

This is a pre print version of the following article:

On the stability loss for an euler beam resting on a tensionless pasternak foundation / Nobili, Andrea; Lanzoni, Luca. - In: ZEITSCHRIFT FUR ANGEWANDTE MATHEMATIK UND PHYSIK. - ISSN 0044-2275. - STAMPA. - 65:4(2014), pp. 797-808. [10.1007/s00033-013-0356-7]

*Terms of use:*

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

18/12/2025 18:01

# On the stability loss for an Euler beam resting on a tensionless Pasternak foundation

Andrea Nobili<sup>a,\*</sup>, Luca Lanzoni<sup>b</sup>

<sup>a</sup>*Department of Mechanical and Civil Engineering, University of Modena and Reggio Emilia, 41122 Modena, Italy*

<sup>b</sup>*University of San Marino, 47890 San Marino*

---

## Abstract

In the present work, the tensionless contact problem of an Euler-Bernoulli beam of finite length resting on a two-parameter Pasternak-type foundation is investigated. Owing to the tensionless character of the contact, the beam may lift-off the foundation and the point where contact ceases and detachment begins, named contact locus, needs be assessed. In this situation, a one-dimensional free boundary problem is dealt with. An extra condition, generally in the form of a homogeneous second-order equation in the displacement and its derivatives, is demanded to set the contact locus and it gives the problem its nonlinear feature. Conversely, the loading and the beam length may be such that the beam rests entirely supported on the foundation, which situation is governed by a classical linear boundary value problem. In this work, the contact evolution is discussed for a continuously varying loading condition, starting from a symmetric layout and at a given beam length, until overturning is eventually reached. In particular, stability is numerically

---

\*Corresponding author

*Email addresses:* `andrea.nobili@unimore.it` (Andrea Nobili),  
`luca.lanzoni@unirsm.sm` (Luca Lanzoni)

assessed through the energy criterion, which is shown to stand for the free boundary situation as well. At overturning, a descending pathway in the system energy appears and stability loss is confirmed.

---

## **1. Introduction**

The problem of a beam or plate resting on an elastic foundation has been extensively investigated in the literature, on account of its implications with the problem of soil-structure interaction [1, 2]. In the vast majority of cases, beam-foundation interaction is modeled through a bilateral constraint, in an attempt to warrant the problem with a linear character. This claim is usually supported on the ground of the beam weight being enough to provide full contact throughout or of the detached region extension being negligible with respect to the contact region (although this does not necessarily entail that its overall effect is indeed negligible). Such claims may be valid in some situations, yet they generally do not stand. In more recent times, the tensionless nature of the contact has been put in the spotlight as a mechanism through which a substantial reduction of the stress in the beam is attained in a natural process [3, 4]. This is especially interesting for design purposes in an earthquake situation [5]. Conversely, beam detachment is a particularly undesired event in the design of railway tracks, as it may lead to derailling [6]. From a mathematical standpoint, tensionless contact brings along non-linearity. Indeed, the governing equations for the beam in contact with and the beam detached from the foundation must be enforced on the relevant regions, which are a problem unknown. On this respect, a free boundary problem (FBP), as opposed to a boundary value problem (BVP), is dealt

with. In particular, for special loading and beam length conditions, the non-contact region, named lift-off region, may vanish. Then, the FBP reduces to a BVP, which is moreover linear. From an operational standpoint, the FBP is recast in terms of a (linear) complementarity problem, i.e. it involves a linear inequality which needs be satisfied on the domain of interest. The usual solution procedure demands setting up a variational formulation on the full domain and then letting a sequence of properly chosen function candidates approximate the solution to the desired degree of accuracy [7]. The situation is much more affordable in the one-dimensional case. Then, it is often convenient to split the domain in two regions (in this framework named the contact and the lift-off region) and directly operate with the regions' boundary (here the contact locus). To this aim, the contact locus needs be set through some extra condition other than the usual boundary conditions, which is named contact locus equation, to be obtained somehow. To retain compatibility with the complementarity problem, one way to obtain the contact locus equation is to set up a variational argument on the action integral, as opposed to the heuristic approach which demands arbitrarily enforcing some property of the contact locus [8]. In so doing, the Erdman-Weierstrass or corner condition provides the contact locus equation [9]. The problem solution is thus an action integral (or free energy) critical point. However, it is remarked that despite the fact that the free energy is the sum of quadratic functionals (in the contact, lift-off and free soil regions) which are, individually, positive-definite, the solution point may not be a minimum, owing to the role of the free boundary.

In this paper, the tensionless contact for an Euler-Bernoulli beam on

an elastic foundation is studied under a point force (whose magnitude is irrelevant [9]) located at midspan and a continuously increasing point couple located at either of the beam ends, which brings the system off the initial symmetric layout. The foundation is a two-parameter Pasternak soil. As well-known, the simplest soil model was proposed by Winkler [10], which represented the foundation through a series of closely-spaced independent bilateral elastic springs. According to this assumption, the reacting pressure at each point of the soil surface is directly proportional to the deflection through a material constant (Winkler modulus), thus endowing the soil behavior with an entirely local character. In order to overcome the limits of the Winkler assumption and in view of obtaining a more accurate description of the soil-structure interaction, many other soil models were proposed. For instance, the soil can be modeled as a homogeneous isotropic elastic half space. Since the half space yields a nonlocal response, the contact problem of an elastic structure resting on a half space is governed by an integro-differential equation, which is difficult to deal with. Therefore, soil models endowed with a weakly nonlocal (e.g. gradient-type) behavior are generally preferred when studying soil-structure interaction problems [11].

In order to introduce some non-locality in the Winkler model, Wieghardt [12] proposed an integral dependence of the soil displacement at a point upon the contact pressure on the whole contact region through an exponentially decaying kernel. When the soil model is recast in terms of the contact pressure, it appears that the latter is related to the deflection and its second derivative at a point, i.e. it is a weakly nonlocal model. Pasternak [13] proposed to model the foundation as an elastic spring layer surmounted by

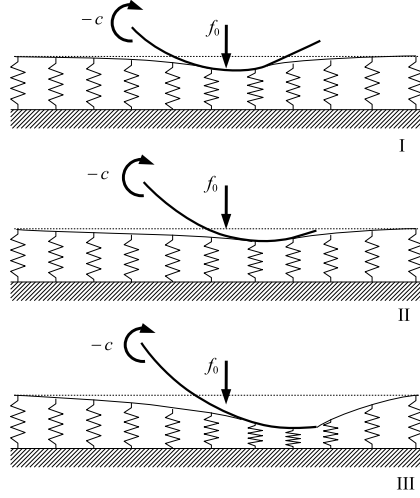


Figure 1: Contact scenarios: I force within the contact interval; II force within either of two lift-off intervals; III force within the single lift-off interval

an elastic incompressible shear-deformable membrane. At the limiting case of vanishing shear modulus, the Winkler model is retrieved. The Pasternak model is governed by an ODE which is analogous to that appearing in the Wieghardt model [14].

In the present work, different contact scenarios are encountered until overturning is eventually reached, namely (I) the force rests within the contact interval, (II) the force rests outside of it, in either of the two lift-off regions, (III) the force stands within the only lift-off region, the remaining part of the beam being supported by the foundation (Fig.1). Mathematically, scenarios I and II involve a FBP while scenario III is a FBP with a partially fixed boundary. Solutions are numerically checked against the energy criterion for stability. The condition on the verge of overturning is also investigated as well as the case of a short beam.

The paper is organized as follows: Sec.2 sets the problem governing equations and boundary conditions (BCs); scenarios I to III are discussed in Secs.2.1–2.3 and Sec.2.4 illustrates the case of a short beam initially fully supported by the foundation. Finally, conclusions are drawn in Sec.3.

## 2. The Free Boundary Problem

Let us consider a rectilinear Euler–Bernoulli beam resting on a tensionless two-parameter elastic soil of the Pasternak type. The beam is subjected to a loading distribution  $q$ , positive downwards, which is equilibrated by a contact pressure distribution  $p$ , that is a problem unknown. The latter, according to the Pasternak model, is given by [2]

$$p(x) = kw - k_G \frac{d^2 w}{dx^2}, \quad (1)$$

where  $k$  is the soil reaction modulus (i.e. Winkler modulus) and  $k_G$  is a measure of the gradient-type nonlocal response. The Pasternak response boils down to the Winkler reaction pressure taking  $k_G = 0$ . Owing to the tensionless character of the foundation, it must be

$$p(x) > 0, x \in [X_1, X_2], \quad (2)$$

being  $[X_1, X_2]$  the contact region, and  $p(x) = 0$  otherwise. In the general case, the contact pattern is unknown, and the contact region may be composed of several intervals of the form (2), in the so called *discontinuous contact scenario*. Here, it is assumed that a single contact region exists, which situation amounts to a *continuous contact scenario* (Fig.2). The problem is entirely recast in terms of the displacement  $w$ , which is common between the beam

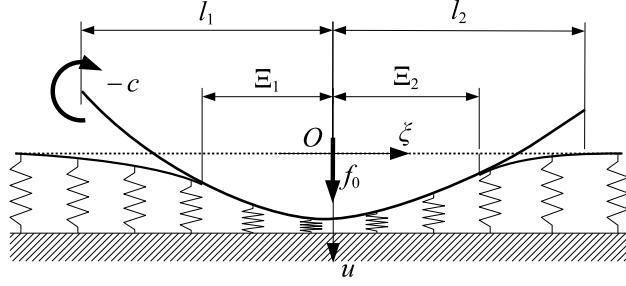


Figure 2: Force within the contact region and lift-off on either sides of it (dimensionless quantities)

and the soil in the contact region (superscript c). In the lift-off region, that is the region where contact no longer holds, two displacement functions are introduced, namely  $w^l$  for the beam and  $w^s$  for the soil, being

$$w^s > w^l, \quad (3)$$

to avoid interpenetration. Determining the boundary of the contact region, i.e. the contact loci, requires solving a second degree homogeneous form in the displacement and its derivatives [9]. In this respect, the problem is nonlinear, even within a small displacement and deformation assumption. In fact, multiple candidates for the contact loci are found, which must be checked against conditions (2, 3).

Introducing the static wavelength  $\beta^{-1}$

$$\beta^4 = \frac{k}{4EI}, \quad (4)$$

the following dimensionless quantities are let:

$u = \beta w, \xi = \beta x$  displacement and abscissa (origin at the point force),

$\sigma = \beta q/k, \pi = \beta p/k$  applied loading and contact pressure,



$\Xi_1 = \beta X_1$ ,  $\Xi_2 = \beta X_2$  left and right contact locus position,

$l_1 = \beta L_1$ ,  $l_2 = \beta L_2$  left and right beam end position.

The governing equations are

$$\frac{1}{4}u_4 = \sigma - \pi, \quad \pi = u - \alpha u_2, \quad \xi \in [\Xi_1, \Xi_2], \quad (5a)$$

$$\frac{1}{4}u_4 = \sigma^l, \quad \xi \in (l_1, \Xi_1) \cup (\Xi_2, l_2) \quad (5b)$$

$$- \alpha u_2 + u = \sigma^s, \quad \xi \in (-\infty, \Xi_1) \cup (\Xi_2, +\infty), \quad (5c)$$

respectively Eqs.(4,7,9) of [9]. Here, the  $n$ -th derivative with respect to  $\xi$  is denoted by the subscript  $n$  and  $\alpha = \beta^2 k_G/k$ . Besides, superscript c has been omitted in (5a) and it is assumed  $\Xi_1 > l_1, \Xi_2 < l_2$  with  $l_2 - l_1 = l = \beta L$  beam total (dimensionless) length. The soil is taken to extend unbounded on either side of the beam, so that it attains zero displacement at infinity. As determined through variational arguments [8, 9], the BCs at the contact locus are

$$u^c = u^l = u^s, \quad u_1^c = u_1^l = u_1^s, \quad u_2^c = u_2^l, \quad u_3^c = u_3^l. \quad (6)$$

As explained in [9] and unlike the general case, the contact locus equation for the Pasternak soil may be expressed as the product of two linear conditions and hereinafter only Kerr's condition is considered, i.e.  $u_1^c = u_1^s$ .

When the beam length is such that the contact locus falls outside the beam span, for instance  $\Xi_2 > l_2$ , then contact extends up to  $l_2$  and it is no longer necessary to determine  $\Xi_2$ . A new set of BCs is enforced at  $l_2$ , namely

$$u^c = u^s, \quad u_1^c = u_1^l, \quad u_2^c = 0, \quad \frac{1}{4}u_3^c = \alpha(u_1^c - u_1^s). \quad (7)$$

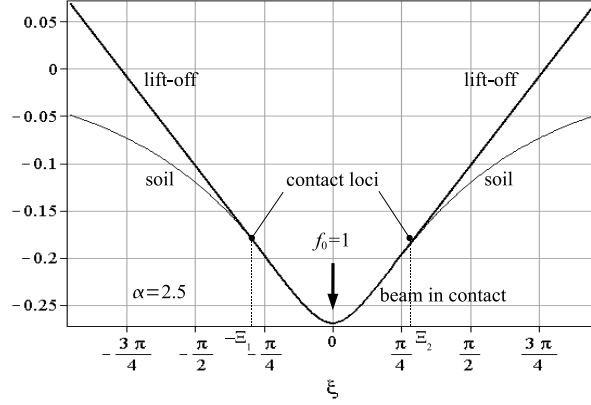


Figure 3: Deformed layout for the the beam-soil system: scenario I ( $\alpha = 2.5, f_0 = 1$ )

The last equation of (7) shows the well-known result that a concentrated force develops at the beam end supported by a Pasternak soil [11]. For a tensionless soil, the extra condition

$$u_1^c - u_1^s > 0 \quad (8)$$

must be enforced along with (2). The governing system of ODEs (5), with the BCs (6) or (7), is hereinafter solved when the loading is given by a unit point force at the beam midspan,  $f_0 = 1$ , together with a point couple at the beam left end  $c$ , positive when counterclockwise (Fig.2). Focus is set upon determining the limiting values of the end couple which trigger the system into a different scenario, i.e. from lift-off on either sides to soil support at one end until, eventually, overturning.

### 2.1. Point force within the contact region

Let us consider the case of a beam subjected to a unit point force acting at midspan with no end couple first. This symmetric layout is the first of a

whole class characterized by the fact that the applied point force falls within the contact interval. By increasing (in absolute value terms) the applied end couple  $c$ , the contact interval drifts away from midspan towards the unloaded end until the point force rests exactly at the left contact locus. We shall refer to this class of loading patterns as scenario I and to  $c^I$  as the limiting couple which sets the point force at  $\Xi_1$ . Besides, within scenario I, the rest of the beam may be either supported by the soil or lifting off it according to its length  $l$ . Indeed, by solving Eqs.(5) together with the BCs (6) in the symmetric layout, the contact loci  $\Xi_1 = \Xi_2 = 0.84239465586$  are found (it is taken  $\alpha = 2.5$  throughout). Assuming  $l > 2\Xi_2$ , lift-off takes place and this condition is represented in Fig.3. Conversely, beams of length  $l < 2\Xi_2$  are completely supported by the foundation, and no lift off occurs. Then, the solution must be rejected and Eqs.(5) solved with the BCs (7). In this circumstance, point reaction forces develop at the contact boundary. This situation is addressed at Sec.2.4.

## 2.2. Point force within either of the two lift-off regions

When the end couple  $c$  is further decreased, i.e.  $c < c^I$ , the contact region drifts away from under the unit force. The latter stands now applied in the lift-off region and it conveys negative work. This contact situation is referred to as scenario II so that  $c^I$  is the limiting couple between scenario I and II. Mathematically,  $c^I$  is characterized by the requirement that  $\Xi_1 = 0$  in either scenario I or scenario II. Solving Eqs.(5) under the BCs (6) and enforcing  $\Xi_1 = 0$ , the pair  $(c^I, \Xi_2)$  is obtained. In particular, Fig.4 plots the solution curves for the last of the BCs (6) evaluated at the left and right contact loci, that is at  $\Xi_1 = 0$  and  $\Xi_2$ : intersections of such curves lend the solution

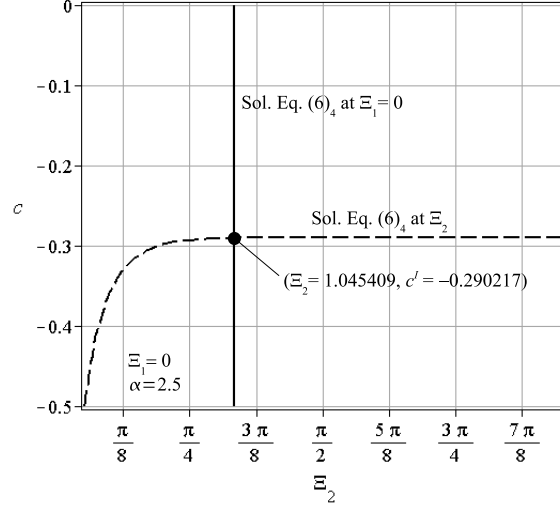


Figure 4: Solution curves for the last of the BCs (6) in terms of  $c$  and  $\Xi_2$ , having set  $\Xi_1 = 0$  ( $\alpha = 2.5, f_0 = 1$ )

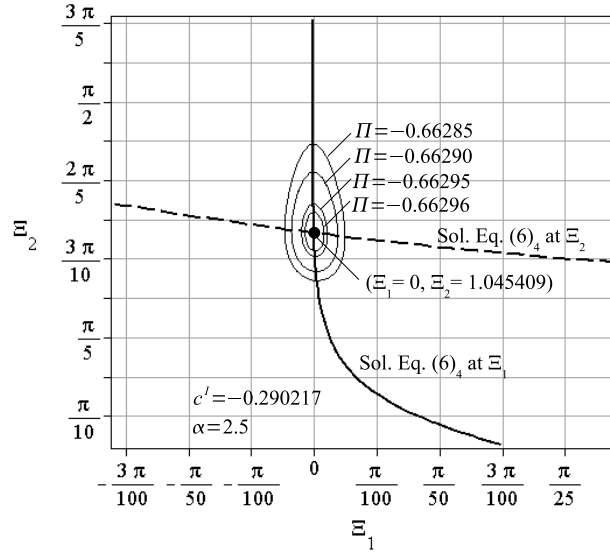


Figure 5: Solution curves for the last of the BCs (6) in terms of contact loci  $\Xi_1, \Xi_2$  and energy level curves

points. The plot is here drawn in terms of the end couple  $c^I$  versus the right contact locus  $\Xi_2$ . For  $\alpha = 2.5$ , it is  $c^I = -0.29021719$  and  $\Xi_2 = 1.04540974$  (it is remarked that couples are positive counterclockwise). As a back-check, the same solution curves are drawn in Fig.5 for  $c = c^I$  in terms of contact loci positions  $\Xi_1, \Xi_2$ . It is easily seen that, as expected, intersection takes place at  $\Xi_1 = 0$ . As already remarked, the contact locus equation is determined through a variational argument so that it makes the free energy  $\Pi$  stationary. In fact, superposed onto the solution curves are the energy level curves, which show that the alleged solution is indeed a minimum. It is observed that the same extremum property does not hold in the  $(c, \Xi_2)$  plane. Furthermore, the solution is acceptable inasmuch as  $l_2 \geq 1.04540974$ , for only then lift-off may take place. The deformed layout is presented in Fig.6 along with slope, dimensionless bending moment, shearing force and (negative) contact pressure distribution. As expected, a linearly varying slope in the left lift-off beam region is retrieved, due to the fact that only a constant bending moment acts at the left end. The corresponding value of the shearing force is zero. It is worth noticing that the shearing force jumps in correspondence of the left contact locus  $\Xi_1$  (i.e. at  $\xi = 0$ ) owing to the presence of the point force  $f_0$ . The reacting pressure  $\pi$  occurring inside the contact region is positive and thereby acceptable; besides, it exhibits a monotonic decreasing trend from left to right.

### 2.3. *Overturning condition*

Decreasing the end couple  $c$  beyond  $c^I$  causes the contact region to move to the right, in an attempt to increase the contact pressure moment resultant. This is possible as long as the beam is long enough. Eventually, the

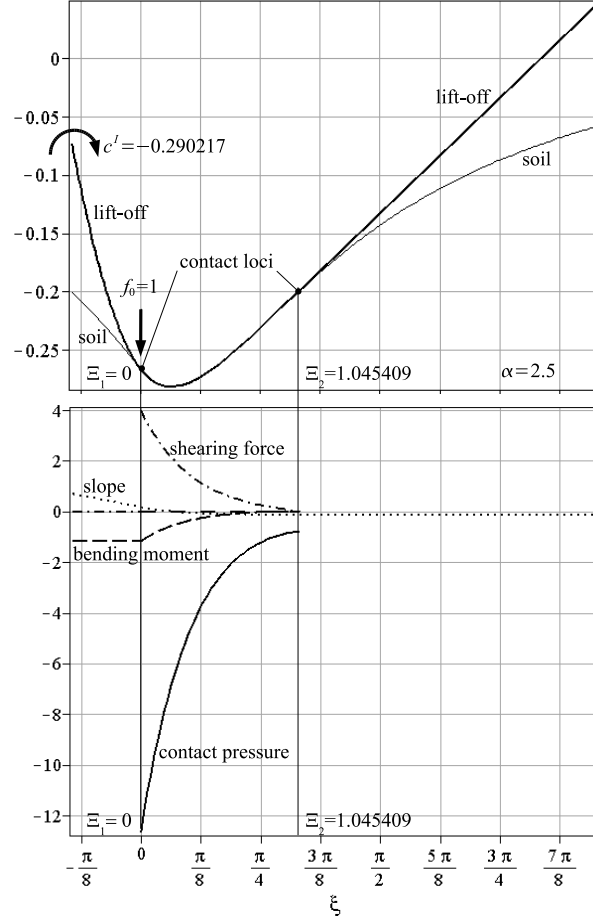


Figure 6: Beam and soil deflection, beam slope, dimensionless bending moment, shearing force and (negative) contact pressure distribution (i.e.  $-\pi$ ) at the boundary between scenario I and II ( $c = c^I$ )

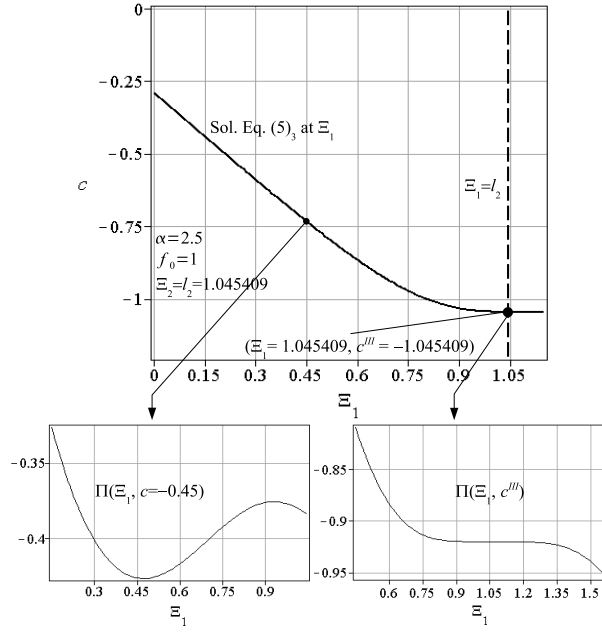


Figure 7:  $c$  vs.  $\Xi_1$  for a beam whose right end rests supported on the foundation; two energy plots  $\Pi$  vs.  $\Xi_1$  are also given: at  $\Xi_1 = l_2$  a descending pathway in the energy appears.

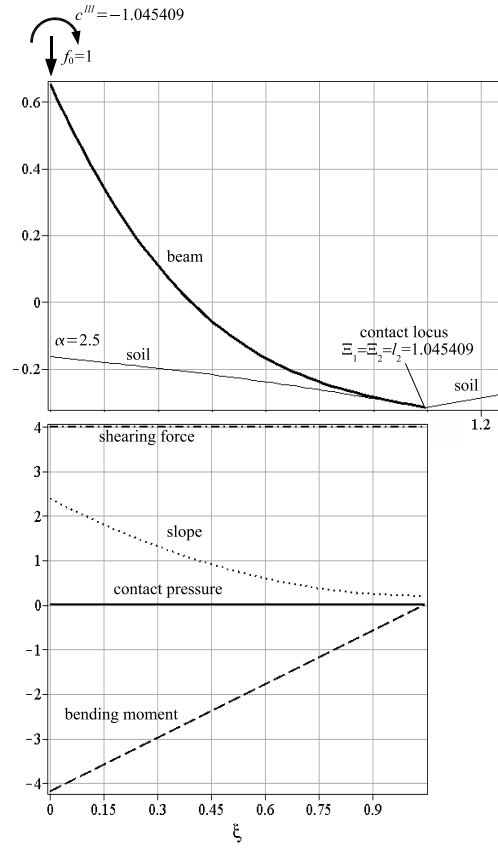


Figure 8: Beam on the verge of overturning ( $c^{III} = -f_0 l_2$ )



beam right end is reached such that  $\Xi_2 = l_2$ . This situation is addressed as scenario III. Therefore, for  $l_2 < 1.04540974$ , scenario III follows directly from scenario I and this happens before  $c$  attains the value  $c^I$ . Scenario III solution candidates are obtained solving Eqs.(5) with BCs (6) at the single contact locus  $\Xi_1$  and with BCs (7) at the right beam end  $l_2$ . Further decrease of the end couple leads to a narrowing of the contact region, i.e.  $\Xi_1$  moves towards  $l_2$ . Fig.7 plots the end couple  $c < c^I$  against the contact locus position  $\Xi_1$ , as well as the right end position  $l_2$ . The curve is obtained evaluating the last of the BCs (6) for  $\Xi_2 = l_2$ . It is observed that the curve is monotonic decreasing and that a solution is always possible for any  $\Xi_1 \leq \Xi_2$ . However, it rests to be determined whether such solutions are also stable or rather overturning sets in at some point. This is accomplished evaluating the free energy as a function of  $\Xi_1$ . It is found that the free energy always exhibits a minimum along the solution curve provided that  $\Xi_1 < l_2$ . In particular, the limiting case  $\Xi_1 = l_2$  corresponds to the onset of overturning for the beam and the corresponding end couple  $c^{III}$  is a minimum beyond which equilibrium no longer stands. This is confirmed by the appearance of a descending pathway in the energy curve, corresponding to increasing  $\Xi_1$ . The beam and soil deformed layouts are presented in Fig.8 together with slope, bending moment, shearing force and contact pressure. As expected, within a linear theory of deformation, the minimum end couple  $c^{III}$  equals the applied force times the beam length, i.e.  $c^{III} = -f_0 l = -1.04540973965$ . Then, it is found that for a beam on the verge of overturning resting on a Pasternak soil, the contact region degenerates into a single point and the contact pressure distribution vanishes. Thus, equilibrium is granted through the concentrated reaction

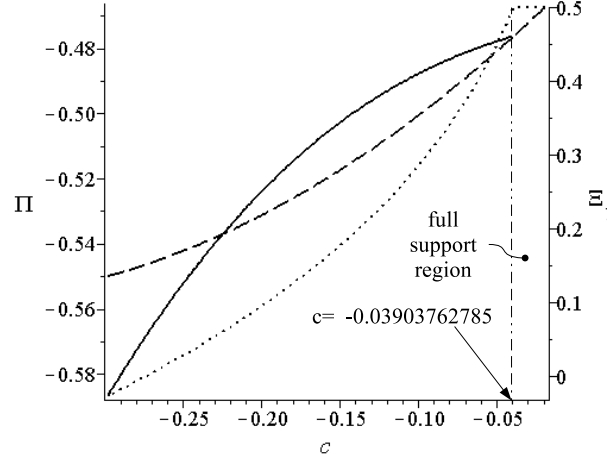


Figure 9: Free energy  $\Pi$  (left axis) and contact locus position  $\Xi_1$  (right axis, dotted curve) vs.  $c$  for a fully supported (dash) and a partially lifting-off (solid) beam ( $l_1 = l_2 = 0.5$ )

which develops at the contact region right boundary, that is where slope discontinuity is met. That the contact pressure distribution role in equilibrating the system evaporates on the verge of overturning is also proved by the fact that the solution curve exhibits a zero slope at  $\Xi_1 = l$ , i.e. small changes in the contact region bear no effect on equilibrium. Finally, Fig.7 shows that for any  $c^{III} < c < c^I$  the solution corresponds to an energy minimum point and it is thereby stable. Conversely, on the verge of overturning, that is for  $c = c^{III}$  the solution is located on a flex point for the energy, which accounts for a loss of stability.

#### 2.4. Short beam

When the beam is short, i.e.  $l < 2\Xi_2$ , no lift-off takes place in the symmetric layout and scenario I sees the beam being initially fully supported by the foundation. For growing values of  $|c|$ , the (left) boundary point reaction

decreases until it eventually reverses its sign. Then, condition (8) no longer holds and lift-off sets in at the left end, while contact extends up to the beam right end. Here, unlike scenario III, the point force is still within the contact region. It is emphasized that failing of condition (8) takes place *before* condition (2) is violated. In this respect, it is remarked that condition (8) is often neglected in the literature. Fig.(9) shows the free energy  $\Pi$  as a function of the point couple  $c$ . The free energy for the beam at partial lift-off (solid curve) is displaced so that it equals the free energy of the fully supported beam (dash curve) when  $\Xi_1 = l_1$  (which happens for  $c = -0.03903762785$ ). In the region  $\Xi_1 > l_1$  full support stands while in the region  $\Xi_1 < l_1$  condition (8) is violated so that partial lift-off sets in. In this latter situation, the free energy of the fully supported beam is lower than the free energy for the beam in partial lift-off. However, it is emphasized that here no branching phenomenon occurs so that the energy criterion cannot be appealed to.

### 3. Discussion

In the present work, three contact scenarios for the problem of a E-B beam loaded by a unit force at midspan and a concentrated couple at one end, resting on a tensionless Pasternak soil have been investigated. The beam is long enough to accommodate lift-off. The tensionless nature of the contact gives the problem a nonlinear nature. Such scenarios amount to three classes of solutions, according to the amount of the external couple. For moderate values, the force rests within the contact region until it stands right on top of either of the contact loci. For higher couple values, the contact region moves away from the force, which now stands in the lift-off region until, eventually,

either of the contact loci hits the beam end. From that point, a final scenario deploys wherein equilibrium holds despite a monotonic decreasing (with the couple) contact region length, owing to a parallel increase of the concentrated force that develops at the beam end supported by the foundation. A limiting value for the couple is found at which equilibrium stability is lost and the beam stands on the verge of overturning. Such limiting configuration occurs in a zero contact length condition. Then, equilibrium is granted only through the concentrated reaction force and the supported beam end. Within a small displacement theory, such limiting couple is given by the distance between the point force and the beam supported end (times the unit force). Stability loss is numerically confirmed by a descending pathway in the free energy. For a short beam, an initial fully supported condition is followed, at increasing values of the end couple, by a partial lift-off condition. Such conditions are ruled by different governing equations so that here the energy criterion is of no avail in determining the limiting couple.

## References

- [1] M. Hetenyi, Beams on elastic foundations, Ann arbor: the university of Michigan press, 1946.
- [2] A. P. S. Selvadurai, Elastic analysis of soil-foundation interaction, vol. 17 of *Developments in geotechnical engineering*, Elsevier scientific publishing company, 1979.
- [3] I. Coskun, The response of a finite beam on a tensionless Pasternak foun-

- dation subjected to a harmonic load, *European Journal of Mechanics* 22 (2003) 151–161.
- [4] Z. Celep, K. Guler, Axisymmetric forced vibrations of an elastic free circular plate on a tensionless two parameter foundation, *Journal of sound and vibrations* 301 (2007) 495–509.
  - [5] C.-S. Yim, K. Chopra, Earthquake response of structures with partial uplift on Winkler foundation, *Earthquake engineering and structural dynamics* 12 (1984) 263–281.
  - [6] A. K. Mallik, S. Chandra, A. B. Singh, Steady-state response of an elastically supported infinite beam to a moving load, *Journal of sound and vibrations* 291 (2006) 1148–1169.
  - [7] N. Kikuchi, Beam bending problems on a Pasternak foundation using reciprocal variational inequalities, *Quarterly of applied mathematics* 34 (1980) 91–108.
  - [8] A. D. Kerr, On the derivation of well posed boundary value problems in structural mechanics, *International Journal of Solids and Structures* 12 (1) (1976) 1–11.
  - [9] A. Nobili, Variational approach to beams resting on two-parameter tensionless elastic foundations, *Journal of Applied Mechanics* 79 (2) (2012) 021010, transactions ASME.
  - [10] E. Winkler, *Die Lehre von der Elastizität und Festigkeit: mit besonderer Rücksicht auf ihre Anwendung in der Technik, für polytechnische*

Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architekten, etc.,  
H. Dominicus, 1867.

- [11] A. D. Kerr, Elastic and viscoelastic foundation models, *Journal of Applied Mechanics* 31 (1964) 491–498.
- [12] K. Wieghardt, Über den Balken auf nachgiebiger Unterlage, *Zeitschrift für angewandte Mathematik und Mechanik (ZAMM)* 2 (3) (1922) 165–184.
- [13] P. L. Pasternak, On a new method of analysis of an elastic foundation by means of two foundation constants, *Gostudarstvennoe Izdatelstvo Literaturi po Stroitelstvu i Arkhitekture* (1954) in russian.
- [14] G. Ruta, I. Elishakoff, Buckling of a column on a Wieghardt foundation, *Zeitschrift für angewandte Mathematik und Mechanik (ZAMM)* 86 (8) (2006) 617–627.