \psi_{\perp}$  satisfies to the following estimates (see (28) and (29) in [13])

$$\|\nabla \psi_1\|_{L^2} \leq Ch^{-1/2} \|\mathbf{c}\|_{\ell^1}$$

and

$$\|\psi_1\|_{L^\infty} \leq Ch^{-1/4} \|\mathbf{c}\|_{\ell^1}.$$

Moreover, from Lemma 1, the following inequalities hold true:

$$\|u_m u_n\|_{L^1} \leq C e^{-(S_0 - \rho')|m-n| - \rho''/h}, \quad m \neq n$$

$$\left\| \sum_n |u_n| \right\|_{L^\infty} \leq Ch^{-1/2}$$

$$\|u_n\|_{L^p} \leq Ch^{-(p-2)/4p} \quad \forall p \in [2, \infty] \quad \text{and} \quad \|\nabla u_n\|_{L^2} \leq Ch^{-1/2}.$$

Furthermore, we recall also the following Sobolev inequality (see Theorem 8.8 in Brezis)

$$\|u\|_{L^\infty(\mathbb{R})} \leq C \|u\|_{H^1(\mathbb{R})}, \quad \forall u \in H^1(\mathbb{R}).$$

Now, let

$$r_4^n = \langle u_n, \psi^3 \rangle - C_1 c_n^3 = f_1^n + f_2^n$$

where we set

$$f_1^n = \langle u_n, \psi_1^3 \rangle - C_1 c_n^3$$

and

$$f_2^n = \langle u_n, \psi^3 - \psi_1^3 \rangle.$$

By the proof of Lemma 3 in [13] we have that for any  $\rho \in (0, S_0)$  there exists  $C := C_\rho > 0$  such that the vector  $\mathbf{f}_1 = \{f_1^n\}_{n \in \mathbb{Z}}$  can be estimated as follows

$$\|\mathbf{f}_1\|_{\ell^1} = \sum_n |f_1^n| \leq C \|\mathbf{c}\|_{\ell^1}^3 e^{-(S_0 - \rho)/h}.$$

For what concerns the term  $\mathbf{f}_2 = \{f_2^n\}_{n \in \mathbb{Z}}$  we observe that

$$f_2^n = \langle u_n, (\psi_1 + \psi_\perp)^3 - \psi_1^3 \rangle = \langle u_n, \psi_\perp^3 \rangle + 3\langle u_n, \psi_\perp^2 \psi_1 \rangle + 3\langle u_n, \psi_\perp \psi_1^2 \rangle$$

where

$$\begin{aligned} \sum_n |\langle u_n, \psi_\perp^3 \rangle| &\leq \left\langle \sum_n |u_n|, |\psi_\perp|^3 \right\rangle \\ &\leq \left\| \sum_n |u_n| \right\|_{L^\infty} \|\psi_\perp\|_{L^2}^2 \|\psi_\perp\|_{L^\infty} \leq Ch^{-1/2} \|\psi_\perp\|_{H^1}^3 \end{aligned}$$

$$\begin{aligned} \sum_n |\langle u_n, \psi_\perp^2 \psi_1 \rangle| &\leq \sum_{n,m} |c_m| |\langle u_n, u_m \psi_\perp^2 \rangle| \\ &\leq \sum_{n,m} |c_m| \|u_n u_m\|_{L^1} \|\psi_\perp\|_{L^\infty}^2 \\ &\leq \sum_{n,m} |c_m| \|u_n u_m\|_{L^1} \|\psi_\perp\|_{H^1}^2 \\ &\leq \sum_m |c_m| \left[ \sum_n \|u_n u_m\|_{L^1} \right] \|\psi_\perp\|_{H^1}^2 \\ &\leq C \|\mathbf{c}\|_{\ell^1} \|\psi_\perp\|_{H^1}^2 \end{aligned}$$

$$\begin{aligned} \sum_n |\langle u_n, \psi_\perp \psi_1^2 \rangle| &\leq \sum_{n,m,\ell} |c_m| |c_\ell| |\langle u_n u_m u_\ell, \psi_\perp \rangle| \\ &\leq \sum_n |c_n|^2 \langle |u_n|^3, |\psi_\perp| \rangle + 2 \sum_n |c_n| \sum_{\ell \neq n} |c_\ell| \langle |u_\ell|^2 |u_n|, |\psi_\perp| \rangle + \\ &\quad + \sum_n \sum_{m,\ell \neq n} |c_m| |c_\ell| \langle |u_m| |u_n| |u_\ell|, |\psi_\perp| \rangle \\ &\leq \max_n [\|u_n\|_{L^2}^2 \|u_n\|_{L^\infty}] \sum_n |c_n|^2 \|\psi_\perp\|_{L^\infty} + \\ &\quad + 2 \sum_n |c_n| \sum_{\ell \neq n} |c_\ell| \|u_n u_\ell\|_{L^1} \|u_\ell\|_{L^\infty} \|\psi_\perp\|_{L^\infty} + \\ &\quad + \sum_n \sum_{m,\ell \neq n} |c_m| |c_\ell| \|u_n u_m\|_{L^1} \|u_\ell\|_{L^\infty} \|\psi_\perp\|_{L^\infty} \\ &\leq \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} \|\psi_\perp\|_{H^1} + 2 \|\mathbf{c}\|_{\ell^1}^2 e^{-(S_0 - \rho)/h} h^{-1/4} \|\psi_\perp\|_{H^1} + \\ &\quad + \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} e^{-(S_0 - \rho)/h} \|\psi_\perp\|_{H^1} \\ &\leq C \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} \|\psi_\perp\|_{H^1} \end{aligned}$$

Therefore

$$\|\mathbf{f}_2\|_{\ell^1} \leq C \left[ h^{-1/2} \|\psi_\perp\|_{H_1}^3 + \|\mathbf{c}\|_{\ell^1} \|\psi_\perp\|_{H_1}^2 + \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} \|\psi_\perp\|_{H^1} \right].$$

Hence,

$$\begin{aligned} \|\mathbf{r}_4\|_{\ell^1} &\leq \|\mathbf{f}_1\|_{\ell^1} + \|\mathbf{f}_2\|_{\ell^1} \\ &\leq C \left[ h^{-1/2} \|\psi_\perp\|_{H_1}^3 + \|\mathbf{c}\|_{\ell^1} \|\psi_\perp\|_{H_1}^2 + \|\mathbf{c}\|_{\ell^1}^2 h^{-1/4} \|\psi_\perp\|_{H^1} + \|\mathbf{c}\|_{\ell^1}^3 e^{-(S_0-\rho)/h} \right]. \end{aligned}$$

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