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## Chapter 1

# A Wiener-Hopf system of equations in the steady-state propagation of a rectilinear crack in an infinite elastic plate

A. Nobili, E. Radi and L. Lanzoni

### 1.1 Introduction

Let us consider a Kirchhoff-Love infinite plate with thickness  $h$ , supported by a Winkler elastic foundation (Fig.1.1). The plate has a linear crack and a Cartesian reference frame is introduced such that the crack corresponds to the negative part of the  $x_1$ -axis. Besides, the crack is linearly propagating at a constant speed  $v$  (steady-state propagation) and the origin of the Cartesian reference frame is attached to and moves along with the crack tip. The governing equation for the transverse displacement of the plate  $w$  reads

$$D\Delta\Delta w + kw = -\rho h\partial_{tt}w + q, \quad (1.1)$$

being  $\Delta = \partial_{x_1x_1} + \partial_{x_2x_2}$  the Laplace operator in two dimensions,  $q$  the transverse distributed load per unit area,  $D$  the plate bending stiffness,  $k$  the Winkler modulus and  $\rho$  the mass density per unit volume [6]. We let  $w = w(x_1 - vt, x_2)$  and Eq.(1.1) may be rewritten as

$$\Delta\Delta w + \kappa^{-2}\partial_{x_1x_1}w + \lambda^{-4}w = \frac{q}{D}, \quad (1.2)$$

having let the *characteristic lengths*

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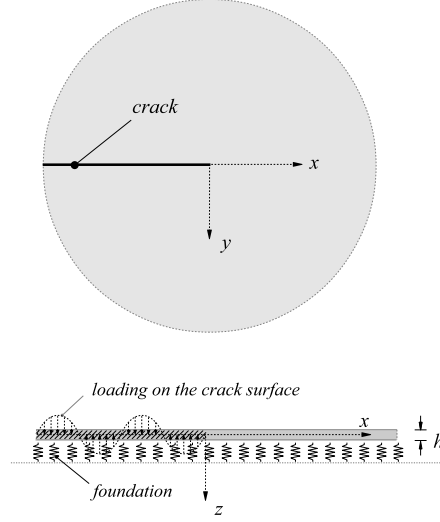
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**Fig. 1.1** Cracked Kirchhoff plate resting on a Winkler elastic foundation. The crack is propagating in the  $x$  direction with speed  $v$

$$\lambda = \sqrt[4]{\frac{D}{k}}, \quad \text{and} \quad \kappa = \sqrt{\frac{D}{\rho h v^2}},$$

together with the positive dimensionless ratio

$$\eta = \lambda / \kappa.$$

We rescale the co-ordinate axes  $(x, y) = \lambda^{-1}(x_1, x_2)$  and take  $q \equiv 0$ , with no loss of generality. Then, Eq.(1.2) becomes

$$\hat{\Delta} \hat{\Delta} w + \eta^2 \partial_{xx} w + w = 0, \quad (1.3)$$

where  $\hat{\Delta} = \partial_{xx} + \partial_{yy}$  is the Laplacian operator in the dimensionless co-ordinates  $x, y$ . The special case  $\eta = 0$  corresponds to the static problem, whose solution is given in [1] and extended in [7]. The Fourier transform of  $w(x, y)$  along  $x$  is defined on the real axis in the usual way

$$\mathcal{F}[w](s, y) = \bar{w}(s, y) \doteq \int_{-\infty}^{+\infty} w(x, y) \exp(isx) dx$$

along with the inverse transform

$$\mathcal{F}^{-1}[\bar{w}](x, y) = w(x, y) \doteq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{w}(s, y) \exp(-isx) ds.$$

In the same fashion, the unilateral (or generalized) transforms are introduced:

$$\mathcal{F}^+[w](s, y) = \bar{w}^+(s, y) \doteq \int_0^{+\infty} w(x, y) \exp(isx) dx$$

which is analytic in the complex half-plane  $\text{Im}(s) > -\alpha_1$  provided that  $\exists \alpha_1 > 0$  such that  $w(x, y) < W_1(y) \exp(-\alpha_1 x)$  as  $x \rightarrow +\infty$  (for a study of the regularity of this transform see [10]), and

$$\mathcal{F}^-[w](s, y) = \bar{w}^-(s, y) \doteq \int_{-\infty}^0 w(x, y) \exp(isx) dx,$$

which is analytic in the complex half-plane  $\text{Im}(s) < \alpha_2$  provided that  $\exists \alpha_2 > 0$  such that  $w(x, y) > W_2(y) \exp(\alpha_2 x)$  as  $x \rightarrow -\infty$ . Consequently, the bilateral Fourier transform is related to the unilateral transforms through

$$\bar{w}(s, y) = \bar{w}^+(s, y) + \bar{w}^-(s, y) \quad (1.4)$$

and it is analytic in the infinite strip  $\mathcal{S} = \{s \in \mathbb{C} : -\alpha_1 < \text{Im}(s) < \alpha_2\}$  containing the real axis. Taking the Fourier transform of Eq.(1.3) in the  $x$  variable, a linear constant coefficient ODE is obtained whose general solution is

$$\begin{aligned} \bar{w}(s, y) = & A_1 \exp(-\sqrt{\Lambda_1}|y|) + B_1 \exp(\sqrt{\Lambda_1}|y|) \\ & + A_2 \exp(-\sqrt{\Lambda_2}|y|) + B_2 \exp(\sqrt{\Lambda_2}|y|), \end{aligned}$$

wherein

$$\Lambda_{1,2} = s^2 \mp \sqrt{\eta^2 s^2 - 1}, \quad (1.5)$$

such that  $\Lambda_1 \Lambda_2 = s^4 - \eta^2 s^2 + 1$ . Hereinafter,  $\text{Re}(s)$  and  $\text{Im}(s)$  denote the real and the imaginary part of  $s \in \mathbb{C}$ , respectively, and a superscript asterisk denotes complex conjugation, i.e.  $s^* = \text{Re}(s) - i \text{Im}(s)$ . Let  $w(x, y^+)$  ( $w(x, y^-)$ ) be the restriction of the displacement  $w(x, y)$  in the upper (lower) half of the  $(x, y)$ -plane, respectively. It is understood that  $y^+ \in (0, +\infty)$  and  $y^- \in (-\infty, 0)$ . The general solution of the ODE (1.3), bounded at infinity, is

$$\bar{w}(s, y^\pm) = A_1^\pm \exp(-\sqrt{\Lambda_1}|y|) + A_2^\pm \exp(-\sqrt{\Lambda_2}|y|), \quad (1.6)$$

where  $A_1^\pm, A_2^\pm$  are four *complex-valued* functions of  $s$  to be determined. The square root in (1.6) is made defined by choosing the Riemann sheet such that  $\text{Re}(\sqrt{\Lambda_{1,2}}) > 0$ . Besides, we let the shorthand notation for the restriction of (1.6) to the  $x$ -axis

$$\bar{w}_{0^\pm}(s) = A_1^\pm + A_2^\pm \quad (1.7)$$

and let  $A_i^\pm$  split into symmetric and skew-symmetric parts

$$A_i^\pm = \frac{1}{2} (\bar{A}_i \pm \Delta A_i), \quad i = 1, 2. \quad (1.8)$$

## 1.2 Boundary conditions

Let the bending moment and equivalent shearing force (deprived of the factor  $D$ )

$$m = -(\partial_{yy} + \nu \partial_{xx})w, \quad v = -\partial_y [\partial_{yy} + (2 - \nu) \partial_{xx}]w,$$

together with the slope

$$\phi = \partial_y w.$$

The boundary conditions (BCs) across the line  $y = 0$  ahead of the crack tip are of the kinematic type as they warrant continuity for displacement and slope

$$[[w_0(x)]] = [[\phi_0(x)]] = 0, \quad x > 0, \quad (1.9)$$

and of the static type, for they demand continuity for bending moment and equivalent shearing force

$$[[m_0(x)]] = [[v_0(x)]] = 0, \quad x > 0. \quad (1.10)$$

Here,  $[[f(0)]]$  denotes the jump of the function  $f(y)$  across  $y = 0$ , namely  $f(0^+) - f(0^-)$ , while a subscript zero stands for evaluation on the real axis. Conversely, it is assumed that the crack flanks are loaded in a continuous fashion by a harmonic loading. Then, the BCs at the crack line  $y = 0$  are

$$m(x, 0^\pm) = M_0 \exp(\imath ax), \quad v(x, 0^\pm) = V_0 \exp(\imath ax), \quad x < 0, \quad (1.11)$$

where  $M_0 = M_0(a)$ ,  $V_0 = V_0(a)$  and

$$\text{Im}(a) < 0 \quad (1.12)$$

to ensure a decay condition as  $x \rightarrow -\infty$ . We observe that the system (1.11), just like the system (1.9, 1.10), entails four conditions, for it applies at both flanks of the crack (denoted by  $y = 0^\pm$ ). As a consequence of loading continuity, Eqs.(1.10) hold on the entire real line. Furthermore, we note that a similar problem arises in the realm of couple-stress theory, although with a different set of boundary conditions [9, 5, 8].

Eq.(1.9) may be written in terms of the generalized plus transform as

$$[[\bar{w}_0^+]] = [[\bar{\phi}_0^+]] = 0,$$

whence, using the general solution (1.6) and in view of Eqs.(1.4, 1.8), it is

$$\Delta A_1 + \Delta A_2 = [[\bar{w}_0^-]], \quad (1.13a)$$

$$-\sqrt{\Lambda_1} \bar{A}_1 - \sqrt{\Lambda_2} \bar{A}_2 = [[\bar{\phi}_0^-]]. \quad (1.13b)$$

Likewise, the Fourier unilateral transform of Eqs.(1.11) gives

$$\bar{m}_0^- = -\imath \frac{M_0}{s+a}, \quad \bar{v}_0^- = -\imath \frac{V_0}{s+a},$$

in the strip  $\mathcal{S}_1 = \{s : \text{Im}(s) < -\text{Im}(a)\}$ . In particular, inequality (1.12) guarantees the existence of the minus transform up to a little above the real axis in the lower complex half-plane. Thus

$$-[(\Lambda_1 - \nu s^2)A_1^\pm + (\Lambda_2 - \nu s^2)A_2^\pm] = \bar{m}_0^+ - \iota \frac{M_0}{s+a}, \quad (1.14a)$$

$$\pm \left\{ \sqrt{\Lambda_1} [\Lambda_1 - (2-\nu)s^2] A_1^\pm + \sqrt{\Lambda_2} [\Lambda_2 - (2-\nu)s^2] A_2^\pm \right\} = \bar{v}_0^+ - \iota \frac{V_0}{s+a}, \quad (1.14b)$$

in the strip  $\mathcal{S}_0 = \mathcal{S} \cap \mathcal{S}_1$ . Finally, the Fourier transforming Eqs.(1.10), which are holding on the entire real line, gives

$$(\Lambda_1 - \nu s^2)\Delta A_1 + (\Lambda_2 - \nu s^2)\Delta A_2 = 0, \quad (1.15a)$$

$$\sqrt{\Lambda_1} [\Lambda_1 - (2-\nu)s^2] \bar{A}_1 + \sqrt{\Lambda_2} [\Lambda_2 - (2-\nu)s^2] \bar{A}_2 = 0, \quad (1.15b)$$

according to which the system (1.14) becomes

$$-\frac{1}{2} [(\Lambda_1 - \nu s^2)\bar{A}_1 + (\Lambda_2 - \nu s^2)\bar{A}_2] = \bar{m}_0^+ - \iota \frac{M_0}{s+a}, \quad (1.16a)$$

$$\frac{1}{2} \left\{ \sqrt{\Lambda_1} [\Lambda_1 - (2-\nu)s^2] \Delta A_1 + \sqrt{\Lambda_2} [\Lambda_2 - (2-\nu)s^2] \Delta A_2 \right\} = \bar{v}_{y0}^+ - \iota \frac{V_0}{s+a}. \quad (1.16b)$$

Conditions (1.15) are immediately fulfilled through letting

$$\Delta A_1 = -(\Lambda_2 - \nu s^2)\Delta A,$$

$$\Delta A_2 = (\Lambda_1 - \nu s^2)\Delta A,$$

$$\bar{A}_1 = -\sqrt{\Lambda_2} [\Lambda_2 - (2-\nu)s^2] \bar{A},$$

$$\bar{A}_2 = \sqrt{\Lambda_1} [\Lambda_1 - (2-\nu)s^2] \bar{A}.$$

Then, Eqs.(1.13) become

$$(\Lambda_1 - \Lambda_2) \Delta A = \llbracket \bar{w}_0^- \rrbracket, \quad (1.18a)$$

$$-\sqrt{\Lambda_1 \Lambda_2} (\Lambda_1 - \Lambda_2) \bar{A} = \llbracket \bar{\phi}_0^- \rrbracket. \quad (1.18b)$$

Likewise, the system (1.16) gives

$$(\Lambda_2 - \Lambda_1) K(s) \bar{A} = \bar{m}_0^+ - \iota \frac{M_0}{s+a}, \quad (1.19a)$$

$$(\Lambda_2 - \Lambda_1) K(s) \Delta A = \bar{v}_0^+ - \iota \frac{V_0}{s+a}, \quad (1.19b)$$

where the *kernel function*  $K(s)$  is let as follows:

$$2(\Lambda_2 - \Lambda_1)K(s) = -\sqrt{\Lambda_1}[\Lambda_1 - (2 - \nu)s^2](\Lambda_2 - \nu s^2) + \sqrt{\Lambda_2}[\Lambda_2 - (2 - \nu)s^2](\Lambda_1 - \nu s^2). \quad (1.20)$$

With the help of Eq.(1.5), Eq.(1.20) may be rewritten as

$$4\sqrt{\eta^2 s^2 - 1}K(s) = \sqrt{s^2 - \sqrt{\eta^2 s^2 - 1}}\left(v_0 s^2 + \sqrt{\eta^2 s^2 - 1}\right)^2 - \sqrt{s^2 + \sqrt{\eta^2 s^2 - 1}}\left(v_0 s^2 - \sqrt{\eta^2 s^2 - 1}\right)^2, \quad (1.21)$$

having let  $v_0 = 1 - \nu$ . In particular, in the limit as  $\eta \rightarrow 0$  and with  $\sqrt{\eta^2 s^2 - 1} \rightarrow \iota$ , the kernel  $4\iota K(s)$  in Eq.(1.21) reduces to Eq.(24) of [1]. Solving the system (1.18) for the unknown functions  $\bar{A}, \Delta A$  and plugging the result in Eqs.(1.19) provides the following two uncoupled Wiener-Hopf (W-H) equations, namely

$$\begin{aligned} K(s)[\bar{w}_0^-] + \bar{v}_0^+ &= \iota \frac{V_0}{s+a}, \\ (\Lambda_1 \Lambda_2)^{-1/2} K(s)[\bar{\phi}_0^-] - \bar{m}_0^+ &= -\iota \frac{M_0}{s+a}. \end{aligned}$$

Making use of Eq.(1.5), this system of functional equations may be rewritten as

$$K(s)[\bar{w}_0^-] + \bar{v}_0^+ = \iota \frac{V_0}{s+a}, \quad (1.22a)$$

$$\frac{K(s)}{\sqrt{s^2 - \beta^2} \sqrt{s^2 - \beta^{*2}}} [\bar{\phi}_0^-] - \bar{m}_0^+ = -\iota \frac{M_0}{s+a}, \quad (1.22b)$$

having taken the factor decomposition  $\Lambda_1 \Lambda_2 = (s^2 - \beta^2)(s^2 - \beta^{*2})$ , with

$$\beta = \sqrt{\frac{1}{2}\eta^2 + \sqrt{\left(\frac{1}{2}\eta^2\right)^2 - 1}}.$$

It is observed that  $\beta$  is a complex number with unit modulus and it is located in the first quadrant of the complex plane inasmuch as  $0 \leq \eta < \sqrt{2}$ . Alternatively, when  $\eta \geq \sqrt{2}$ , we define the positive real numbers  $\beta_1 \geq \beta_2$ ,

$$\beta_{1,2} = \sqrt{\frac{1}{2}\eta^2 \pm \sqrt{\left(\frac{1}{2}\eta^2\right)^2 - 1}}$$

and the following manipulations still hold formally, with the understanding that  $\beta$  stands for  $\beta_1$  and  $\beta^*$  stands for  $\beta_2$ . Furthermore, we observe that

$$\beta\beta^* = \beta_1\beta_2 = 1.$$

### 1.3 Wiener-Hopf factorization

The kernel  $K(s)$  is an even function and it possesses 6 roots

$$K(s) = 0 \text{ for } s = \pm s_1, \pm s_1^*, \text{ and } s = \pm i r_1, \quad (1.23)$$

all of which are of multiplicity 1, in the general case. Here,  $s_1$  is taken to sit in the first quadrant of the complex plane,

$$s_1 = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}, \quad (1.24)$$

having let  $\eta_R = \sqrt{2}\gamma_e$ , with

$$\gamma_e = \sqrt[4]{(1-\nu)(3\nu-1+2\sqrt{2\nu^2-2\nu+1})} \quad (1.25)$$

and  $\gamma_e \in [\sqrt{2}(\sqrt{5}-2)^{1/4}, 1]$  is a well-known bending edge-wave constant [4, 3, 2]. The double roots in the kernel  $K(s)$  bring in four non-straight branch cuts which extend from  $s = \pm \zeta_1, \pm \zeta_1^*$  to  $\pm i\infty$ .

It is observed that for  $\nu = 0$ , we have  $\gamma_e = 1$  and the roots  $\pm s_1, \pm s_1^*$  coincide with the branch points for the double roots  $\pm \beta, \pm \beta^*$ , whence their multiplicity goes down to 1/2. Besides, it is easy to see that  $s_1$  is complex-valued for  $\eta < \eta_R$  (subsonic regime) and it sits in the complex plane on the circle of radius  $\gamma_e^{-1}$ , i.e.

$$s_1 = \gamma_e^{-1} \exp(i\theta/2), \quad \tan \theta = \frac{1 - (\eta/\eta_R)^4}{(\eta/\eta_R)^2}.$$

In a similar fashion, letting  $\eta_S = \sqrt{2}\gamma_m$ , we find the location of the purely imaginary roots  $\pm i r_1$

$$r_1 = \gamma_m^{-1} \sqrt{\left(\frac{\eta}{\eta_S}\right)^2 + \sqrt{\left(\frac{\eta}{\eta_S}\right)^4 + 1}},$$

where we have defined the monotonic decreasing function of  $\nu$

$$\gamma_m = \sqrt[4]{(1-\nu)(-3\nu+1+2\sqrt{2\nu^2-2\nu+1})},$$

We note that  $\gamma_m \in [\frac{1}{\sqrt{2}}(2\sqrt{2}-1)^{1/4}, \sqrt{2}(\sqrt{5}+2)^{1/4}]$  and  $r_1$  is a real-valued monotonic decreasing (increasing) function of  $\nu$  (of  $\eta$ ), whose minimum  $r_1 = \gamma_m^{-1}$  is attained in the static case  $\eta = 0$ , i.e. unlike  $s_1$ , this root never reaches the real axis. In the special case of  $\eta = 0$  (stationary crack), we have

$$r_1 = \gamma_e^{-1} \quad \text{and} \quad s_1 = \frac{\sqrt{2}}{2\gamma_e}(1+i),$$



whence  $s_1$  sits on the bisector of the first-third quadrants of the complex plane. In contrast, for  $\eta > \eta_R$  (hypersonic regime)  $s_1$  turns real-valued and the root landscape (1.23) switches to

$$\pm s_-, \pm s_+, \pm ir_1, \quad (1.26)$$

where now  $0 < s_- < s_+$  are real values

$$s_{\mp} = \gamma_e^{-1} \sqrt{\left(\frac{\eta}{\eta_R}\right)^2 \mp \sqrt{\left(\frac{\eta}{\eta_R}\right)^4 - 1}}. \quad (1.27)$$

Let us define, for any value  $b > \alpha_1 = \max(\text{Im}(s_1), r_1)$ ,

$$F(s) = \frac{\sqrt{s-ib}\sqrt{s+ib}}{c(s-s_1)(s+s_1)(s-s_1^*)(s+s_1^*)} K(s), \quad v \neq 0, \quad (1.28)$$

along with the constant  $c = v_0(3+v)/4$ . In the special case  $v = 0$ , we need set

$$F(s) = \frac{\sqrt{s-ib}\sqrt{s+ib}}{c\sqrt{s-s_1}\sqrt{s+s_1}\sqrt{s-s_1^*}\sqrt{s+s_1^*}(s-ir_1)(s+ir_1)} K(s), \quad v = 0.$$

The function  $F(s)$  is deprived of zeros in an semi-infinite strip of analyticity about the real axis  $\mathcal{S}$ , extending along the imaginary axis up to  $\alpha_1 = \alpha_2 = \text{Im}(s_1)$ , and it is such that  $\lim_{|s| \rightarrow \infty} F(s) = 1$  in this strip. The Cauchy integral theorem gives

$$\ln F(s) = \frac{1}{2\pi i} \oint_C \frac{\ln F(z)}{z-s} dz,$$

where  $C$  may be taken as the close path in the analyticity strip consisting of two parallel infinite lines a little above and a little below the real axis while  $s$  sits within this closed path. The former contribution brings along a minus function,  $F^-(s)$ , the latter a plus function,  $F^+(s)$ , for we may define

$$F^+(s) = \exp R(s) \quad \text{and} \quad F^-(s) = F^+(-s),$$

where

$$R(s) = \frac{1}{2\pi i} \int_{-\infty-ic}^{\infty-ic} \frac{\ln F(z)}{z-s} dz, \quad 0 < c < \alpha_1. \quad (1.29)$$

Then, provided  $|\text{Im}(s)| < c$ , we have

$$F(s) = F^+(s)F^-(s),$$

and the system (1.22) reads

$$K^-(s) \llbracket \bar{w}_0^- \rrbracket + \frac{\bar{v}_0^+}{K^+(s)} = iV_0 \frac{1}{(s+a)K^+(s)}, \quad (1.30a)$$

$$\frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}\llbracket\bar{\phi}_0^-\rrbracket - \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\bar{m}_0^+ = -iM_0\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{(s+a)K^+(s)}, \quad (1.30b)$$

where,

$$K^\pm(s) = \sqrt{c}\frac{(s \pm s_1)(s \mp s_1^*)}{\sqrt{s \pm ib}}F^\pm(s)\exp(\pm i\pi/4),$$

with the property that  $K^-(s) = K^+(-s)$ . Clearly, for large values of  $|s|$ , we get the asymptotic behavior

$$K^\pm(s) \sim \sqrt{c}\exp(\pm i\pi/4)|s|^{3/2}.$$

Finally, the RHS' are split in terms of a plus and a minus function, thus giving

$$\frac{\bar{v}_0^+}{K^+(s)} - i\frac{V_0}{s+a}\left(\frac{1}{K^+(s)} - \frac{1}{K^-(a)}\right) = i\frac{V_0}{(s+a)K^-(a)} - K^-(s)\llbracket\bar{w}_0^-\rrbracket, \quad (1.31a)$$

$$\begin{aligned} & \frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)}\bar{m}_0^+ - i\frac{M_0}{s+a}\left(\frac{\sqrt{s+\beta}\sqrt{s-\beta^*}}{K^+(s)} - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\right) \\ &= i\frac{M_0}{s+a}\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)} + \frac{K^-(s)}{\sqrt{s-\beta}\sqrt{s+\beta^*}}\llbracket\bar{\phi}_0^-\rrbracket, \end{aligned} \quad (1.31b)$$

Since the LHSs (RHSs) represent two analytic functions in the upper (lower) half complex plane with a common strip of regularity, they can be analytically continued to the whole complex plane giving two entire functions  $E_1(s)$  and  $E_2(s)$ , i.e. they are holomorphic over the whole complex plane. It is observed that both hands of Eqs.(1.31) behave like  $s^{-1}$  as  $s \rightarrow \infty$ , whereupon  $E_1(s) \equiv E_2(s) \equiv 0$ , by Liouville's theorem. Indeed,

$$\begin{aligned} w(x) \sim x^{3/2} & \Rightarrow \bar{w}^-(s) \sim s^{-5/2}, & \phi(x) \sim x^{1/2} & \Rightarrow \bar{\phi}^-(s) \sim s^{-3/2}, \\ m(x) \sim x^{-1/2} & \Rightarrow \bar{m}^+(s) \sim s^{-1/2}, & v(x) \sim x^{-3/2} & \Rightarrow \bar{v}^+(s) \sim s^{1/2}, \end{aligned}$$

the latter being meaningful in a distributional sense. Thus

$$\llbracket\bar{w}_0^-\rrbracket = i\frac{V_0}{K^-(a)}\frac{1}{(s+a)K^-(s)}, \quad (1.32)$$

and

$$\llbracket\bar{\phi}_0^-\rrbracket = iM_0\frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{K^-(a)}\frac{\sqrt{s-\beta}\sqrt{s+\beta^*}}{(s+a)K^-(s)}. \quad (1.33)$$

Likewise, we obtain a direct expression for the unilateral Fourier transform of bending moment and shearing force along the co-ordinate axis  $y = 0$ , namely

$$\bar{m}_0^+ = i\frac{M_0}{s+a}\left(1 - \frac{\sqrt{a-\beta}\sqrt{a+\beta^*}}{\sqrt{s+\beta}\sqrt{s-\beta^*}}\frac{K^+(s)}{K^-(a)}\right) \quad (1.34)$$

and

$$\bar{v}_0^+ = i \frac{V_0}{s+a} \left( 1 - \frac{K^+(s)}{K^-(a)} \right). \quad (1.35)$$

It is observed that, according to Jordan's lemma [10], Eqs.(1.32) and (1.33) satisfy both BCs (1.9) and, by the same argument, Eqs.(1.35) and (1.34) convey the conditions (1.10).

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