Explicit formulation for the Rayleigh wave field induced by surface stresses in an orthorhombic half-plane

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Abstract

We develop an explicit asymptotic model for the Rayleigh wave field arising in case of stresses prescribed on the surface of an orthorhombic elastic half-plane. The model consists of an elliptic equation governing the behaviour within the half-plane, with boundary values given on the half-plane surface by a wave equation. Consequently, propagation along the surface is entirely accounted for by the hyperbolic equation, which, besides, may be immediately recast in terms of the associated surface displacement. The model readily solves otherwise involved dynamic problems for prescribed surface stresses, and its effectiveness is demonstrated for the classical Lamb's problem, as well as for the steady-state moving load problem. The latter example shows that the proposed model is really obtained by perturbation around the steady-state solution for a load moving at the Rayleigh speed.

Keywords: Asymptotic model, Rayleigh wave, Moving load

1. Introduction

Surface waves in anisotropic elastic media have been well investigated on the grounds of their wide range of application in defect detection, waveguide scattering and earthquake analysis. A large degree of interest has been raised by surface waves propagating in crystals, starting from the work of R. Stoneley [29], which generalizes the classical result of Lord Rayleigh concerning isotropic

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media [31] to solid-solid interfaces. Propagation of harmonic surface waves in orthorhombic media was investigated, among others, by Sveklo [32], Chadwick [4] and Destrade [6]. Existence and uniqueness of surface waves in anisotropic media has been provided by Barnett & Lothe [1] on the basis of the Stroh formalism [30].

Most of the literature concerning surface waves really deals with harmonic surface waves of sinusoidal profile. A more general approach to surface waves of arbitrary shape originates from the works of Sobolev [28], Friedlander [10], and Chadwick [3], which consider eigen-solutions in the form of arbitrary plane harmonic functions. Such an approach has been recently extended to 3D as well as to interfacial waves [19, 23]. The effect of anisotropy has also been incorporated, see [24, 26].

Another viewpoint in modelling near-surface dynamics is given in [17], where a hyperbolic-elliptic formulation for the Rayleigh wave induced stresses field is provided. Focus is set on singling out the contribution of Rayleigh waves to the overall dynamics and consequently neglect bulk waves. This approach relies on a slow time perturbation of the eigen-solution obtained in [3]. An important feature of the described formulation takes the shape of a hyperbolic equation for either elastic potential, that is valid over the surface and governs surface wave propagation. The same approach has been extended, for linear isotropic elasticity, to 3D wave propagation, thin coating layers and to mixed boundary conditions [5], [7] and [8]. Besides, interesting results concerning the near-resonant regime for loads moving on an elastic half-space have been reported in [9, 12, 15]. In parallel, parabolic-elliptic models were constructed for bending edge Rayleigh-type waves [16]. Some more results on the asymptotic theories for Rayleigh and Rayleigh-type waves may be found in [13] and [14].

In this paper, we extend the results of [17] to orthorhombic elastic materials. This generalisation is non-trivial due to the fact that anisotropy prevents the Helmholtz decomposition of the displacement field into a longitudinal and a transverse potential. A slow time perturbation procedure is carried out which, at leading order, sustains the representation of the wave field in terms of a sin-

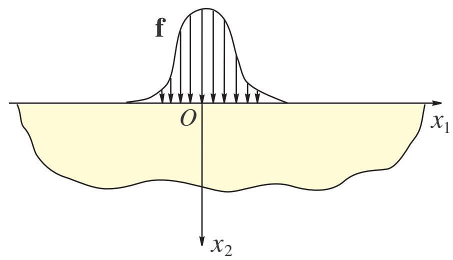

Figure 1: A line load $\mathbf{f}(x_1, t)$ acting on an orthorhombic half-plane

gle plane harmonic function, see [26]. Consideration of the first correction to the leading order solution brings in the hyperbolic equation governing surface displacements. The complete formulation includes an elliptic equation for the auxiliary plane harmonic function, complemented by the boundary condition on the surface in the form of the aforementioned hyperbolic equation. In order to illustrate the efficacy of the asymptotic solution we consider two model examples, namely Lamb's normal load problem and the steady-state moving point load. In the case of Lamb's problem, the result of the asymptotic formulation corresponds to the Rayleigh pole contribution to the solution obtained by means of integral transforms. In the case of a steady-state moving load, it is shown that the exact solution for the subsonic regime is easily obtained in terms of harmonic functions, a fact observed in [2] for isotropic elasticity. As expected, the near-resonant behaviour coincides with that given by the derived hyperbolic-elliptic model.

2. Governing equations

Consider an orthorhombic linear elastic half-plane occupying the region $\mathcal{E}_2^+ = \{(x_1, x_2) : x_2 \geq 0\}$ of a two-dimensional Euclidean space. In the context of wave propagation, the plane (x_1, x_2) is often named *the sagittal plane*. The half-plane is acted upon, on its boundary $x_2 = 0$, by a line load distribution $\mathbf{f}(x_1, t)$ (Fig.1). Let $\mathbf{u} = (u_1, u_2)$ be the displacement vector of a material particle with co-ordinates $\mathbf{x} = (x_1, x_2)$ in the half-plane. In light of the half-plane being orthorhombic, its constitutive equations are taken in planar orthotropic form, see e.g. [11, §2.5],

$$\sigma_{11} = c_{11}\partial_1 u_1 + c_{12}\partial_2 u_2,$$
$$\sigma_{22} = c_{12}\partial_1 u_1 + c_{22}\partial_2 u_2,$$
$$\sigma_{12} = c_{66}\left(\partial_2 u_1 + \partial_1 u_2\right),$$

where ∂_j stands for partial differentiation along x_j and c_{ij} are material constants such that

$$c_{11} > 0, \quad c_{66} > 0, \quad c_{11}c_{22} - c_{12}^2 > 0,$$

ensuring that the strain-energy density is *positive definite*. We let the shorthand notation

$$\beta = c_{12} + c_{66}.$$

The constitutive parameters c_{11}, c_{22}, c_{12} and c_{66} are often expressed in terms of the technical material constants $E_1, E_2, \nu_{12}, \nu_{21}, G_{12}$, there holding the reciprocal relation

$$\nu_{12}E_2 = \nu_{21}E_1.$$

For instance, for unidirectionally reinforced glass-epoxy composite material [11, Table 2-3b]

$$E_1 = 54 \text{ GPa}, \qquad E_2 = 18 \text{ GPa}, \qquad G_{12} = c_{66} = 9 \text{ GPa},$$
$$\nu_{12} = 0.25, \qquad \rho = 1900 \text{ kg/m}^3,$$

whence

$$c_{11} = 55.15 \text{ GPa}, \quad c_{22} = 18.38 \text{ GPa}, \quad c_{12} = 4.60 \text{ GPa}. \tag{1}$$

Following [27], three wave speeds are defined

$$c_1 = \sqrt{\frac{c_{11}}{\rho}}, \quad c_2 = \sqrt{\frac{c_{22}}{\rho}} \quad \text{and} \quad c_6 = \sqrt{\frac{c_{66}}{\rho}},$$

together with the combination

$$c_c = \sqrt{\frac{c_{11}c_{22} - c_{12}^2}{\rho c_{22}}} < c_1. \tag{2}$$

Motion in the half-plane is governed by the following pair of PDEs

$$c_{11}\partial_{11}^2 u_1 + c_{66}\partial_{22}^2 u_1 + \beta\partial_{12}^2 u_2 = \rho\partial_{tt}^2 u_1, \tag{3a}$$

$$\beta\partial_{12}^2 u_1 + c_{66}\partial_{11}^2 u_2 + c_{22}\partial_{22}^2 u_2 = \rho\partial_{tt}^2 u_2. \tag{3b}$$

The boundary conditions (BCs) on the surface $x_2 = 0$ are in the form of imposed traction

$$\sigma_{12} = f_1(x_1, t), \quad \sigma_{22} = f_2(x_1, t), \tag{4}$$

where we have let $\mathbf{f} = -(f_1, f_2)$ to abide with tradition (force positive downwards). Clearly, displacement is expected to decay depth-wise at infinity

$$(u_1, u_2) \to (0, 0), \quad \text{as } x_2 \to +\infty. \tag{5}$$

Let us focus on waves radiating from the load with phase speed c close to the Rayleigh wave speed c_R, i.e.

$$\epsilon = \left| \frac{c}{c_R} - 1 \right| \ll 1. \tag{6}$$

Examples of such occurrence include the near-resonance regime of a moving load or a far-field approximation, for more details see [14]. For the sake of definiteness, we consider waves propagating in the positive direction according to the moving co-ordinate $\xi = x_1 - c_R t$, $c_R > 0$, associated with the Rayleigh wavefront. Then, a slow time perturbation procedure is readily established, see e.g. [17] and [18], which allows for the following arbitrary profile travelling-wave solution

$$u_j = U_j(\xi, x_2, \tau), \quad j = 1, 2,$$

where $\tau = \epsilon t$ is the so-called slow time.

Plugging this solution into the governing equations (3), one gets the pair of linear PDEs

$$(c_{11} - \rho c_R^2)\partial_{\xi\xi}^2 U_1 + c_{66}\partial_{22}^2 U_1 + \beta\partial_{\xi 2}^2 U_2 = \rho\epsilon\partial_\tau\left(\epsilon\partial_\tau U_1 - 2c_R\partial_\xi U_1\right), \tag{7a}$$

$$\beta\partial_{\xi 2}^2 U_1 + (c_{66} - \rho c_R^2)\partial_{\xi\xi}^2 U_2 + c_{22}\partial_{22}^2 U_2 = \rho\epsilon\partial_\tau\left(\epsilon\partial_\tau U_2 - 2c_R\partial_\xi U_2\right). \tag{7b}$$

As well-known, this system of two second order PDEs may be recast in terms of a single fourth order PDE, for instance in terms of U_1,

$$a_0\partial^4_{\xi\xi\xi\xi}U_1 + a_1\partial^4_{\xi\xi 22}U_1 + a_2\partial^4_{2222}U_1 + \epsilon\partial^2_{\tau\xi}\left(b_1\partial^2_{\xi\xi}U_1 + d_1\partial^2_{22}U_1\right) - \epsilon^2\partial^2_{\tau\tau}\left(b_2\partial^2_{\xi\xi}U_1 + d_2\partial^2_{22}U_1\right) - \epsilon^3 b_3\partial^4_{\tau\tau\tau\xi}U_1 + \epsilon^4 b_4\partial^4_{\tau\tau\tau\tau}U_1 = 0, \quad (8)$$

where the coefficients are

$$\begin{aligned}
a_0 &= \left(c_{11} - \rho c_R^2\right)\left(c_{66} - \rho c_R^2\right), \\
a_1 &= c_{11}c_{22} + c_{66}^2 - \beta^2 - (c_{22} + c_{66})\,\rho c_R^2, \\
a_2 &= c_{22}c_{66}, \\
b_1 &= 2\rho c_R\left[c_{11} + c_{66} - 2\rho c_R^2\right], \\
d_1 &= 2\rho c_R(c_{22} + c_{66}), \\
b_2 &= \rho\left[c_{11} + c_{66} - 6\rho c_R^2\right], \\
d_2 &= \rho(c_{22} + c_{66}) \\
b_3 &= 4\rho^2 c_R, \\
b_4 &= \rho^2.
\end{aligned}$$

Once U_1 is determined, $\partial^2_{\xi 2}U_2$ may be readily found from Eq.(7a) and possibly integrated. Similarly, boundary conditions (4) give, at $x_2 = 0$,

$$c_{66}\left(\partial_\xi U_2 + \partial_2 U_1\right) = f_1, \quad (9a)$$

$$c_{12}\partial_\xi U_1 + c_{22}\partial_2 U_2 = f_2. \quad (9b)$$

3. Two-term solution

We begin by expanding displacements as asymptotic series

$$U_j(\xi, x_2, \tau) = \epsilon^{-1}U_j^{(0)}(\xi, x_2, \tau) + U_j^{(1)}(\xi, x_2, \tau) + \epsilon U_j^{(2)}(\xi, x_2, \tau) + \ldots,$$

where $j = 1, 2$. It should be emphasized that the presence of the leading terms of order ϵ^{-1} is due to the near-resonant nature of the excitation, given that a homogeneous problem is expected at leading order [14].

3.1. Leading order analysis

At leading order, Eq.(8) reduces to the homogeneous fourth-order pseudo-static PDE already discussed in [26, Eq.(8)], namely

$$\left(a_0\partial^4_{\xi\xi\xi\xi} + a_1\partial^4_{\xi\xi 22} + a_2\partial^4_{2222}\right) U_1^{(0)} = 0, \tag{10}$$

while Eq.(7a) gives

$$\beta\partial^2_{\xi 2}U_2^{(0)} = \left(\rho c_R^2 - c_{11}\right)\partial^2_{\xi\xi}U_1^{(0)} - c_{66}\partial^2_{22}U_1^{(0)}. \tag{11}$$

Eq.(10) may be formally rewritten in operator form as the product of two second-order bi-dimensional Laplace operators[1], suitably stretched along the x_2 coordinate,

$$a_2\Delta_1\Delta_2 U_1^{(0)} = 0, \tag{12}$$

being

$$\Delta_j = \left(\partial^2_{22} + \lambda_j^2\partial^2_{\xi\xi}\right), \quad j = 1, 2,$$

and where the dimensionless coefficients λ_1 and λ_2 are obtained from solving

$$\lambda_1^2 + \lambda_2^2 = \frac{a_1}{a_2} \quad \text{and} \quad \lambda_1^2\lambda_2^2 = \frac{a_0}{a_2}. \tag{13}$$

In the general case, λ_1 and λ_2 are solution of the secular equation for Rayleigh wave propagating in orthorhombic crystals (see [27, §5.3.3.1] and observe that λ_j here is $\chi_j = \imath q_j$ there). In this paper, we restrict attention to the case when λ_1 and λ_2 are positive real numbers, which requires $(a_0, a_1) > 0$, with this implying

$$c_R < \min\left(c_6, c_h\right), \quad c_h = \sqrt{\frac{c_c^2 - 2c_6^2\frac{c_{12}}{c_{22}}}{1 + \frac{c_{66}}{c_{22}}}} < c_c. \tag{14}$$

The solution of Eq.(12) may be obtained superposing a pair of plane harmonic functions (with physical dimension of length), stretched along x_2 according to the factors λ_1 and λ_2 [26, Eq.(12)] and decaying along x_2 to comply with conditions (5),

$$U_1^{(0)}(\xi, x_2, \tau) = \phi_1^{(0)}(\xi, \lambda_1 x_2, \tau) + \phi_2^{(0)}(\xi, \lambda_2 x_2, \tau). \tag{15}$$

[1] This factorizing is most often met in dealing with the bi-harmonic operator of plate theory, cf[21].

This solution may be inserted into Eq.(11) to give

$$U_2^{(0)}(\xi, x_2, \tau) = \alpha_1 \phi_1^{(0)*}(\xi, \lambda_1 x_2, \tau) + \alpha_2 \phi_2^{(0)*}(\xi, \lambda_2 x_2, \tau), \tag{16}$$

having let the dimensionless coefficients

$$\alpha_j = \frac{c_{66}\lambda_j^2 + \rho c_R^2 - c_{11}}{\beta \lambda_j} < 0. \tag{17}$$

Here, calling upon the harmonic character of $\phi_j^{(0)}(\xi, \lambda_j x_2, \tau)$, use has been made of the Cauchy-Riemann identities according to which we have (no summation over $j = 1, 2$ is implied)

$$\partial_\xi \phi_j^{(0)} = \lambda_j^{-1} \partial_2 \phi_j^{(0)*}, \quad \partial_2 \phi_j^{(0)} = -\lambda_j \partial_\xi \phi_j^{(0)*}, \tag{18}$$

where a superscript asterisk denotes the harmonic conjugated function. We recall that the operation of harmonic conjugation is an involution, in the sense that

$$\phi_j^{(0)**} = -\phi_j^{(0)}.$$

Plugging the solutions (15) and (16) into the BCs (9) gives, at $x_2 = 0$,

$$(\alpha_1 - \lambda_1)\partial_\xi \phi_1^{(0)^*} + (\alpha_2 - \lambda_2)\partial_\xi \phi_2^{(0)^*} = 0, \tag{19a}$$

$$(c_{12} + c_{22}\alpha_1\lambda_1)\partial_\xi \phi_1^{(0)} + (c_{12} + c_{22}\alpha_2\lambda_2)\partial_\xi \phi_2^{(0)} = 0. \tag{19b}$$

Taking harmonic conjugation of Eq.(19a), a homogeneous algebraic linear system in the unknowns $\partial_\xi \phi_1^{(0)}(\xi, 0, \tau)$ and $\partial_\xi \phi_2^{(0)}(\xi, 0, \tau)$ is found, which admits non-trivial solutions provided that c_R satisfies the Rayleigh wave speed equation, namely

$$\det \begin{bmatrix} \alpha_1 - \lambda_1 & \alpha_2 - \lambda_2 \\ c_{12} + c_{22}\alpha_1\lambda_1 & c_{12} + c_{22}\alpha_2\lambda_2 \end{bmatrix} = 0.$$

Expanding the determinant and making use of Eqs.(13), we get the velocity equation

$$R(c^2) = 0, \tag{20}$$

where $R(c^2)$ is the Rayleigh function

$$R(c^2) = \left[c_{22}(c_{11} - \rho c^2) - c_{12}^2\right] \sqrt{\frac{(c_{11} - \rho c^2)(c_{66} - \rho c^2)}{c_{22}c_{66}}} - \rho c^2 \left(c_{11} - \rho c^2\right), \tag{21}$$

which possesses a unique real solution, $c = c_R < c_1$. Indeed, the square root in Eq.(20) may be eliminated by squaring (see also [27, §5.3.3.1])

$$c_6^2(c_1^2 - c^2)c^4 - c_2^2(c_6^2 - c^2)(c_c^2 - c^2)^2 = 0,$$

which shows that the root $c = c_R$ satisfies

$$c_R < \min(c_6, c_c)\,. \tag{22}$$

We observe that this property is already implied by the requirement that λ_1 and λ_2 be real, through the condition (14). There may be several roots of Eq.(20), but only one complies with the constraints (14) and (22). For the parameter set (1), we find

$$c_R = 2084.13 \text{ m/s}.$$

Furthermore, either of the boundary conditions (19), once integrated along ξ, yields a connection between $\phi_1^{(0)}$ and $\phi_2^{(0)}$ on the surface $x_2 = 0$ (cf[26, Eq.(18)])

$$\phi_2^{(0)}(\xi, 0, \tau) = -\vartheta\phi_1^{(0)}(\xi, 0, \tau), \tag{23}$$

where

$$\vartheta = \frac{\alpha_1 - \lambda_1}{\alpha_2 - \lambda_2} = \frac{c_{12} + c_{22}\alpha_1\lambda_1}{c_{12} + c_{22}\alpha_2\lambda_2}.$$

Consequently, since the harmonic functions $\phi_2^{(0)}(\xi, x_2, \tau)$ and $-\vartheta\phi_1^{(0)}(\xi, x_2, \tau)$ coincide on the boundary $x_2 = 0$, they coincide throughout and, therefore, the displacement field may be conveniently expressed in terms of a single plane harmonic function, say $\phi_1^{(0)}$, as

$$U_1^{(0)}(\xi, x_2, \tau) = \phi_1^{(0)}(\xi, \lambda_1 x_2, \tau) - \vartheta\phi_1^{(0)}(\xi, \lambda_2 x_2, \tau), \tag{24a}$$

$$U_2^{(0)}(\xi, x_2, \tau) = \alpha_1\phi_1^{(0)*}(\xi, \lambda_1 x_2, \tau) - \vartheta\alpha_2\phi_1^{(0)*}(\xi, \lambda_2 x_2, \tau). \tag{24b}$$

Thus, the leading order analysis brings out the eigen-solution for the Rayleigh wave field, already discussed in [26].

3.2. First order correction

Following [17], now consider the first order correction to the leading order solution. Proceeding to the next order, Eq.(8) gives

$$a_2 \triangle_1 \triangle_2 U_1^{(1)} = -2\rho c_R \left[(c_{11} + c_{66} - 2\rho c_R^2)\partial^2_{\xi\xi} + (c_{22} + c_{66})\partial^2_{22}\right] \partial^2_{\xi\tau} U_1^{(0)}, \tag{25}$$

while Eq.(7a) lends

$$\beta\partial^2_{\xi 2}U_2^{(1)} = \left(\rho c_R^2 - c_{11}\right)\partial^2_{\xi\xi}U_1^{(1)} - c_{66}\partial^2_{22}U_1^{(1)} - 2\rho c_R\partial^2_{\xi\tau}U_1^{(0)}. \qquad (26)$$

Clearly, $U_1^{(1)}$ may be found from Eq.(25) as the sum of the solution to the homogeneous equation

$$\triangle_1 \triangle_2 U^{(1)} = 0,$$

which, as for the leading order problem, is readily obtained through superposition of a pair of plane harmonic functions $\phi_1^{(1)}$ and $\phi_2^{(1)}$, with any particular solution $U_{1p}^{(1)}$ of the inhomogeneous equation. In order to find the latter, we exploit linearity and consider the functions $\phi_1^{(0)}$ and $\phi_2^{(0)}$ appearing in Eq.(15) one at a time. Thus, we let $U_{1p}^{(1)} = U_{1a}^{(1)} + U_{1b}^{(1)}$, where $U_{1a}^{(1)}$ is the particular solution of Eq.(25) where $U_1^{(0)}$ has been replaced by $\phi_1^{(0)}$ and, similarly, $U_{1b}^{(1)}$ is the particular solution of the same equation in which $U_1^{(0)}$ has been substituted with $\phi_2^{(0)}$. Thus, for the first contribution, making use of the Cauchy-Riemann identities (18), we get

$$a_2 \triangle_1 \triangle_2 U_{1a}^{(1)} = e_1\partial^4_{\xi\xi\xi\tau}\phi_1^{(0)}, \qquad (27)$$

wherein

$$e_1 = -2\rho c_R\left[c_{11} + c_{66} - (c_{22} + c_{66})\lambda_1^2 - 2\rho c_R^2\right].$$

The same linear decomposition goes with ${U_2}_p^{(1)} = {U_{2a}}^{(1)} + {U_{2b}}^{(1)}$ in Eq.(26) and, for the first contribution, we have

$$\beta\partial^2_{\xi 2}U_{2a}^{(1)} = \left(\rho c_R^2 - c_{11}\right)\partial^2_{\xi\xi}U_{1a}^{(1)} - c_{66}\partial^2_{22}U_{1a}^{(1)} - 2\rho c_R\partial^2_{\xi\tau}\phi_1^{(0)}.$$

The particular solution of Eq.(27) may be conveniently sought in the form[2]

$$U_{1a}^{(1)} = x_2\psi_1(\xi, \lambda_1 x_2, \tau)$$

where ψ_1 is harmonic in the first two arguments. Then, noting that

$$\triangle_1 U_{1a}^{(1)} = 2\partial_2\psi_1,$$

[2]Indeed, this comes from the general form of the solution for the bi-harmonic equation, see [33]. The alternative choice ${U_{1a}}^{(1)}(\xi, x_2, \tau) = \xi\psi_1(\xi, \lambda_1 x_2, \tau)$ is likewise possible, but it is less advantageous when it comes to writing the BCs at $x_2 = 0$.

and, using once more the Cauchy-Riemann identities, we have, from Eq.(27),

$$2a_2 \triangle_2 \partial_2 \psi_1 = -2a_2\lambda_1(\lambda_2^2 - \lambda_1^2)\partial^3_{\xi\xi\xi}\psi_1^* = e_1 \partial^4_{\xi\xi\xi\tau}\phi_1^{(0)},$$

whence

$$U_{1a}^{(1)} = \frac{e_1}{2a_2\lambda_1(\lambda_2^2 - \lambda_1^2)} x_2 \partial_\tau \phi_1^{(0)*}. \tag{28}$$

In likewise manner, the second particular solution associated with $\phi_2^{(0)}$ satisfies

$$a_2 \triangle_1 \triangle_2 U_{1b}^{(1)} = e_2 \partial^4_{\xi\xi\xi\tau}\phi_2^{(0)}, \tag{29}$$

where

$$e_2 = -2\rho c_R \left[c_{11} + c_{66} - (c_{22} + c_{66})\lambda_2^2 - 2\rho c_R^2\right],$$

and, similarly, Eq.(26) becomes

$$\beta \partial^2_{\xi 2} U_{2b}^{(1)} = \left(\rho c_R^2 - c_{11}\right) \partial^2_{\xi\xi} U_{1b}^{(1)} - c_{66}\partial^2_{22} U_{1b}^{(1)} - 2\rho c_R \partial^2_{\xi\tau}\phi_2^{(0)}.$$

Proceeding as before, we get for the solution of Eq.(29)

$$U_{1b}^{(1)} = \frac{e_2}{2a_2\lambda_2(\lambda_1^2 - \lambda_2^2)} x_2 \partial_\tau \phi_2^{(0)*}. \tag{30}$$

Hence, putting together the homogeneous and the particular solutions (28,30), the general solution is arrived at

$$U_1^{(1)}(\xi, x_2, \tau) = \phi_1^{(1)}(\xi, \lambda_1 x_2, \tau) + \phi_2^{(1)}(\xi, \lambda_2 x_2, \tau) + \frac{1}{2a_2(\lambda_2^2 - \lambda_1^2)} x_2 \left[\frac{e_1}{\lambda_1}\,\partial_\tau\phi_1^{(0)*} - \frac{e_2}{\lambda_2}\,\partial_\tau\phi_2^{(0)*}\right], \tag{31}$$

where $\phi_j^{(1)}(\xi, \lambda_j x_2)$, $j = 1,2$ are plane harmonic functions which decay as $x_2 \to +\infty$.

Plugging Eq.(31) into Eq.(26), employing the Cauchy-Riemann identities (18) and integrating along x_2 gives

$$\begin{aligned}\partial_2 U_2^{(1)} &= \alpha_1\lambda_1\partial_\xi\phi_1^{(1)} + \alpha_2\lambda_2\partial_\xi\phi_2^{(1)} \\ &- \beta^{-1}\left\{2\rho c_R + \frac{e_1}{c_{22}(\lambda_2^2 - \lambda_1^2)}\right\}\partial_\tau\phi_1^{(0)} - \beta^{-1}\left\{2\rho c_R + \frac{e_2}{c_{22}(\lambda_1^2 - \lambda_2^2)}\right\}\partial_\tau\phi_2^{(0)} \\ &+ \frac{x_2}{2a_2(\lambda_2^2 - \lambda_1^2)}\left[e_1\alpha_1\partial^2_{\xi\tau}\phi_1^{(0)*} - e_2\alpha_2\partial^2_{\xi\tau}\phi_2^{(0)*}\right],\end{aligned} \tag{32}$$

and also

$$\partial_\xi U_2^{(1)} = \alpha_1 \partial_\xi \phi_1^{(1)*} + \alpha_2 \partial_\xi \phi_2^{(1)*} - \frac{1}{\beta\lambda_1}\left\{2\rho c_R + \frac{e_1(2c_{66}\lambda_1 - \alpha_1\beta)}{2a_2\lambda_1(\lambda_2^2-\lambda_1^2)}\right\}\partial_\tau\phi_1^{(0)*}$$
$$- \frac{1}{\beta\lambda_2}\left\{2\rho c_R + \frac{e_2(2c_{66}\lambda_2 - \alpha_2\beta)}{2a_2\lambda_2(\lambda_1^2-\lambda_2^2)}\right\}\partial_\tau\phi_2^{(0)*}$$
$$+ \frac{x_2}{2a_2(\lambda_2^2-\lambda_1^2)}\left[\frac{e_2\alpha_2}{\lambda_2}\partial^2_{\xi\tau}\phi_2^{(0)} - \frac{e_1\alpha_1}{\lambda_1}\partial^2_{\xi\tau}\phi_1^{(0)}\right]. \quad (33)$$

At this order, the BCs (9) are inhomogeneous

$$c_{66}\left(\partial_2 U_1^{(1)} + \partial_\xi U_2^{(1)}\right) = f_1, \quad (34a)$$

$$c_{12}\partial_\xi U_1^{(1)} + c_{22}\partial_2 U_2^{(1)} = f_2, \quad (34b)$$

where the left hand sides (LHSs) are evaluated at $x_2 = 0$. Substituting into Eqs.(34) the expressions (31,32,33) gives

$$(\alpha_1-\lambda_1)\,\partial_\xi\phi_1^{(1)*} + (\alpha_2-\lambda_2)\,\partial_\xi\phi_2^{(1)*}$$
$$= \frac{f_1}{c_{66}} + (\alpha_1-\lambda_1)\left(Z_{11}\partial_\tau\phi_1^{(0)*}(\xi,0,\tau) + Z_{12}\partial_\tau\phi_2^{(0)*}(\xi,0,\tau)\right) \quad (35)$$

having let

$$(\alpha_1-\lambda_1)Z_{1j} = \frac{1}{\beta\lambda_j}\left(e_j\frac{\alpha_j\beta + (\beta-2c_{66})\lambda_j}{2a_2\lambda_j(\lambda_j^2-\lambda_k^2)} + 2\rho c_R\right), \quad j,k\in\{1,2\} \text{ and } k\neq j.$$

Similarly, Eq.(34b) becomes

$$(c_{12}+c_{22}\alpha_1\lambda_1)\,\partial_\xi\phi_1^{(1)} + (c_{12}+c_{22}\alpha_2\lambda_2)\,\partial_\xi\phi_2^{(1)}$$
$$= f_2 + (c_{12}+c_{22}\alpha_1\lambda_1)\left(Z_{21}\partial_\tau\phi_1^{(0)}(\xi,0,\tau) + Z_{22}\partial_\tau\phi_2^{(0)}(\xi,0,\tau)\right), \quad (36)$$

where

$$(c_{12}+c_{22}\alpha_1\lambda_1)\,Z_{2j} = \frac{1}{\beta}\left(2\rho c_R c_{22} - \frac{e_j}{\lambda_j^2-\lambda_k^2}\right), \quad j,k\in\{1,2\} \text{ and } k\neq j.$$

It is observed that the entries of the square matrix Z_{jk} have dimension of slowness, i.e. inverse of speed.

Let us consider the case of normal load first, i.e. $f_1 = 0$. Taking harmonic conjugation of Eq.(35) and considering Eq.(36) lends a singular linear system in the unknowns $\partial_\xi \phi_1^{(1)}$ and $\partial_\xi \phi_2^{(1)}$ which admits a unique solution provided that the following compatibility condition is satisfied by the RHS

$$(Z_{21} - Z_{11})\partial_\tau \phi_1^{(0)} + (Z_{22} - Z_{12})\partial_\tau \phi_2^{(0)} + \frac{1}{c_{12} + c_{22}\alpha_1\lambda_1} f_2 = 0, \quad \text{at } x_2 = 0.$$

This equation may be recast entirely in terms of a single harmonic function, say $\phi_1^{(0)}$, through the connection (23) and then differentiated with respect to ξ,

$$\partial^2_{\xi\tau}\phi_1^{(0)} = \frac{c_R}{2}\gamma_1 \partial_\xi f_2,$$

where

$$\gamma_1 = -\frac{2}{c_R}\frac{1}{(c_{12} + c_{22}\alpha_1\lambda_1)\left[Z_{21} - Z_{11} - \vartheta(Z_{22} - Z_{12})\right]}, \tag{37}$$

with dimension of elastic compliance, i.e. inverse of stress. Upon returning to the original variables by means of the approximate operator relation [17, Eq.(3.20)]

$$\partial^2_{\xi\tau} \approx \frac{c_R}{2\epsilon}\left(\partial^2_{11} - c_R^{-2}\partial^2_{tt}\right)$$

and introducing the auxiliary harmonic function $\phi_1(x_1, x_2, t) = \epsilon^{-1}\phi_1^{(0)}(\xi, \lambda_1 x_2, \tau)$, we have

$$\left(\partial^2_{11} - c_R^{-2}\partial^2_{tt}\right)\phi_1(x_1, 0, t) = \gamma_1 \partial_1 f_2(x_1, t). \tag{38}$$

Therefore, the case of normal loading reduces to the following hyperbolic-elliptic formulation for the auxiliary function ϕ_1 which

- satisfies the homogeneous elliptic equation

$$\left(\partial^2_{22} + \lambda_1^2\partial^2_{11}\right)\phi_1 = 0 \tag{39}$$

  in the domain $\mathcal{E}_2^+$,

- with boundary values on the surface $x_2 = 0$ given by the hyperbolic equation (38) and the depth-wise decay condition (5).

In particular, boundary values may be found independently from (38), i.e. without solving the problem in the domain.

A similar formulation may be derived for shear loading, through the compatibility condition

$$(Z_{21} - Z_{11})\partial_\tau \phi_1^{(0)*} + (Z_{22} - Z_{12})\partial_\tau \phi_2^{(0)*} - \frac{1}{c_{66}(\alpha_1 - \lambda_1)} f_1 = 0, \quad \text{at } x_2 = 0.$$

Consequently, the boundary condition for the elliptic equation (39) takes the form

$$\partial^2_{\xi\tau} \phi_1^{(0)*}(\xi, 0, t) = \frac{c_R}{2} \gamma_2 \partial_1 f_1(\xi, t),$$

or, equivalently, we get the following hyperbolic equation for the harmonic conjugate $\phi_1^*(x_1, x_2, t) = \epsilon^{-1} \phi_1^{(0)*}(\xi, \lambda_1 x_2, \tau)$

$$\left(\partial^2_{11} - c_R^{-2} \partial^2_{tt}\right) \phi_1^*(x_1, 0, t) = \gamma_2 \partial_1 f_1(x_1, t), \tag{40}$$

where we have let the compliance

$$\gamma_2 = \frac{2}{c_R} \frac{1}{c_{66}(\alpha_1 - \lambda_1)\left[Z_{21} - Z_{11} - \vartheta(Z_{22} - Z_{12})\right]}.$$

Finally, we note that in light of Eqs.(24), Eqs.(38) and (40) may be reinterpreted as hyperbolic equations for the surface displacement. Indeed, to leading order, we have

$$u_1(x_1, 0, t) = (1 - \vartheta)\phi_1(x_1, 0, t),$$
$$u_2(x_1, 0, t) = (\alpha_1 - \vartheta\alpha_2)\phi_1^*(x_1, 0, t),$$

whence Eq.(38) may be rewritten as

$$\left(\partial^2_{11} - c_R^{-2} \partial^2_{tt}\right) u_1(x_1, 0, t) = (1 - \vartheta)\gamma_1 \partial_1 f_2(x_1, t), \tag{41}$$

whereas, for shear loading, Eq.(40) becomes

$$\left(\partial^2_{11} - c_R^{-2} \partial^2_{tt}\right) u_2(x_1, 0, t) = (\alpha_1 - \vartheta\alpha_2)\gamma_2 \partial_1 f_1(x_1, t). \tag{42}$$

4. Isotropic case

In the case of an isotropic half-plane, we have

$$c_{11} = c_{22} = \lambda + 2G, \quad c_{66} = G, \quad c_{12} = \lambda, \quad \beta = \lambda + G,$$

where λ is Lamé's constant and G is the shear elastic modulus. Consequently, in a linearly isotropic solid, the elastic wave speeds collapse into either $c_d = \sqrt{(\lambda + 2G)/\rho}$ or $c_s = \sqrt{G/\rho}$, respectively the longitudinal and the transversal wave speed,

$$c_1 = c_2 = c_d > c_6 = c_s.$$

Then, the constants a_0 and a_1 allow for a simple interpretation

$$a_0 = \rho^2(c_d^2 - c_R^2)(c_s^2 - c_R^2), \quad a_1 = \rho^2(c_d^2 + c_s^2)\left(c_h^2 - c_R^2\right),$$

with $c_h^2 = 2/(c_d^{-2} + c_s^{-2})$ being the harmonic mean of the speeds squared. Besides, given that Eq.(2) reduces to

$$c_c = 2\frac{c_s}{c_d}\sqrt{c_d^2 - c_s^2}$$

and in light of $c_d^2 > 2c_s^2$, it is a straightforward matter to prove that $c_h < c_c$, as expected. Using the property of the harmonic mean, we conclude

$$c_s < c_h < c_c < c_d,$$

from which the condition (14) for having real λ reduces to

$$c_R < c_s,$$

that is indeed guaranteed by Eq.(22). Therefore, in the isotropic case, only purely real attenuation indices are possible.

Besides, we have (λ_1 and λ_2 may be freely swapped)

$$\lambda_{1,2}^2 = 1 - \frac{c_R^2}{c_{d,s}^2}, \quad \alpha_1 = -\lambda_1^{-1} \quad \alpha_2 = -\lambda_2,$$

and

$$\vartheta = (\lambda_1\lambda_2)^{-1}\left(1 - \frac{c_R^2}{2c_s^2}\right).$$

Eq.(41) gives, in the isotropic case,

$$\left(\partial_{11}^2 - c_R^{-2}\partial_{tt}^2\right) u_1(x_1, 0, t) = \frac{1 - k_2^4}{4GB}\partial_1 f_1(x_1, t),$$

being (we use the notation in [17])

$$k_1 = \sqrt{1 - \frac{c_R^2}{c_d^2}} = \lambda_2, \quad k_2 = \sqrt{1 - \frac{c_R^2}{c_s^2}} = \lambda_1,$$

and

$$B = \frac{k_2}{k_1}(1 - k_1^2) + \frac{k_1}{k_2}(1 - k_2^2) - 1 + k_2^4.$$

In the same fashion, Eq.(42), in the isotropic case, reduces to

$$\left(\partial_{11}^2 - c_R^{-2}\partial_{tt}^2\right) u_2(x_1, 0, t) = -\frac{1 - k_2^4}{4GB}\partial_1 f_1(x_1, t).$$

5. Illustrative examples

In this Section, we consider two classical problems of elastodynamics and demonstrate the efficacy of the developed asymptotic model. First, we consider Lamb's problem for a normal load and then we present the analysis of a steady-state moving load problem in the near-resonant regime.

5.1. Lamb's problem for a normal load

Lamb's problem considers the effect of a normal (or tangential) point load suddenly applied onto the surface of an elastic half-space [20]. To this aim, we let $f_1 \equiv 0$ and $f_2(x_1, t) = P_0\delta(x_1)\delta(t)$ in the boundary conditions (4) and begin by considering the conventional approach to the problem by means of integral transforms. On applying the Fourier-Laplace transform pair to Eqs.(3), we get

$$c_{66}\frac{\mathrm{d}^2 u_1^{FL}}{\mathrm{d}x_2^2} - \left(s^2 c_{11} + \rho p^2\right) u_1^{FL} + \imath s\beta\frac{\mathrm{d}u_2^{FL}}{\mathrm{d}x_2} = 0, \tag{43a}$$

$$\imath s\beta\frac{\mathrm{d}u_1^{FL}}{\mathrm{d}x_2} + c_{22}\frac{\mathrm{d}^2 u_2^{FL}}{\mathrm{d}x_2^2} - \left(s^2 c_{66} + \rho p^2\right) u_2^{FL} = 0, \tag{43b}$$

where $u_j^{FL}(x_2)$ are the Fourier-Laplace transforms of the displacements $u_j(x_1, x_2, t)$, $j = 1, 2$, along x_1 and t, i.e.

$$u_j^{FL}(s, x_2, p) = \int_0^{+\infty} \exp(-pt)\mathrm{d}t \int_{-\infty}^{+\infty} u_j(x_1, x_2, t)\exp(\imath s x_1)\mathrm{d}x_1.$$

Here, $\imath$ is the imaginary unit and s and p denote the Fourier and Laplace transform parameters, respectively. Eqs.(43) may be recast into the single fourth order ODE

$$a_2\frac{\mathrm{d}^4u_1^{FL}}{\mathrm{d}x_2^4}-s^2\left(c_{11}c_{22}-c_{12}^2-2c_{12}c_{66}+(c_{22}+c_{66})\rho\frac{p^2}{s^2}\right)\frac{\mathrm{d}^2u_1^{FL}}{\mathrm{d}x_2^2}$$
$$+s^4\left(c_{11}+\rho\frac{p^2}{s^2}\right)\left(c_{66}+\rho\frac{p^2}{s^2}\right)u_1^{FL}=0,$$

that clearly corresponds to the Fourier transform of the bi-harmonic operator (12) in which we substitute c_R^2 with $-p^2/s^2$. The decaying solution of this fourth order ODE is found in the form

$$u_1^{FL}(x_2)=C_1\exp\left(-q_1sx_2\right)+C_2\exp\left(-q_2sx_2\right), \tag{44}$$

where C_1 and C_2 are complex-valued functions of s and p. To warrant decay, the attenuation coefficients appearing in the exponentials need have positive real part, i.e. $\Re(sq_j)>0$, $j=1,2$. Besides, they are solution of the secular equation for Rayleigh wave propagating in orthorhombic crystals [27, §5.3.3.1] (cfEqs.(13))

$$q_1^2+q_2^2=\frac{c_{11}c_{22}-c_{12}^2-2c_{12}c_{66}+(c_{22}+c_{66})\rho\frac{p^2}{s^2}}{a_2},$$
$$q_1^2q_2^2=\frac{\left(c_{11}+\rho\frac{p^2}{s^2}\right)\left(c_{66}+\rho\frac{p^2}{s^2}\right)}{a_2}.$$

Hence, from Eq.(43a), the transformed vertical displacement reads

$$u_2^{FL}=C_1F(q_1)\exp\left(-q_1sx_2\right)+C_2F(q_2)\exp\left(-q_2sx_2\right), \tag{45}$$

wherein we let the dimensionless functions

$$F(q_j)=-\imath\,\frac{c_{66}q_j^2-\rho\frac{p^2}{s^2}-c_{11}}{\beta q_j},\qquad j=1,2.$$

The Fourier-Laplace transform of the boundary conditions (4) yields

$$\frac{\mathrm{d}u_1^{FL}}{\mathrm{d}x_2}(0)+\imath su_2^{FL}(0)=0,\qquad \imath sc_{12}u_1^{FL}(0)+c_{22}\frac{\mathrm{d}u_2^{FL}}{\mathrm{d}x_2}(0)=P_0,$$

which, upon substituting Eqs.(44) and (45), gives a linear algebraic system in the unknowns C_1 and C_2. After some algebraic manipulations, the transform of the horizontal displacement on the surface is found

$$u_1^{FL}(0)=C_1+C_2=-\imath\,P_0\frac{Q(-p^2/s^2)}{sR\left(-p^2/s^2\right)}, \tag{46}$$

where

$$Q\left(z\right)=c_{11}-\rho\,z-c_{12}\sqrt{\frac{\left(c_{11}-\rho\,z\right)\left(c_{66}-\rho\,z\right)}{a_2}},$$

and $R(c^2)$ has been defined in (21). It is clear that the function $R(c^2)$ possesses a single zero, associated with the Rayleigh wave speed, at $c_R^2=-p^2/s^2<c_1^2$, which is a pole for Eq.(46). The near-resonance regime is to be found right in the neighborhood of this pole, wherein the RHS of Eq.(46) may be well approximated by the leading order term of its Taylor expansion

$$R\left(-\frac{p^2}{s^2}\right)\approx-R'(c_R^2)\left(\frac{p^2}{s^2}+c_R^2\right)\quad\text{and}\quad Q\left(-\frac{p^2}{s^2}\right)\approx Q(c_R^2).$$

In this context, the speed c appearing in the asymptotic approach is a measure of the distance from this pole, i.e. $c^2=-p^2/s^2$. Whence, in the vicinity of the Rayleigh pole,

$$u_1^{FL}(0)=\imath\gamma_3\frac{sc_R^2P_0}{p^2+c_R^2s^2},$$

where the constant

$$\gamma_3=-\frac{2}{c_R^2}\frac{\chi_1^2\chi_6\left(\sqrt{a_2}\chi_1-c_{12}\chi_6\right)}{\rho c_{22}\chi_1^2(\chi_1^2+3\chi_6^2)-c_{12}^2(\chi_1^2+\chi_6^2)+2\sqrt{a_2}\chi_1\chi_6(2\chi_1^2-\rho c_{11})},$$

is expressed in terms of the density, stiffness components, and the Rayleigh wave speed, with the positive quantities

$$\chi_1=\sqrt{c_1^2-c_R^2},\quad\chi_6=\sqrt{c_6^2-c_R^2}.$$

It may be verified from (15) and (23) that

$$\gamma_3=(1-\vartheta)\gamma_1,$$

where γ_1 given by Eq.(37). Hence, on the surface $x_2=0$, for the displacement u_1^{FL}

$$\left[\partial_{11}^2u_1-c_R^{-2}\partial_{tt}^2u_1\right]^{FL}=-\imath\gamma_3sP_0=\left[\gamma_3P_0\delta'(x_1)\delta(t)\right]^{FL}$$

the inversion of which is straightforward

$$\left(\partial_{11}^2-c_R^{-2}\partial_{tt}^2\right)u_1=\gamma_3P_0\delta'(x_1)\delta(t).$$

This hyperbolic equation is in fact Eq.(41), with $f_2 = P_0\delta(x_1)\delta(t)$. Thus, not surprisingly, the contribution of the Rayleigh pole matches the derived asymptotic formulation.

We now derive the Rayleigh wave field for the Lamb problem through the asymptotic formulation. We begin by solving on the surface $x_2 = 0$ the hyperbolic equation (38) specified for Lamb's normal load, namely

$$\left(\partial_{11}^2 - \frac{1}{c_R^2}\,\partial_{tt}^2\right)\phi_1(x_1, 0, t) = \gamma_1 P_0 \delta'(x_1)\delta(t).$$

Employing the fundamental solution of the 1D wave operator, see e.g. [25], we deduce

$$\phi_1(x_1, 0, t) = \frac{c_R}{2}\gamma_1 P_0\left[\delta(x_1 - c_R t) - \delta(x_1 + c_R t)\right].$$

Then, using Poisson's formula for Laplace operator in the half-plane [33, §5.4.4], the harmonic function ϕ_1 is immediately retrieved

$$\begin{aligned}\phi_1(x_1, x_2, t) = \frac{x_2}{\pi}\int\limits_{-\infty}^{\infty}\frac{1}{(x_1 - r)^2 + x_2^2}\,\phi_1(r, 0, t)\mathrm{d}r \\ = \frac{\gamma_1 P_0 c_R}{2\pi}x_2\left[\frac{1}{(x_1 - c_R t)^2 + x_2^2} - \frac{1}{(x_1 + c_R t)^2 + x_2^2}\right].\end{aligned}$$

The full-field displacement is determined accordingly

$$u_{1,2}(x_1, x_2, t) = \frac{\gamma_1 P_0 c_R}{2\pi}(\hat{U}_{1,2}^- - \hat{U}_{1,2}^+),$$

where

$$\hat{U}_1^{\mp} = \lambda_1\frac{x_2}{(x_1 \mp c_R t)^2 + (\lambda_1 x_2)^2} - \vartheta\lambda_2\frac{x_2}{(x_1 \mp c_R t)^2 + (\lambda_2 x_2)^2}$$

and

$$\hat{U}_2^{\mp} = \alpha_1\frac{x_1 \mp c_R t}{(x_1 \mp c_R t)^2 + (\lambda_1 x_2)^2} - \vartheta\alpha_2\frac{x_1 \mp c_R t}{(x_1 \mp c_R t)^2 + (\lambda_2 x_2)^2}.$$

Figs.2 and 3 present the displacement profiles $U_{1,2}^-$ in terms of the dimensionless co-ordinate $\eta = \xi/x_2$.

5.2. Steady-state moving load problem

We now consider the action of a normal line load in steady motion along the surface $x_2 = 0$ with constant speed c, this time not necessarily close to

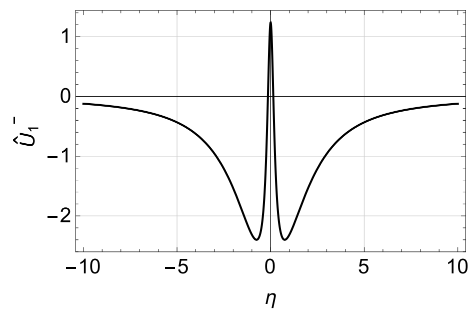

Figure 2: Displacement profile $\hat{U}_1^-$ for Lamb's normal load problem as obtained from the asymptotic analysis for carbon epoxy composite

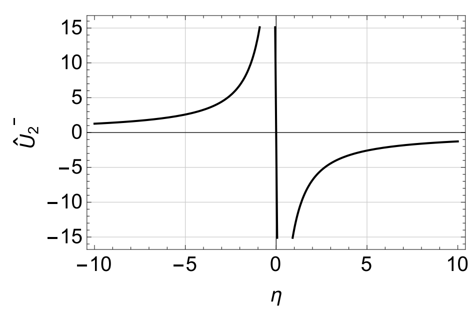

Figure 3: Displacement profile $\hat{U}_2^-$ for Lamb's normal load problem as obtained from the asymptotic analysis for carbon epoxy composite

the Rayleigh wave speed c_R. For this setup, we let for the loading $f_1 = 0$ and $f_2 = f(x_1 - ct)$. Following the original approach in [2], we first derive the problem general solution in the subsonic regime $c < c_R$ and then draw a comparison with the results of the asymptotic formulation letting the speed of the load approach the resonant surface wave speed (see also [14]). In going through the derivations, it will become apparent that the asymptotic model is basically obtained by perturbation of this steady-state moving load problem solution around Rayleigh's critical speed.

First of all, Eqs.(3) are re-written in terms of the moving coordinate $\xi = x_1 - ct$

$$(c_{11} - \rho c^2)\partial^2_{\xi\xi}u_1 + c_{66}\partial^2_{22}u_1 + \beta\partial^2_{\xi 2}u_2 = 0, \tag{47a}$$

$$\beta\partial^2_{\xi 2}u_1 + (c_{66} - \rho c^2)\partial^2_{\xi\xi}u_2 + c_{22}\partial^2_{22}u_2 = 0, \tag{47b}$$

subject to the boundary conditions

$$\partial_\xi u_2 + \partial_2 u_1 = 0, \tag{48a}$$

$$c_{12}\partial_\xi u_1 + c_{22}\partial_2 u_2 = f(\xi). \tag{48b}$$

According to [10] and [3], in the subsonic regime Eqs.(47) allow a solution in terms of a single plane harmonic function. Indeed, the system (47) may be rewritten as the product of two stretched Laplace operators for either unknown, say

$$\Delta_1\Delta_2 u_1 = 0.$$

The solution to this PDE is readily found by superposition of a pair of plane harmonic functions

$$u_1(\xi, x_2) = \phi_1(\xi, \lambda_1 x_2) + \phi_2(\xi, \lambda_2 x_2), \tag{49}$$

and plugging this back into Eq.(47a) lends

$$u_2(\xi, x_2) = \alpha_1\phi_1^*(\xi, \lambda_1 x_2) + \alpha_2\phi_2^*(\xi, \lambda_2 x_2), \tag{50}$$

wherein the coefficients α_j are defined as in Eq.(17) provided that c_R is replaced by the moving load speed c. Unlike the leading order analysis of Sec.3.1, substitution of this solution into the boundary conditions (48) yields a inhomogeneous linear system that is regular inasmuch as $c \neq c_R$. The solution to this system is readily obtained by Cramer's rule

$$\partial_\xi \phi_1 = Q_1(c^2) f, \tag{51}$$

where

$$Q_1(c^2) = \frac{\lambda_1 \left(c_{12}\lambda_2^2 + c_{11} - \rho c^2 \right)}{\left(\lambda_1 - \lambda_2 \right) R(c^2)}, \tag{52}$$

with $R(c^2)$ defined in Eq.(21). Clearly, the resonant nature of the Rayleigh wave is seen from (52).

The connection between the harmonic functions is given by

$$\phi_2(\xi, 0) = -\vartheta \phi_1(\xi, 0),$$

Integrating Eq.(51) provides a Dirichlet-type boundary condition for the elliptic equation (39), namely

$$\phi_1(\xi, 0) = Q_1(c^2) \int f(\xi) \mathrm{d}\xi. \tag{53}$$

Thus, making use of the harmonic form of the solution allows significant simplification, since the vector problem of elastodynamics is now reduced to the solution of a scalar elliptic equation.

For the sake of illustration, we consider the case of a point normal load travelling at constant speed c, i.e. $f(\xi) = P_0 \delta(\xi)$. Then, integration of Eq.(53) gives

$$\phi_1(\xi, 0) = Q_1(c^2) P_0 \left[H(\xi) - \frac{1}{2} \right],$$

where the arbitrary integration constant 1/2 (stemming in light of the steady-state nature of the problem) is chosen as to preserve symmetry and $H(\xi)$ is Heaviside's step function. Then, the harmonic function $\phi_1 = \phi_1(\xi, x_2)$ may be determined over the interior of the half-plane by means of Poisson's formula

$$\phi_1(\xi, x_2) = \frac{P_0}{\pi} Q_1(c^2) \arctan \frac{\xi}{x_2}.$$

The displacement field immediately follows from Eqs.(49)

$$u_1(\xi, x_2) = \frac{P_0}{\pi} Q_1(c^2) \hat{U}_1, \quad \hat{U}_1 = \arctan \frac{\xi}{\lambda_1 x_2} - \vartheta \arctan \frac{\xi}{\lambda_2 x_2},$$

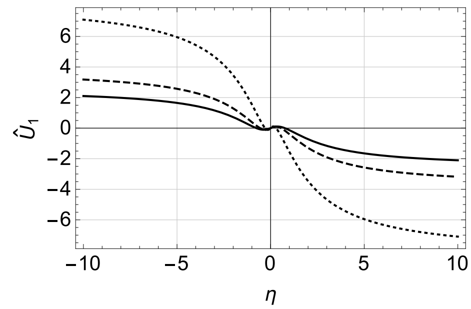

Figure 4: Displacement profile $\hat{U}_1$ for the steady state motion of a point load acting on carbon epoxy composite material and moving at speed $c = 1000$ (solid), 1500 (dashed) and 2000 (dotted) m/s

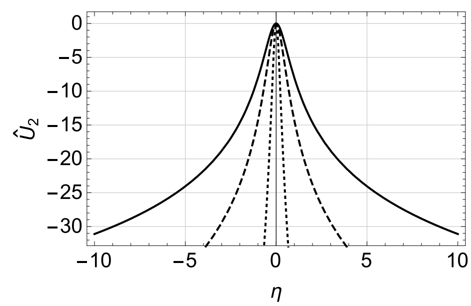

Figure 5: Displacement profile $\hat{U}_2$ for the steady state motion of a point load acting on carbon epoxy composite material and moving at speed $c = 1000$ (solid), 1500 (dashed) and 2000 (dotted) m/s

and from Eq.(50)

$$u_2(\xi, x_2) = \frac{P_0}{2\pi} Q_1(c^2)\hat{U}_2 \quad \hat{U}_2 = \alpha_1 \ln\left(\frac{\xi^2}{\lambda_1^2 x_2^2} + 1\right) - \vartheta\alpha_2 \ln\left(\frac{\xi^2}{\lambda_2^2 x_2^2} + 1\right).$$

Indeed, on account of the fact that $\ln z = \frac{1}{2}\ln|z|^2 + \imath \arg z$ is an analytic function of z (in the cut plane), its real and imaginary parts are harmonic conjugated functions and, letting $z = 1 + \imath\xi/x_2$, we have $|z|^2 = \xi^2/x_2^2 + 1$, $\arg z = \arctan \xi/x_2$, together with the required decay condition.

Figs.4 and 5 plot the displacement profiles $\hat{U}_{1,2}$ in terms of the dimensionless variable $\eta = \xi/x_2$ at different load speeds.

Let us now implement the asymptotic formulation in the case of near-resonant speed for the moving load, i.e. when condition (6) is satisfied. Then, from Eq.(38) we have at $x_2 = 0$

$$\left(1 - \frac{c^2}{c_R^2}\right)\partial^2_{\xi\xi}\phi_1 = \gamma_1 \partial_\xi f,$$

from which we immediately deduce (compare with Eq.(53))

$$\phi_1(\xi, 0) = Q_2(c^2)\int f(\xi)\mathrm{d}\xi,$$

with

$$Q_2(c^2) = \frac{c_R^2\gamma_1}{c_R^2 - c^2}.$$

Similarly to the analysis given for Lamb's problem, it may be seen that the expression for $Q_2(c^2)$ corresponds to the leading order term in the Taylor expansion about the Rayleigh speed of Eq.(52), which gives the amplitude coefficient $Q_1(c^2)$ for the exact solution.

6. Conclusions

In this paper, an explicit asymptotic model is derived which accounts for the Rayleigh wave contribution to the dynamic field induced by stresses acting on the surface of an orthorhombic half-plane. The model is set in terms of a single plane harmonic function and it allows to reduce the vector problem of elastodynamics to a scalar elliptic equation with boundary values given by an hyperbolic equation. Indeed, the model incorporates time dependence through the 1D wave equation (Eqs.(38) and (40) for the normal and shear load, respectively), which, once solved on the half-plane surface, provides the boundary values for a Laplace equation in the interior. This approach allows to easily tackle generally involved propagation problems and obtain their far-field approximation, which is sensitive to the propagated field. Moreover, the hyperbolic equations (41) and (42) immediately lend the surface displacement.

In order to illustrate the proposed asymptotic formulation, two classical problems are considered, namely Lamb's problem for normal load and the steady-state propagation of a line load in the subsonic regime. In the case of Lamb's problem, a pair of opposite moving Rayleigh waves are generated on the surface and their effect in the interior is immediately determined by Poisson's integral formula. This displacement field matches the behaviour found by using the integral transform method and then approximating the solution in the transformed domain by a Taylor series centered at the Rayleigh pole (i.e. the residue of the Rayleigh pole). Conversely, the steady-state propagation problem lends itself most easily to a solution in terms of a single plane harmonic function and, in the limit as the speed approaches the Rayleigh wave speed, the results of the asymptotic model are retrieved. Thus, it is shown that the proposed approach is obtained by perturbing the steady-state solution for a load moving at the critical speed.

Finally, we note that this approach can be generalized to include complex conjugate attenuation parameters associated with oscillatory decay (although this situation is non-trivial for it is set outside the scope of harmonic solutions), to 3D problems and surface waves in a coated half-space, transient moving load problems treated similarly to [12], surface waves in case of impedance boundary conditions [34] and also extensions of the parabolic-elliptic model for bending edge waves to anisotropic plates [16, 22].

Acknowledgment

AN gratefully acknowledges financial support from the National Group of Mathematical Physics (GNFM-INdAM) through the Young researchers fellowship programme 2017, Prot. n. U UFMBAZ-2017/0000259 08/06/2017.

DP acknowledges support through the Erasmus+ EU framework.

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