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Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems

Simona Bonvicini

Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/A, 41125 Modena, Italy

Marco Buratti

Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy

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Abstract

We give a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens.

Keywords: Hamilton-Waterloo problem, group action, octahedral binary group, dicyclic group, special linear group.

Math. Subj. Class.: 05C70, 05E18, 05B10

1 Introduction

A cycle decomposition of a simple graph \( \Gamma = (V, E) \) is a set \( \mathcal{D} \) of cycles whose edges partition \( E \). A partition \( \mathcal{F} \) of \( \mathcal{D} \) into classes (2-factors) each of which covers all \( V \) exactly once is said to be a 2-factorization of \( \Gamma \). The type of a 2-factor \( F \) is the partition \( \pi = [\ell_1^{n_1}, \ldots, \ell_t^{n_t}] \) (written in exponential notation) of the integer \( |V| \) into the lengths of the cycles of \( F \).

A 2-factorization \( \mathcal{F} \) of \( K_v \) (the complete graph of order \( v \)) or \( K_v - I \) (the cocktail party graph of order \( v \)) whose 2-factors are all of the same type \( \pi \) is a solution of the so-called Oberwolfach Problem \( \text{OP}(v; \pi) \). If instead the 2-factors of \( \mathcal{F} \) are of two different types \( \pi \) and \( \psi \), then \( \mathcal{F} \) is a solution of the so-called Hamilton-Waterloo Problem \( \text{HWP}(v; \pi, \psi; r, s) \) where \( r \) and \( s \) denote the number of 2-factors of \( \mathcal{F} \) of type \( \pi \) and \( \psi \), respectively.

A complete solution of the OPs whose 2-factors are uniform, namely of the form \( \text{OP}(\ell n; [\ell^n]) \), has been given in [1] and [12]. Other important classes of OPs has been
solved in [4, 15]. For the time being, to look for a solution to all possible OPs and, above all, HWPs is too ambitious. Anyway it is reasonable to believe that we are not so far from a complete solution of the HWPs whose 2-factors are uniform, namely of the form HWP\( (v; [h^{v/h}], [w^{v/w}]; r, s) \). We can say this especially because of the big progress recently done in [10].

Danziger, Quattrocchi and Stevens [11] treated the HWPs whose 2-factors are either triangle-factors or quadrangle-factors, they namely studied HWP\( (12n; [3^{4n}], [4^{3n}]; r, s) \). In the following such an HWP will be denoted, more simply, by HWP\( (12n; 3, 4; r, s) \). They solved this problem for all possible triples \( (n, r, s) \) except the following ones:

(i) \( (4, r, 23 - r) \) with \( r \in \{5, 7, 9, 13, 15, 17\} \);
(ii) \( (2, r, 11 - r) \) with \( r \in \{5, 7, 9\} \).

Six of the nine above problems have been recently solved in [14] where it was pointed out that all nine problems were also solved in a work still in preparation [2] by the authors of the present paper. Meanwhile, a solution for each of the remaining three problems not considered in [14] have been given in [16]. Notwithstanding, in the present paper we want to present our solutions to the nine HWPs left open by Danziger, Quattrocchi and Stevens in detail. These solutions, differently from those of [14, 16], are full of symmetries since they are \( G \)-regular for a suitable group \( G \). We recall that a cycle decomposition (or 2-factorization) of a graph \( \Gamma \) is said to be \( G \)-regular when it admits \( G \) as an automorphism group acting sharply transitively on all vertices. Here is explicitly our main result:

**Theorem 1.1.** There exists a \( O \)-regular 2-factorization of \( K_{48} - I \) having \( r \) triangle-factors and \( 23 - r \) quadrangle-factors where \( O \) is the binary octahedral group and \( r \in \{5, 7, 9, 13, 15, 17\} \).

There exists a \( Q_{24} \)-regular 2-factorization of \( K_{24} - I \) having \( r \) triangle-factors and \( 11 - r \) quadrangle-factors where \( Q_{24} \) is the dicyclic group of order 24 and \( r \in \{7, 9\} \).

There exists a \( SL_2(3) \)-regular 2-factorization of \( K_{24} - I \) having six triangle-factors and five quadrangle-factors where \( SL_2(3) \) is the 2-dimensional special linear group over \( \mathbb{Z}_3 \).

2 Some preliminaries

The use of the classic method of differences allowed to get cyclic (namely \( Z_v \)-regular) solutions of some HWPs in [8, 9, 13]. Now we summarize, in the shortest possible way, the method of partial differences. This method, explained in [7] and successfully applied in many papers (see, especially, [6]), has been also useful for the investigation of \( G \)-regular 2-factorizations of a complete graph of odd order [9]. The \( G \)-regular 2-factorizations of a cocktail party graph can be treated similarly.

Throughout this paper any group \( G \) will be assumed to be written multiplicatively and its identity element will be denoted by 1. Let \( \Omega \) be a symmetric subset of a group \( G \); this means that \( 1 \not\in \Omega \) and that \( \omega \in \Omega \) if and only if \( \omega^{-1} \in \Omega \). The Cayley graph on \( G \) with connection-set \( \Omega \), denoted by \( \text{Cay}[G : \Omega] \), is the simple graph whose vertices are the elements of \( G \) and whose edges are all 2-subsets of \( G \) of the form \( \{g, \omega g\} \) with \( (g, \omega) \in G \times \Omega \).

**Remark 2.1.** If \( \lambda \) is an involution of a group \( G \), then \( \text{Cay}[G : G \setminus \{1, \lambda\}] \) is isomorphic to \( K_{|G|} - I \). So, in the following, such a Cayley graph will be always identified with the cocktail party graph of order \( |G| \).
Let $\text{Cycle}(G)$ be the set of all cycles with vertices in $G$ and consider the natural right action of $G$ on $\text{Cycle}(G)$ defined by $(c_1, c_2, \ldots, c_n)^g = (c_1g, c_2g, \ldots, c_ng)$ for every $C = (c_1, c_2, \ldots, c_n) \in \text{Cycle}(G)$ and every $g \in G$. The stabilizer and the orbit of any $C \in \text{Cycle}(G)$ under this action will be denoted by $\text{Stab}(C)$ and $\text{Orb}(C)$, respectively. The list of differences of $C \in \text{Cycle}(G)$ is the multiset $\Delta C$ of all possible quotients $xy^{-1}$ with $(x,y)$ an ordered pair of adjacent vertices of $C$. One can see that the multiplicity $m_{\Delta C}(g)$ of any element $g \in G$ in $\Delta C$ is a multiple of the order of $\text{Stab}(C)$. Thus it makes sense to speak of the list of partial differences of $C$ as the multiset $\partial C$ on $G$ in which the multiplicity of any $g \in G$ is defined by

$$m_{\partial C}(g) := \frac{m_{\Delta C}(g)}{|\text{Stab}(C)|}.$$  

We underline the fact that $\partial C$ is, in general, a multiset. Note that if $\partial C$ is a set, namely without repeated elements, then it is symmetric so that it makes sense to speak of the Cayley graph $\text{Cay}[G : \partial C]$. The following elementary but crucial result holds.

**Lemma 2.2.** If $C \in \text{Cycle}(G)$ and $\partial C$ does not have repeated elements, then $\text{Orb}(C)$ is a $G$-regular cycle-decomposition of $\text{Cay}[G : \partial C]$.

By Remark 2.1, as an immediate consequence of the above lemma we can state the following result.

**Theorem 2.3.** Let $\lambda$ be an involution of a group $G$. If $\{C_1, \ldots, C_t\}$ is a subset of $\text{Cycle}(G)$ such that $\bigcup_{i=1}^{t} \partial C_i = G \setminus \{1, \lambda\}$, then $\bigcup_{i=1}^{t} \text{Orb}(C_i)$ is a $G$-regular cycle-decomposition of $K[G] - I$.

We need, as last ingredient, the following easy remarks.

**Remark 2.4.** If $C \in \text{Cycle}(G)$ and $V(C)$ is a subgroup of $G$, then $\text{Orb}(C)$ is a 2-factor of the complete graph on $G$ whose stabilizer is the whole $G$.

If $C_1, \ldots, C_t$ are cycles of $\text{Cycle}(G)$ and $\bigcup_{i=1}^{t} V(C_i)$ is a complete system of representatives for the left cosets of a subgroup $S$ of $G$, then $\bigcup_{i=1}^{t} \text{Orb}_S(C_i)$ is a 2-factor of the complete graph on $G$ whose stabilizer is $S$.

3 Octahedral solutions of six Hamilton-Waterloo problems

Throughout this section $G$ will denote the so-called binary octahedral group which is usually denoted by $\overline{O}$. This group, up to isomorphism, can be viewed as a group of units of the skew-field $\mathbb{H}$ of quaternions introduced by Hamilton, that is an extension of the complex field $\mathbb{C}$. We recall the basic facts regarding $\mathbb{H}$. Its elements are all real linear combinations of $1$, $i$, $j$ and $k$. The sum and the product of two quaternions are defined in the natural way under the rules that

$$i^2 = j^2 = k^2 = ijk = -1.$$  

If $q = a + bi + cj + dk \neq 0$, then the inverse of $q$ is given by

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$
The 48 elements of the multiplicative group $G$ are the following:

$$\pm 1, \pm i, \pm j, \pm k;$$
$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k);$$
$$\frac{1}{\sqrt{2}}(\pm x \pm y), \quad \{x, y\} \in (\{1, i, j, k\}).$$

The use of the octahedral group $G$ was crucial in [3] to get a Steiner triple system of any order $v = 96n + 49$ with an automorphism group acting sharply transitively an all but one point. Here $G$ will be used to get a $G$-regular solution of each of the six Hamilton-Waterloo problems of order 48 left open in [11]. We will need to consider the following subgroups of $G$ of order 16 and 12, respectively:

- $K = \langle k, \frac{1}{\sqrt{2}}(j - k) \rangle$;
- $L = \langle \frac{1}{\sqrt{2}}(j - k), \frac{1}{2}(-1 - i + j + k) \rangle$.

### 3.1 An octahedral solution of HWP(48; 3, 4; 5, 18)

Consider the nine cycles of $\text{Cycle}(G)$ defined as follows.

$$C_1 = (1, -\frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j - k))$$
$$C_2 = (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k))$$
$$C_3 = (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 - i - j + k))$$
$$C_4 = (1, k, -1, -k)$$
$$C_5 = (1, j, -1, -j)$$
$$C_6 = (1, \frac{1}{\sqrt{2}}(-i + k), -\frac{1}{2}(1 + i + j + k), -\frac{1}{\sqrt{2}}(j + k))$$
$$C_7 = (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{2}(1 + i), \frac{1}{2}(1 - i - j + k))$$
$$C_8 = (1, \frac{1}{2}(1 - i + j - k), k, -\frac{1}{\sqrt{2}}(1 + j))$$
$$C_9 = (1, \frac{1}{\sqrt{2}}(-1 - i), -\frac{1}{\sqrt{2}}(1 + i), \frac{1}{2}(-1 - i + j - k))$$

We note that $\text{Stab}(C_i) = V(C_i)$ for $2 \leq i \leq 5$ while all other $C_i$’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that $\text{Orb}(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \{-\frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j - k), -\frac{1}{\sqrt{2}}(1 + i)\}^{\pm 1}$$
$$\Omega_2 = \{\frac{1}{2}(-1 - i + j + k)\}^{\pm 1}$$
$$\Omega_3 = \{\frac{1}{2}(-1 + i + j - k)\}^{\pm 1}$$
$$\Omega_4 = \{k\}^{\pm 1}$$
$$\Omega_5 = \{j\}^{\pm 1}$$
$$\Omega_6 = \{\frac{1}{\sqrt{2}}(-i + k), \frac{1}{\sqrt{2}}(j - k), \frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(j + k)\}^{\pm 1}$$
that $C^\ell$ length 3 or 1 or 4 according to whether $i = 1$, or $2 \leq i \leq 5$, or $6 \leq i \leq 9$, respectively. The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^9 Orb(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 5 triangle-factors and 18 quadrangle-factors, namely a $G$-regular solution of HWP(48; 3, 4; 7, 16).

3.2 An octahedral solution of HWP(48; 3, 4; 7, 16)

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$C_1 = (1, \frac{-1}{\sqrt{2}}(i + j), \frac{1}{\sqrt{2}}(1 - i + j + k))$$

$$C_2 = (1, \frac{1}{\sqrt{2}}(-1 - i + j + k), \frac{1}{\sqrt{2}}(1 - i - j - k))$$

$$C_3 = (1, \frac{1}{\sqrt{2}}(-1 + i + j + k), \frac{1}{\sqrt{2}}(-1 - i - j + k))$$

$$C_4 = (1, \frac{1}{\sqrt{2}}(-i + k), \frac{1}{\sqrt{2}}(1 + i + j - k), \frac{-1}{\sqrt{2}}(j + k))$$

$$C_5 = (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{\sqrt{2}}(1 - k), \frac{1}{\sqrt{2}}(1 + i))$$

$$C_6 = (1, \frac{1}{\sqrt{2}}(1 + k), \frac{-1}{\sqrt{2}}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j))$$

$$C_7 = (1, \frac{-1}{\sqrt{2}}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - i - j + k))$$

We note that $Stab(C_3) = V(C_3)$ while all other $C_i$’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $Cay[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \{-\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(1 - i + j + k), \frac{1}{\sqrt{2}}(-j + k)\}^{\pm 1}$$

$$\Omega_2 = \{\frac{1}{2}(-1 - i + j + k), \frac{1}{2}(1 - i - j - k), \frac{1}{2}(-1 - i + j - k)\}^{\pm 1}$$

$$\Omega_3 = \{\frac{1}{2}(-1 + i + j - k)\}^{\pm 1}$$

$$\Omega_4 = \{\frac{1}{\sqrt{2}}(-i + k), -\frac{1}{\sqrt{2}}(1 - k), \frac{1}{\sqrt{2}}(i + k), -\frac{1}{\sqrt{2}}(j + k)\}^{\pm 1}$$

$$\Omega_5 = \{\frac{1}{\sqrt{2}}(i - j) - j, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 + i)\}^{\pm 1}$$

$$\Omega_6 = \{\frac{1}{\sqrt{2}}(1 + k), \frac{1}{\sqrt{2}}(-1 + j), -\frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 + j)\}^{\pm 1}$$

$$\Omega_7 = \{-\frac{1}{2}(1 + i + j + k), -i, -k, \frac{1}{2}(1 - i - j + k)\}^{\pm 1}$$
One can see that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} \text{Orb}_G(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = \text{Orb}_{S_i}(C_i)$ where
$$S_i = \begin{cases} K & \text{for } i = 1, 2; \\ G & \text{for } i = 3; \\ L & \text{for } 4 \leq i \leq 7. \end{cases}$$

By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $4 \leq i \leq 7$, respectively.

The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 3$.

Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} \text{Orb}_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 7 triangle-factors and 16 quadrangle-factors, namely a $G$-regular solution of HWP($48; 3, 4; 7, 16$).

### 3.3 An octahedral solution of HWP($48; 3, 4; 9, 14$)

Consider the eight cycles of $\text{Cycle}(G)$ defined as follows.

$$C_1 = \{1, \frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i-j-k)\}$$

$$C_2 = \{1, -\frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(1+j)\}$$

$$C_3 = \{1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(1+i-j+k)\}$$

$$C_4 = \{1, \frac{1}{\sqrt{2}}(-i+k), \frac{1}{\sqrt{2}}(1-i), \frac{1}{2}(-1-i+j-k)\}$$

$$C_5 = \{1, \frac{1}{\sqrt{2}}(i-j), \frac{1}{2}(-1+i+j+k), \frac{1}{2}(j+k)\}$$

$$C_6 = \{1, \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1-i), \frac{1}{2}(1-i-j+k)\}$$

$$C_7 = \{k, -1, -k\}$$

$$C_8 = \{1, j, -1, -j\}$$

We note that $\text{Stab}(C_i) = V(C_i)$ for $i = 7, 8$ while all other $C_i$’s have trivial stabilizer. By Lemma 2.2, one can check that $\text{Orb}(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \{\frac{1}{\sqrt{2}}(i+j), \frac{1}{2}(1-i-j-k), \frac{1}{\sqrt{2}}(-1+i)\}^{\pm 1}$$

$$\Omega_2 = \{-\frac{1}{\sqrt{2}}(1-k), \frac{1}{\sqrt{2}}(1+j), \frac{1}{2}(1+i+j+k)\}^{\pm 1}$$

$$\Omega_3 = \{\frac{1}{2}(-1-i+j+k), \frac{1}{2}(1+i-j+k), \frac{1}{2}(-1-i-j+k)\}^{\pm 1}$$

$$\Omega_4 = \{\frac{1}{\sqrt{2}}(-i+k), \frac{1}{2}(1-i+j+k), \frac{1}{\sqrt{2}}(i+k), \frac{1}{2}(-1-i+j-k)\}^{\pm 1}$$

$$\Omega_5 = \{\frac{1}{\sqrt{2}}(i-j), \frac{1}{\sqrt{2}}(j-k), -\frac{1}{\sqrt{2}}(1+j), -\frac{1}{\sqrt{2}}(j+k)\}^{\pm 1}$$

$$\Omega_6 = \{\frac{1}{\sqrt{2}}(1+i), i, \frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i-j+k)\}^{\pm 1}$$

$$\Omega_7 = \{k\}^{\pm 1}$$

$$\Omega_8 = \{j\}^{\pm 1}$$

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{8} \text{Orb}(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = \text{Orb}_{S_i}(C_i)$
where

\[
S_i = \begin{cases} 
  K & \text{for } 1 \leq i \leq 3; \\
  L & \text{for } 4 \leq i \leq 6; \\
  G & \text{for } i = 7, 8.
\end{cases}
\]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{48} - I \) with \( \text{Stab}_G(F_i) = S_i \), hence \( \text{Orb}_G(F_i) \) has length 3 or 4 or 1 according to whether \( 1 \leq i \leq 3 \) or \( 4 \leq i \leq 6 \) or \( i = 7, 8 \), respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 3 \). Thus, recalling that \( C \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^{8} \text{Orb}_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{48} - I \) with 9 triangle-factors and 14 quadrangle-factors, namely a \( G \)-regular solution of HWP\((48; 3, 4; 9, 14)\).

### 3.4 An octahedral solution of HWP\((48; 3, 4; 13, 10)\)

Consider the nine cycles of \( \text{Cycle}(G) \) defined as follows.

\[
\begin{align*}
C_1 &= (1, -\frac{1}{\sqrt{2}}(i+j), -\frac{1}{\sqrt{2}}(1+j)) \\
C_2 &= (1, \frac{1}{2}(1-i+j-k), -\frac{1}{\sqrt{2}}(i+k)) \\
C_3 &= (1, \frac{1}{\sqrt{2}}(-i+j), \frac{1}{2}(1-i-j-k)) \\
C_4 &= (1, \frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(i-k)) \\
C_5 &= (1, \frac{1}{2}(-1-i+j+k), \frac{1}{2}(-1+i-j-k)) \\
C_6 &= (1, k, -1, -k) \\
C_7 &= (1, j, -1, -j) \\
C_8 &= (1, -\frac{1}{2}(1+i+j+k), \frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(1+j)) \\
C_9 &= (1, -\frac{1}{\sqrt{2}}(1+k), -k, \frac{1}{2}(-1+i+j-k))
\end{align*}
\]

We note that \( \text{Stab}(C_i) = V(C_i) \) for \( 5 \leq i \leq 7 \) while all other \( C_i \)'s have trivial \( G \)-stabilizer. Thus, by Lemma 2.2, one can check that \( \text{Orb}(C_i) \) is a \( \ell_i \)-cycle decomposition of \( \text{Cay}[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)'s are the symmetric subsets of \( G \) listed below.

\[
\begin{align*}
\Omega_1 &= \{-\frac{1}{\sqrt{2}}(i+j), -\frac{1}{\sqrt{2}}(1+j), \frac{1}{2}(1+i+j-k)\}^{\pm 1} \\
\Omega_2 &= \{\frac{1}{2}(1-i+j-k), -\frac{1}{\sqrt{2}}(i+k), \frac{1}{\sqrt{2}}(1+i)\}^{\pm 1} \\
\Omega_3 &= \{\frac{1}{\sqrt{2}}(-i+j), \frac{1}{2}(1-i-j-k), \frac{1}{\sqrt{2}}(j-k)\}^{\pm 1} \\
\Omega_4 &= \{\frac{1}{2}(-1+i-j+k), \frac{1}{\sqrt{2}}(i-k), -\frac{1}{\sqrt{2}}(j+k)\}^{\pm 1} \\
\Omega_5 &= \{\frac{1}{2}(-1-i+j+k)\}^{\pm 1} \\
\Omega_6 &= \{k\}^{\pm 1} \\
\Omega_7 &= \{j\}^{\pm 1} \\
\Omega_8 &= \{-\frac{1}{2}(1+i+j+k), i, \frac{1}{\sqrt{2}}(-1+i), \frac{1}{\sqrt{2}}(1+j)\}^{\pm 1} \\
\Omega_9 &= \{-\frac{1}{\sqrt{2}}(1+k), \frac{1}{\sqrt{2}}(1-k), \frac{1}{2}(1-i+j+k), \frac{1}{2}(-1+i+j-k)\}^{\pm 1}
\end{align*}
\]
Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $C := \bigcup_{i=1}^{9} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$
S_i = \begin{cases}
K & \text{for } 1 \leq i \leq 4; \\
G & \text{for } 5 \leq i \leq 7; \\
L & \text{for } i = 8, 9.
\end{cases}
$$

By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 4$ or $5 \leq i \leq 7$ or $i = 8, 9$, respectively.

The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $C$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 13 triangle-factors and 10 quadrangle-factors, namely a $G$-regular solution of HWP(48; 3, 4; 13, 10).

### 3.5 An octahedral solution of HWP(48; 3, 4; 15, 8)

Consider the seven cycles of $Cycle(G)$ defined as follows.

\begin{align*}
C_1 &= (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{\sqrt{2}}(i + k)) \\
C_2 &= (1, -\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j)) \\
C_3 &= (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(1 - i + j + k)) \\
C_4 &= (1, \frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j)) \\
C_5 &= (1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(i - k)) \\
C_6 &= (1, -j, k, -\frac{1}{\sqrt{2}}(1 - k)) \\
C_7 &= (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k))
\end{align*}

Here, every $C_i$ has trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

\begin{align*}
\Omega_1 &= \{\frac{1}{2}(-1 - i + j + k), \frac{1}{\sqrt{2}}(i + k), \frac{1}{\sqrt{2}}(-j + k)\}^{\pm 1} \\
\Omega_2 &= \{-\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{2}(1 + i + j - k)\}^{\pm 1} \\
\Omega_3 &= \{\frac{1}{2}(-1 + i + j - k), \frac{1}{2}(1 - i + j + k), \frac{1}{2}(-1 - i + j - k)\}^{\pm 1} \\
\Omega_4 &= \{\frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j), \frac{1}{\sqrt{2}}(1 + i)\}^{\pm 1} \\
\Omega_5 &= \{\frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(i - k), \frac{1}{\sqrt{2}}(j + k)\}^{\pm 1} \\
\Omega_6 &= \{-j, +i, \frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(1 - k)\}^{\pm 1} \\
\Omega_7 &= \{\frac{1}{\sqrt{2}}(i - j), -\frac{1}{\sqrt{2}}(1 + i), +k, \frac{1}{2}(-1 + i + j + k)\}^{\pm 1}
\end{align*}

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $C := \bigcup_{i=1}^{7} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$ where

$$
S_i = \begin{cases}
K & \text{for } 1 \leq i \leq 5; \\
L & \text{for } i = 6, 7.
\end{cases}
$$
By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 4 according to whether $1 \leq i \leq 5$ or $i = 6, 7$, respectively. The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $C$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{10} Orb_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 15 triangle-factors and 8 quadrangle-factors, namely a $G$-regular solution of HWP(48; 3, 4; 15, 8).

### 3.6 An octahedral solution of HWP(48; 3, 4; 17, 6)

Consider the ten cycles of $Cycle(G)$ defined as follows.

- $C_1 = (1, -\frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(i + k))$
- $C_2 = (1, -\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(-1 + i + j + k))$
- $C_3 = (1, \frac{1}{2}(1 + i - j - k), -\frac{1}{\sqrt{2}}(1 + j))$
- $C_4 = (1, \frac{1}{\sqrt{2}}(-i + j), \frac{1}{\sqrt{2}}(-i + k))$
- $C_5 = (1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - j))$
- $C_6 = (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k))$
- $C_7 = (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 - i - j + k))$
- $C_8 = (1, k, -1, -k)$
- $C_9 = (1, j, -1, -j)$
- $C_{10} = (1, \frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 - i), \frac{1}{2}(1 - i - j + k))$

We note that $Stab(C_i) = V(C_i)$ for $6 \leq i \leq 9$ while all other $C_i$’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G: \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

- $\Omega_1 = \{-\frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(i + k), \frac{1}{2}(-1 - i + j - k)\}^{\pm 1}$
- $\Omega_2 = \{-\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(-1 + i + j + k), \frac{1}{\sqrt{2}}(-1 + i)\}^{\pm 1}$
- $\Omega_3 = \{\frac{1}{2}(1 + i - j - k), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{\sqrt{2}}(j + k)\}^{\pm 1}$
- $\Omega_4 = \{\frac{1}{\sqrt{2}}(-i + j), \frac{1}{\sqrt{2}}(-i + k), \frac{1}{2}(1 - i - j)\}^{\pm 1}$
- $\Omega_5 = \{\frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - j), \frac{1}{\sqrt{2}}(j - k)\}^{\pm 1}$
- $\Omega_6 = \{\frac{1}{2}(-1 - i + j + k)\}^{\pm 1}$
- $\Omega_7 = \{\frac{1}{2}(-1 + i + j - k)\}^{\pm 1}$
- $\Omega_8 = \{k\}^{\pm 1}$
- $\Omega_9 = \{j\}^{\pm 1}$
- $\Omega_{10} = \{\frac{1}{\sqrt{2}}(1 + i), i, \frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j + k)\}^{\pm 1}$

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Lemma 2.2 we can say that $C := \bigcup_{i=1}^{10} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$
where

\[ S_i = \begin{cases} 
K & \text{for } 1 \leq i \leq 5; \\
G & \text{for } 6 \leq i \leq 9; \\
L & \text{for } i = 10.
\end{cases} \]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{48} \) with \( \text{Stab}_G(F_i) = S_i \), hence \( \text{Orb}_G(F_i) \) has length 3 or 1 or 4 according to whether \( 1 \leq i \leq 5 \) or \( 6 \leq i \leq 9 \) or \( i = 10 \), respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 7 \). Thus, recalling that \( C \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( F := \bigcup_{i=1}^{10} \text{Orb}_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{48} - I \) with 17 triangle-factors and 6 quadrangle-factors, namely a \( G \)-regular solution of HWP(48; 3, 4; 17, 6).

4 Dicyclic solutions of two Hamilton-Waterloo problems

In this section \( G \) will denote the dicyclic group of order 24 which is usually denoted by \( Q_{24} \). Thus \( G \) has the following presentation:

\[ G = \langle a, b \mid a^{12} = 1, b^2 = a^6, b^{-1}ab = a^{-1} \rangle \]

Note that the elements of \( G \) can be written in the form \( a^i b^j \) with \( 0 \leq i \leq 11 \) and \( j = 0, 1 \). The group \( G \) has a unique involution which is \( a^6 \) and we will need to consider the following subgroups of \( G \):

- \( H = \langle b \rangle = \{1, b, a^6, a^6b\} \);
- \( K = \langle a^2 \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\} \);
- \( L = \langle a^2b, a^3 \rangle = \{1, a^3, a^6, a^9, a^2b, a^8b, a^5b, a^{11}b\} \).

4.1 A dicyclic solution of HWP(24; 3, 4; 7, 4)

Consider the four cycles of \( Cycle(G) \) defined as follows.

\[ C_1 = (1, a^3b, a^5) \]
\[ C_2 = (1, a^{10}, a^7b) \]
\[ C_3 = (1, a^4, a^8) \]
\[ C_4 = (1, b, a^3b, a) \]

We note that the \( \text{Stab}(C_3) = V(C_3) \) while all other \( C_i \)’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that \( \text{Orb}(C_i) \) is a \( \ell_i \)-cycle decomposition of \( \text{Cay}[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)’s are the symmetric subsets of \( G \) listed below.

\[ \Omega_1 = \{a^3b, a^5, a^2b\}^{\pm 1} \]
\[ \Omega_2 = \{a^2, ab, a^5b\}^{\pm 1} \]
\[ \Omega_3 = \{a^4\}^{\pm 1} \]
\[ \Omega_4 = \{b, a^3, a^4b, a\}^{\pm 1} \]

Now note that the \( \Omega_i \)’s partition \( G \setminus \{1, a^6\} \). Thus, by Theorem 2.3 we can say that \( C := \bigcup_{i=1}^{4} \text{Orb}(C_i) \) is a \( G \)-regular cycle-decomposition of \( K_{24} - I \). Now set \( F_i = \text{Orb}_{S_i}(C_i) \)
where
\[
S_i = \begin{cases} 
  L & \text{for } i = 1, 2; \\
  G & \text{for } i = 3; \\
  K & \text{for } i = 4.
\end{cases}
\]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{24} - I \) with \( Stab_G(F_i) = S_i \), hence \( Orb_G(F_i) \) has length 3 or 1 or 4 according to whether \( i = 1, 2 \) or \( i = 3 \) or \( i = 4 \), respectively.

The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 3 \). Thus, recalling that \( \mathcal{C} \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^{4} Orb_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{24} - I \) with 7 triangle-factors and 4 quadrangle-factors, namely a \( G \)-regular solution of HWP(24; 3, 4; 9, 2).

### 4.2 A dicyclic solution of HWP(24; 3, 4; 9, 2)

Consider the four cycles of \( Cycle(G) \) defined as follows.

\[
C_1 = (1, b, a^6, a^6b) \\
C_2 = (1, a^4b, a^6, a^{10}b) \\
C_3 = (1, a^4, a^7b) \\
C_4 = (1, a^3b, a^8b) \\
C_5 = (a^4, a^7, a^5)
\]

We note that \( Stab(C_i) = V(C_i) \) for \( i = 1, 2 \) while all other \( C_i \)'s have trivial stabilizer. By Lemma 2.2, one can check that \( Orb(C_i) \) is a \( \ell_i \)-cycle decomposition of \( Cay[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)'s are the symmetric subsets of \( G \) listed below.

\[
\begin{align*}
  \Omega_1 &= \{ b \}^\pm 1 \\
  \Omega_2 &= \{ a^4b \}^\pm 1 \\
  \Omega_3 &= \{ a^4, ab, a^5b \}^\pm 1 \\
  \Omega_4 &= \{ a^3b, a^2b, a^5 \}^\pm 1 \\
  \Omega_5 &= \{ a^4, a^2, a^3 \}^\pm 1
\end{align*}
\]

Also here the \( \Omega_i \)'s partition \( G \setminus \{ 1, a^6 \} \), hence \( \mathcal{C} := \bigcup_{i=1}^{5} Orb_G(C_i) \) is a \( G \)-regular cycle-decomposition of \( K_{24} - I \) by Theorem 2.3. Now set:

\[
F_1 = Orb_G(C_1), \quad F_2 = Orb_G(C_2), \\
F_3 = Orb_L(C_3), \quad F_4 = Orb_H(C_4) \cup Orb_H(C_5).
\]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{24} - I \) and we have

\[
Stab_G(F_1) = Stab_G(F_2) = G; \quad Stab_G(F_3) = L; \quad Stab_G(F_4) = H
\]

so that the lengths of the \( G \)-orbits of \( F_1, \ldots, F_4 \) are 1, 1, 3 and 6, respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \geq 3 \). Thus, recalling that \( \mathcal{C} \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^{5} Orb_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{24} - I \) with 9 triangle-factors and 2 quadrangle-factors, namely a \( G \)-regular solution of HWP(24; 3, 4; 9, 2).
5 A special linear solution of HWP(24; 3, 4; 5, 6)

In this section $G$ will denote the 2-dimensional special linear group over $\mathbb{Z}_3$, usually denoted by $SL_2(3)$, namely the group of $2 \times 2$ matrices with elements in $\mathbb{Z}_3$ and determinant one. The only involution of $G$ is $2E$ where $E$ is the identity matrix of $G$. The 2-Sylow subgroup $Q$ of $G$, isomorphic to the group of quaternions, is the following:

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We will also need to consider the subgroup $H$ of $G$ of order 6 generated by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Hence we have:

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

The use of the special linear group $G$ was crucial in [5] to get a Steiner triple system of any order $v = 144n + 25$ with an automorphism group acting sharply transitively an all but one point. Here $G$ will be used to get a $G$-regular solution of the last Hamilton-Waterloo problem left open in [11].

Consider the six cycles of $Cycle(G)$ defined as follows.

- $C_1 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right)$
- $C_2 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \right)$
- $C_3 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \right)$
- $C_4 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right)$
- $C_5 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right)$
- $C_6 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$

Here the stabilizer of $C_i$ is trivial for $i = 1, 6$ while it coincides with $V(C_i)$ for $2 \leq i \leq 5$. By Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $Cay[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

- $\Omega_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}^{\pm 1}$
- $\Omega_2 = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$, $\Omega_3 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right\}^{\pm 1}$
- $\Omega_4 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}^{\pm 1}$, $\Omega_5 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$
- $\Omega_6 = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}^{\pm 1}$
Once again we see that the $\Omega_i$’s partition $G \setminus \{E, 2E\}$, therefore $\mathcal{C} := \bigcup_{i=1}^{6} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = Orb_{S_i}(C_i)$ with

$$S_i = \begin{cases} Q & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ H & \text{for } i = 6. \end{cases}$$

By Remark 2.4, each $F_i$ is a 2-factor of $K_{24} - I$ and we have $Stab_G(F_i) = S_i$ so that the lengths of the $G$-orbits of $F_1, \ldots, F_6$ are 3, 1, 1, 1, 1 and 4, respectively.

The cycles of $F_i$ have length 3 or 4 according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{24} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{6} Orb_G(F_i)$ is a $G$-regular 2-factorization of $K_{24} - I$ with 5 triangle-factors and 6 quadrangle-factors, namely a $G$-regular solution of HWP $(24; 3, 4; 5, 6)$.

References


