SUPERPOSITION PRINCIPLE FOR THE TENSIONLESS CONTACT OF A BEAM RESTING ON A WINKLER OR A PASTERNAK FOUNDATION

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ABSTRACT

A Green function based approach is presented to address the nonlinear tensionless contact problem for beams resting on either a Winkler or a Pasternak two-parameter elastic foundation. Unlike the traditional solution procedure, this approach allows determining the contact locus position independently from the deflection curves. In so doing, a general nonlinear connection between the loading and the contact locus is found which enlightens the specific features of the loading that affect the position of the contact locus. It is then possible to build load classes sharing the property that their application leads to the same contact locus. Within such load classes, the problem is linear and a superposition principle holds. Several applications of the method are presented, including symmetric and non-symmetric contact layouts, which can be hardly tackled within the traditional solution procedure. Whenever possible, results are compared with the existing literature.

Keywords: Tensionless contact, Green function, two-parameter elastic foundation

INTRODUCTION

The contact problem for beams resting on elastic foundations has long attracted considerable attention, given its relevance in describing soil-structure interaction (Hetenyi 1946; Selvadurai 1979). In particular, a very extensive literature exists concerning beams resting on one, two and three-parameter elastic foundations (Kerr 1964). The existing literature is for the most part devoted to considering contact as a bilateral constraint, which fact limits the validity of the analysis to situations where lift-off plays a minor role. However, in so doing, the problem retains a valuable linear character and the superposition principle holds.

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When lift-off becomes an important feature, tensionless contact must be reverted to at the expense of the problem linearity. From a mathematical standpoint, tensionless contact determines a free-boundary problem (Kerr 1976; Nobili 2012).

Historically, interest in tensionless contact between a beam and a foundation arose in connection with railway systems. In this respect, Weitsman (1971), Lin and Adams (1987) and recently Chen and Chen (2011) considered detachment and stability for the problem of tensionless contact under a moving load. Besides, much research on tensionless structure-foundation contact is devoted to assessing its role in reducing the structural stress in a seismic event (Celep and Güler 1991; Psycharis 2008). Recently, Coskun (2003) studied forced harmonic vibrations of a finite beam supported by a tensionless Pasternak soil, while Zhang and Murphy (2004) studied a finite beam in tensionless contact in a non-symmetric contact scenario. Tensionless contact for an infinite beam in a multiple contact scenario was investigated by Ma et al. (2009a) and Ma et al. (2009b). An extensive body of literature exists regarding numerical strategies specifically devised to deal with tensionless contact. Recently, Sapountzakis and Kampitsis (2010) considered a boundary element method for beam-columns partly supported on a Winkler and, later (2011), a three-constant soil model.

The classic approach to solving a tensionless contact problem for a beam on an elastic foundation consists of integrating the deflection curves for the beam in contact, the beam in lift-off and the soil, and then matching the solutions at the yet unknown contact locus, that is the point where contact ceases and lift-off begins (Weitsman 1970; Kerr and Coffin 1991). This approach suffers from two major shortcomings. On the one hand, the procedure initially assumes a contact layout and then proceeds to determining the relevant quantities within such layout. It then remains to be checked that results are consistent with the assumptions. On the other hand, contact loci positions are determined through deflection curves integration. Since the general integrals of the governing equations depend on the loading, it appears that results are restricted to one particular loading.

In this paper, a Green function approach is adopted. Unlike the classic approach, this method consists of first determining the contact locus through a nonlinear equation and then solving the linear problem for the deflection curves. In fact, only the first stage is here presented, the second being a classic problem. Although the method still requires some assumptions concerning the lay-out of the contact, nonetheless such assumptions are somewhat relaxed and a general connection between the contact locus and a family of loadings is obtained, so much so that a form of superposition is also retrieved. It is emphasized that this procedure differs from the integral approach of Tsai and Westmann (1967), which is still based on the Green function and yet it aims at determining the deflection curves and the contact locus in one stage.

THE FREE-BOUNDARY PROBLEM

The tensionless contact problem for a Euler–Bernoulli (E-B) beam resting on a tensionless elastic foundation is first stated in its simplest form, concerning a Winkler soil in a symmetric contact scenario (Fig.1). Let $[-X, X]$ denote the contact interval and $X > 0$ be the *contact locus*, i.e. the beam rests supported on the soil up to abscissa X and then it detaches from it. The beam detached from the soil is often addressed as lifting off the soil. The free soil extends beyond X to infinity. Here, the inverse of a reference length is introduced as the ratio between the soil modulus k and the beam flexural rigidity EI, i.e. $\beta^4 = k(4EI)^{-1}$. Then, the problem is cast in dimensionless form: $\Xi = \beta X$ is the dimensionless contact locus position and $u = \beta w$ denotes the beam dimensionless displacement. The beam displacement function, u, restricted to the contact interval $I^c = [0, \Xi]$ and to the lift-off interval $I^l = (\Xi, l]$, is denoted by u^c and u^l, respectively. $2l = 2\beta L$ is the beam dimensionless length and u^s is the soil dimensionless displacement in the unbounded region $I^s = [\Xi, +\infty)$, which is relevant for the Pasternak soil alone. Besides, $\sigma^c = \beta q^c/k$ and $\sigma^l = \beta q^l/k$ are the dimensionless loadings acting in I^c and I^l, respectively. In the contact interval I^c, the beam rests entirely supported on the soil and the governing equation reads

$$\frac{1}{4}\left(u^c\right)^{(iv)} + u^c = \sigma^c, \tag{1}$$

where superscripts within parenthesis denote the differentiation order with respect to ξ. To shorten notation, it is expedient to write the k-th derivative $\left(u^c\right)^{(k)}$ with respect to ξ as u^c_k. The problem boundary conditions (BCs) due to symmetry are

$$u_1^c(0) = 0, \qquad u_3^c(0) = 0, \tag{2}$$

while the BCs at the contact locus Ξ, enforce continuity for the beam of the bending moment and of the shearing force

$$u_2^c(\Xi) = u_2^l(\Xi), \quad u_3^c(\Xi) = u_3^l(\Xi). \tag{3}$$

However, unlike an ordinary boundary value problem (BVP), here the contact locus is a problem unknown, whence a further condition is demanded for its placing. This condition, named *contact locus equation*, enforces displacement continuity with the Winkler foundation (which is here assumed load free), i.e.

$$u^c(\Xi) = 0. \tag{4}$$

In more general terms, the problem may be rewritten formally as

$$D^c u^c = \sigma^c \tag{5}$$

where D^c denotes the differential operator embodying the dimensionless governing equation in the contact region I^c, with its boundary conditions.

THE GREEN FUNCTION APPROACH

In this paper, a new solution procedure is introduced which takes advantage of the Green function to obtain an explicit connection between the loading and the contact locus position. Let the adjoint problem for Eq.(5) be considered

$$\tilde{D}^c G(\xi, \zeta) = \delta(\xi, \zeta), \tag{6}$$

where $\delta(\xi, \zeta)$ is Dirac's delta function about $\xi = \zeta$ and $\tilde{D}^c$ the adjoint operator. Let n indicate the order of the operator D^c, i.e. $n = 4$ for both the Pasternak and the Winkler models. It is worth recalling that the Green function G is determined assuming homogeneous boundary conditions at the boundary ∂I^c and it is thereby independent of the behavior in the lift-off region. The latter comes into play in the form of a boundary term $BT(\xi, \zeta)$. Furthermore, a over-determined system becomes an under-determined problem for the Green function. It is then possible to write the displacement at a point ζ in the contact region as

$$u^c(\zeta) = \int_{I^c} \sigma^c(\xi) G(\xi, \zeta) d\xi \tag{7}$$

and, accordingly, the condition setting the contact locus. For instance, for a Winkler foundation, it is

$$u^c(\Xi) = \lim_{\zeta \to \Xi} \int_{I^c} \sigma^c(\xi) G(\xi, \zeta) d\xi - [BT(\xi, \Xi)]_{\xi=0}^{\Xi} = 0. \tag{8}$$

Here, boundary terms are algebraic and have been gathered in $BT(\xi, \Xi)$. Eq.(8) sets an integral connection between the applied loading and the contact locus Ξ which has a three-fold purpose. First, it may be employed to test a given load distribution against the contact locus Ξ. Second, it may be employed to build the loading classes $\mathcal{Q}_{\mathcal{X}}$, whose elements share the property that their application produces the same set of contact loci $\mathcal{X} = \{\Xi_j\}$. Then, the nonlinear contact problem of a beam resting on a tensionless two-parameters elastic soil may be actually solved for any one representative of the load class, the solution for the other load members of that class being obtained by linear combination. The third purpose of the condition is to provide the contact locus without recurring to the actual integration of the deflection curves.

TENSIONLESS WINKLER-TYPE SOIL

Let us first consider the case of a E-B beam resting on a tensionless Winkler soil and acted upon by a line load σ^c (the resultant of which is indeed irrelevant owing to the homogeneous nature of the BC setting the contact locus) possibly extending up to (though vanishing at) the contact locus Ξ, in a symmetric continuous contact scenario. Here, the BCs (3) are homogeneous. The boundary term reads

$$BT = \frac{1}{4} \left[u_3^c G - u_2^c G' + u_1^c G'' - u^c G''' \right]_0^{\Xi}. \tag{9}$$

Here, prime denotes differentiation with respect to ξ, while G is shorthand for $G(\xi, \zeta)$. It is easily seen that to warrant the vanishing of the boundary term, the Green function has to be subjected to symmetric conditions at $\xi = 0$

$$G'(0, \zeta) = G'''(0, \zeta) = 0 \tag{10}$$

and to the single condition

$$G''(\Xi, \zeta) = 0. \tag{11}$$

such that the beam slope $u_1^c(\Xi)$ drops out the boundary term. This result holds in general, even when the loading extends beyond the contact locus, which amounts to saying that the Green function is entirely independent of the lift-off part. The problem for the Green function is underdetermined and it possesses one free integration parameter.

The ODE for the Green function is

$$\frac{1}{4}G^{(iv)}(\xi, \zeta) + G(\xi, \zeta) = \delta(\xi, \zeta), \tag{12}$$

whose general solution is written as

$$G(\xi, \zeta) = \left\{ \begin{array}{ll} a_i(\zeta, \Xi), & \xi < \zeta \\ b_i(\zeta, \Xi), & \xi > \zeta \end{array} \right\} \eta_i(\xi), \quad i = 1, \ldots, n. \tag{13}$$

Here, $\{\eta_i(\xi)\}$ is the fundamental set and, for a Winkler soil,

$$\{\eta_i(\xi)\} = \{e^{\xi}\cos\xi, e^{\xi}\sin\xi, e^{-\xi}\cos\xi, e^{-\xi}\sin\xi\}. \tag{14}$$

Hereinafter, a summation convention is assumed for twice repeated subscripts, ranging from 1 to n. Let us further enforce the BC

$$G'''(\Xi, \zeta) = 0, \tag{15}$$

whence a self-adjoint formulation for G is set. Since the problem is self adjoint, the Green function is symmetric as it allows exchanging the role of ξ and ζ. Through Eq.(13), the contact zone displacement is given by

$$u^c(\zeta) = a_i(\zeta,\Xi)\int_0^\zeta \sigma^c(\xi)\eta_i(\xi)d\xi + b_i(\zeta,\Xi)\int_\zeta^\Xi \sigma^c(\xi)\eta_i(\xi)d\xi, \quad \zeta \in [0,\Xi]. \tag{16}$$

In particular, letting $\zeta \to \Xi$, it is $u^c(\zeta) \to 0$ according to Eq.(4). Letting

$$F(\Xi) = \alpha_i(\Xi)A_i(\Xi), \tag{17}$$

where

$$A_i(\Xi) = a_i(\Xi,\Xi), \qquad \alpha_i(\Xi) = \int_0^\Xi \sigma^c(\xi)\eta_i(\xi)d\xi, \tag{18}$$

it is $F(\Xi) = 0$. It is remarked that Eqs.(18) should be taken in a limiting sense as $\zeta \to \Xi$, although direct substitution is equally permitted for the Winkler foundation. In particular, explicit expressions are available for the functions A_i, namely

$$A_1 = A_3 = 2\Lambda^{-2}\cos(\Xi)\cosh(\Xi), \tag{19a}$$

$$A_2 = -A_4 = 2\Lambda^{-2}\sin(\Xi)\sinh(\Xi), \tag{19b}$$

having let the nonnegative quantity $\Lambda^2 = \sin(2\Xi) + \sinh(2\Xi)$. Eq.(17), with Eqs.(14) and (19), may be rewritten as

$$F(\Xi) = \alpha_+(\Xi)\cos(\Xi)\cosh(\Xi) + \alpha_-(\Xi)\sin(\Xi)\sinh(\Xi), \tag{20}$$

where $2\alpha_+(\Xi) = \alpha_2(\Xi) + \alpha_4(\Xi)$, $2\alpha_-(\Xi) = \alpha_1(\Xi) - \alpha_3(\Xi)$. The dependence from the loading is completely embedded in the functions $\alpha_+(\Xi), \alpha_-(\Xi)$ and it is clear that different loadings giving the same functions are equivalent inasmuch as the contact locus is concerned. Eq.(20) acquires a particularly simple form when it exists $\rho^c < \Xi$ such that the loading vanishes outside the interval

$[0, \rho^c]$, for then

$$\tan(\Xi)\tanh(\Xi) = -\frac{\alpha_+(\rho^c)}{\alpha_-(\rho^c)} = r \tag{21}$$

and the RHS r is a constant with respect to Ξ. It is observed that for r positive the contact locus sits in the interval $(\pi/2, \pi)$ and, by solution continuity, for r negative in $(\pi, \frac{3}{2}\pi)$. In this situation, loadings are equivalent inasmuch as they exhibit the same ratio r. For instance, in the case of two symmetric pairs of concentrated forces, placed at Δ_1 and $\Delta_2 > \Delta_1$, it is

$$r = -\frac{\cos(\Delta_1)\cosh(\Delta_1) + \cos(\Delta_2)\cosh(\Delta_2)}{\sin(\Delta_1)\sinh(\Delta_1) + \sin(\Delta_2)\sinh(\Delta_2)} \tag{22}$$

such that solving the implicit equation $r = k$, k being a real constant, gives the set of pairs Δ_1, Δ_2 yielding the same contact locus $\Xi(k)$. Fig.2 shows the curves $\Delta_2 - \Delta_1$ vs. Δ_1 for $k = 1, 5, 10$. The curves may be taken as a graphical representation of the sets $\mathcal{Q}_k$. Indeed, Fig.3 shows that for $k = 1$, the deformed beam profiles for the cases $\Delta_2 - \Delta_1 = 0.1$ and $\Delta_2 - \Delta_1 = 1$, to which it pertains respectively $\Delta_1 = 0.8857167949$ and $\Delta_1 = 0.2529526456$, exhibit the same contact locus position $\Xi(1) = 2.347045566$. Among such loadings the superposition principle does hold. Eq.(20) is generally nonlinear in Ξ owing to both the functions α_i and A_i.

Let us now investigate the contribution of the boundary term and consider the situation where the beam is loaded beyond the contact locus through the line load $\sigma^l(\xi), \Xi < \xi < l$. Then, a boundary term enters the function F. Exploiting the symmetry of the Green function and the continuity of its first derivative, Eq.(17) becomes

$$F(\Xi) = \left\{\alpha_i(\Xi) - \frac{1}{4}u_3^c(\Xi)\eta_i(\Xi) + \frac{1}{4}u_2^c(\Xi)\eta_i'(\Xi)\right\} A_i(\Xi) \tag{23}$$

where, in analogy with the first of Eqs.(18), it is let $B_i(\Xi) = b_i(\Xi, \Xi)$. With a bit of work, Eq.(20)

is now

$$\alpha_+(\Xi)\cos(\Xi)\cosh(\Xi)+\alpha_-(\Xi)\sin(\Xi)\sinh(\Xi)=\frac{1}{8}u_3^c(\Xi)\left[\cosh(2\Xi)+\cos(2\Xi)\right] \\ -\frac{1}{8}u_2^c(\Xi)\left[\sinh(2\Xi)-\sin(2\Xi)\right]. \quad (24)$$

Eq.(24) provides a nonlinear equation relating the loading and the contact locus, in a symmetric layout, which gathers all the nonlinear feature of the unilateral contact problem. It also provides a mean of determining whether the beam lifts off the foundation or, rather, rests entirely supported on it. To this aim, solutions of Eq.(24) are checked against the beam length l and when it is found that $\Xi > l$, then the beam rests entirely supported by the foundation.

Applications for a Winkler soil

Symmetric case

Let us consider the case of a beam loaded at midspan by a unit force. Then, it is $\alpha_+ = 1$, $\alpha_- = 0$ and Eq.(20) reduces to the simple relation

$$\cosh\Xi\cos\Xi = 0, \quad (25)$$

which corresponds to Eq.(7) of Weitsman (1970) and yields the well-known result $\Xi = \pi/2$. We are interested in adding an end force f^l and an end couple c^l such that the contact locus remains unchanged. To this aim, a relationship between $u_2^c(\Xi)$ and $u_3^c(\Xi)$ needs be sought in order that the boundary contribution drops out. Writing the latter as at the RHS of Eq.(24) and considering that

$$\frac{1}{4}u_2^c(\Xi) = c^l + f^l(l-\Xi), \qquad \frac{1}{4}u_3^c(\Xi) = -f^l, \quad (26)$$

given that f^l is positive when downwards and c^l when clockwise, a connection is found between c^l and f^l as follows:

$$c^l = -f^l\left(l-\Xi+R^W(\Xi)\right), \quad (27)$$

where the positive function is let

$$R^W(\Xi) = \frac{\cosh(2\Xi) + \cos(2\Xi)}{\sinh(2\Xi) - \sin(2\Xi)}. \tag{28}$$

In particular, for $\Xi = \pi/2$, it is $R^W(\Xi) = 0.6536439910$.

As a second application, the case of a pair of concentrated forces, symmetric about $\xi = 0$ and placed at a distance $2\Delta > 0$ apart, is considered. Then, it is $\alpha_i = \eta_i(\Delta)$ and Eq.(21) gives a connection between the contact locus and the distance $\Delta < \Xi$, namely

$$\tan\Xi \tanh\Xi = -\frac{1}{\tan\Delta \tanh\Delta}. \tag{29}$$

It is immediate to see that the sign of both the left and the right hand side is given by the tangent terms: for $\Delta \in [0, \pi/2)$, the RHS is negative and solutions are to be found in the interval $\Xi \in [\pi/2, \pi)$. By the same token, for $\Delta \in [\pi/2, \pi)$, continuity of the solution suggests taking $\Xi \in [\pi, \frac{3}{2}\pi)$. It is further observed that the situation $\Delta = \Xi$ is not allowed. If the applied forces are far apart beyond a limiting spacing $2\tilde{\Delta}$, lift-off takes place in the neighborhood of the origin as well, in a discontinuous contact scenario. Such limiting spacing occurs when

$$u^c(0) = b_i(0,\Xi) \int_0^{\Xi} \sigma^c(\xi)\eta_i(\xi) d\xi = b_i(0,\Xi)\eta_i(\tilde{\Delta}) = 0 \tag{30}$$

and the grazing condition $u_1^c(0) = 0$ follows directly from the symmetry requirement. Here, it is

$$b_1(0,\Xi) = \frac{1}{2\Lambda^2}\left[\cos(2\Xi) + \cosh(2\Xi) - \sinh(2\Xi) - \sin(2\Xi) + 2\right]$$
$$b_2(0,\Xi) = \frac{1}{2\Lambda^2}\left[\cos(2\Xi) - \cosh(2\Xi) + \sinh(2\Xi) + \sin(2\Xi)\right]$$

and $b_2(0,\Xi) + b_4(0,\Xi) = 1$, $b_1(0,\Xi) - b_3(0,\Xi) = -1$. For a general Δ, Eq.(30) with Eq.(29) yields

$$b_2(0,\Xi) - b_1(0,\Xi)\tan\Xi \tanh\Xi = f(\Delta), \tag{31}$$

being

$$f(\Delta) = -\frac{e^{-\Delta}}{2\sinh\Delta}\left(1+\cot\Delta\right). \tag{32}$$

Eq.(31) lends a connection between the contact locus and the spacing Δ. Since $\Delta > 0$ demands $\Xi > \pi/2$, the LHS of (31) is positive and to get a positive value for the RHS it must be $\Delta > \tilde{\Delta} = 2.356194490$. Fig.4 shows the beam bending moment, shearing force and contact pressure in the contact interval. As on the verge of lifting-off, the latter vanishes at midspan.

As a third example, Eq.(24) is put to advantage for the case of a constant line loading q extending up to the abscissa l_q and a concentrated force $2f_0$ at midspan. When $l_q = l$ the classic solution for a concentrated load $2f_0$ acting at midspan of a beam with weight per unit length q is obtained. This situation is generally more involved than the previous ones because, for l_q large enough, the contact locus sits within the loaded interval. Eq.(24) gives

$$\begin{aligned} 2f_0\cos\Xi\cosh\Xi + \frac{q}{2}\left[\sinh 2\Xi + \sin 2\Xi\right] = -q(l_q-\Xi)\left[\cosh 2\Xi + \cos 2\Xi\right] \\ -\frac{1}{2}q(l_q-\Xi)^2\left[\sinh 2\Xi - \sin 2\Xi\right], \end{aligned} \tag{33}$$

provided that $l_q > \Xi$. When $l_q < \Xi$ it is

$$\begin{aligned} 2f_0\cos\Xi\cosh\Xi + q\left[\cos l_q \sinh l_q + \sin l_q \cosh l_q\right]\cos\Xi\cosh\Xi \\ + q\left[-\cos l_q\sinh l_q + \sin l_q\cosh l_q\right]\sin\Xi\sinh\Xi = 0. \end{aligned} \tag{34}$$

For $f_0 = 1$, Fig.5 plots both Eqs.(33,34) in their realms of validity, the boundary between them being represented by the bisector. It is seen that for q small ($q = 0.01$), the contact locus tends to the classic result $\pi/2$ in a wide range of l_q. At $q = 0.05$, it is observed that for a given l_q multiples solutions for Ξ are found and a maximum value for $l_q > \Xi$ appears. Beyond such maximum, a second branch of solution exists with $\Xi > l_q$. It rests to be seen whether the beam is long enough to warrant the admissibility of such solution. In order to discuss the multiplicity of solutions, Fig.6 shows the beam profiles for $q = 0.05$ and $l_q = 3$, when the solution $\Xi < l_q$, curve (a), and $\Xi > l_q$, curve (b), are considered. It is seen that the solution (b) leads to interpenetration and must be discarded. However, above the maximum value for l_q, solution (a) disappears and solution (b) becomes admissible.

Non-symmetric case

Let us now drop the symmetry assumption and deal with a general continuous contact scenario (Fig.7). Then, two contact loci, $\Xi_1 < \Xi_2$, are expected and Eq.(16) becomes

$$u^c(\zeta) = a_i(\zeta, \Xi_1, \Xi_2) \int_{\Xi_1}^{\zeta} \sigma^c(\xi)\eta_i(\xi)d\xi + b_i(\zeta, \Xi_1, \Xi_2) \int_{\zeta}^{\Xi_2} \sigma^c(\xi)\eta_i(\xi)d\xi. \tag{35}$$

Likewise, two limits are now considered

$$\lim_{\zeta \to \Xi_2} u^c(\zeta) = 0 \quad \Leftrightarrow \quad \alpha_i(\Xi_1, \Xi_2) A_i(\Xi_1, \Xi_2) = 0, \tag{36a}$$

$$\lim_{\zeta \to \Xi_1} u^c(\zeta) = 0 \quad \Leftrightarrow \quad \alpha_i(\Xi_1, \Xi_2) B_i(\Xi_1, \Xi_2) = 0, \tag{36b}$$

having let $A_i(\Xi_1, \Xi_2) = a_i(\Xi_2, \Xi_1, \Xi_2)$, $B_i(\Xi_1, \Xi_2) = b_i(\Xi_1, \Xi_1, \Xi_2)$ and $\alpha_i = \int_{\Xi_1}^{\Xi_2} \sigma^c(\xi)\eta_i(\xi)d\xi$. Despite the fact that the analysis follows along the same path as in the symmetric situation, the increased mathematical complication suggests to limit the discussion to a single concentrated force. Then, $\sigma^c = \delta(\xi, \Delta)$ and it is expedient to set the ξ-axis origin at $\xi = \Delta$ without loss of generality. Eqs.(36) become

$$A_1(\Xi_1^*, \Xi_2^*) + A_3(\Xi_1^*, \Xi_2^*) = 0, \tag{37a}$$

$$B_1(\Xi_1^*, \Xi_2^*) + B_3(\Xi_1^*, \Xi_2^*) = 0, \tag{37b}$$

with the understanding that $\Xi_1^* = \Xi_1 - \Delta$ and likewise $\Xi_2^* = \Xi_2 - \Delta$. It is easy to show that for a symmetric disposition of the contact loci, i.e. $\Xi_1^* = -\Xi_2^*$, Eqs.(37) collapse into a single equation, which corresponds to Eq.(25). Indeed, every time a solution exists with $\Xi_1^* = -\Xi_2^*$ for either of the Eqs.(37), then it complies with both. It is natural to introduce $d = \Xi_1^* + \Xi_2^*$, the deviation with respect to a symmetric condition (Fig.7). Fig.8 draws the solution curves d vs. Ξ_2^* for the first (dash curve) and the second (solid curve) of Eqs.(37). In this plot, each intersection point is a possible solution of the system. The shaded area, bounded from below by the dotted curve $d = 2\Xi_2^*$, is ruled out as it leads to a contact locus $\Xi_1^* > \Xi_2^*$. It is seen that a discrete number of solutions is available yet the ones with minimum Ξ_2^* and d are specially interesting. As long as $l_2 \geq \Delta + \pi/2$, which means that the solution points at $\Xi_2^* \geq \pi/2$ are admissible, the classic solution $d = 0$, corresponding to a symmetric layout, is retrieved (point A in Fig.8). When such condition no longer holds, one of the beam ends plunges into the foundation, say the right end, whence it is $\Xi_2 = l_2$ fixed. Then, only the second equation of (37) survives (solid curve) and it provides d vs. $\Xi_2^* = l_2 - \Delta$. Note that $\Xi_1^* = d - \Xi_2^*$ or, equivalently, $\Xi_1 = d - l_2 + 2\Delta$.

It is interesting to describe the system behavior as Δ increases and the loading is brought closer and closer to the beam end. Then, d is found moving along the solid curve from point A to point B and beyond, until the origin is reached. It is seen that d acquires decreasing (with Δ) negative values until the point B is reached, where the layout with maximum deviation from symmetry $|d|$ is found. Since, for the most part, the solid curve possesses unit slope, in the neighborhood of A it is $d \approx -\Delta$ and the left contact locus moves rightwards proportionally with Δ, i.e. $\Xi_1 \approx -l_2 + \Delta$. The contact imprint, however, is given by $l_c = \Xi_2 - \Xi_1$ and it shrinks as

$$l_c \approx 2l_2 - \Delta. \tag{38}$$

Fig.9 plots the position of the left contact locus against the loading offset Δ for a beam with $l_2 = \pi/2$, that is starting from point A. Since both the absolute position Ξ_1 and the relative position Ξ_1^* are given, the difference between the curves equals Δ, while the distance $l_2 - \Xi_1$ gives the contact imprint length l_c. The deviation from symmetry, d, is also shown as the difference from the dash-dot curve and $-l_2$. Dotted curves show the positive and negative unit slope, which confirm the behavior previously inferred for Ξ_1, l_c and d. Beyond B, a substantial rotation of the beam occurs which leads to a very small contact imprint and an almost symmetric situation. Here,

d increases towards zero again.

It is easy to obtain the results numerically developed in Zhang and Murphy (2004) for a beam of varying length l loaded symmetrically and non-symmetrically by a unit force. Three regimes are considered:

1. when $\Xi_2 < l_2$ and $|\Xi_1| < l_1$,
2. when, say, $\Xi_2 > l_2$ and $|\Xi_1| < l_1$,
3. when $\Xi_2 > l_2$ and $|\Xi_1| > l_1$.

In regime 1, both left and right contact loci sit inside the beam, the symmetric solution $d = 0$ is admitted and the contact imprint length $l_c = \Xi_2 - \Xi_1 = 2\Xi_2^*$ is constant. In regime 2, the beam right length l_2 is too short to warrant that the right contact locus sits inside the beam. Conversely, the left length l_1 accommodates the left contact locus. It is observed that this regime demands a non-symmetric loading situation. Having let $l_2 = k_2 l$ and $\Delta = k_\Delta l$, where $k_2, k_\Delta < 1$, Eq.(38) shows that the contact imprint length scales linearly with l with a proportionality coefficient $2k_2 - k_\Delta$. Finally, regime 3 is such that both contact loci exceed the beam left and right length. The beam rests entirely supported by the soil and the contact imprint length corresponds to the beam length. In a symmetric layout, beam length scaling brings the system from regime 1 to regime 3 or vice versa and the contact imprint length is either constant or equal to l, as numerically found in Zhang and Murphy (2004). In a non-symmetric layout, the system undergoes all three regimes and, from 1 to 3, the contact imprint length is constant, decreases with coefficient $2k_2 - k_\Delta$ and finally equals the beam length, i.e. coefficient 1.

PASTERNAK SOIL

Let us consider the case of a E-B beam resting on a tensionless Pasternak soil in a symmetric continuous contact scenario. The governing ODE reads, in the contact interval,

$$\frac{1}{4}u_4^c - \alpha u_2^c + u^c = \sigma^c, \tag{39}$$

where the dimensionless quantity $\alpha = \beta^2 k_G/k$ is introduced and k_G is the shear modulus of the foundation (Selvadurai 1979). The BCs enforce symmetry at $\xi = 0$, as at Eqs.(2), and bending moment and shearing force continuity at the contact locus Ξ, as in (3). Furthermore, two BCs involve the soil profile through setting displacement and slope continuity (Kerr 1976), i.e.

$$u^c(\Xi) = u^s(\Xi), \quad u_1^c(\Xi) = u_1^s(\Xi). \tag{40}$$

The problem is only formally self-adjoint, as the Green function for the beam in the contact region is determined by symmetric BCs at $\xi = 0$, as at Eqs.(10), and homogeneous conditions at $\xi = \Xi$ as follows:

$$\frac{1}{4}G''(\Xi,\zeta) - \alpha G(\Xi,\zeta) = 0, \quad \frac{1}{4}G'''(\Xi,\zeta) - \alpha G'(\Xi,\zeta) = 0. \tag{41}$$

It is observed that, in the limit for $\alpha \to 0$, the BCs (11,15) for the Winkler soil are retrieved. The displacement in the contact region at $\zeta < \Xi$ is given by

$$u^c(\zeta) = \int_0^\Xi \sigma^c(\xi)G(\xi,\zeta)d\xi - \frac{1}{4}[u_3^c G - u_2^c G']_0^\Xi \\ + \left[\left(-\frac{1}{4}G'' + \alpha G\right)u_1^c + \left(\frac{1}{4}G''' - \alpha G'\right)u^c\right]_0^\Xi, \tag{42}$$

where $G(\xi,\zeta)$ takes the shape (13) and, of course, the functions $a_i(\zeta,\Xi)$ and $b_i(\zeta,\Xi)$ differ from the case of the Winkler foundation. The fundamental set is taken in even/odd fashion

$$\{\eta_i(\xi)\} = \Big\{ \cosh\left(\sqrt{\lambda_1}\xi\right), \sinh\left(\sqrt{\lambda_1}\xi\right), \cosh\left(\sqrt{\lambda_2}\xi\right), \sinh\left(\sqrt{\lambda_2}\xi\right) \Big\}.$$

Here, for the sake of definiteness, it is assumed $\alpha > 1$ whence $\lambda_{1,2} = 2\left(\alpha \pm \sqrt{\alpha^2 - 1}\right)$. The term in square brackets at RHS of (42) vanishes owing to the BCs (2,10,41).

In the absence of a soil loading σ^s, it is $u_1^s = -\alpha^{-1/2}u^s$, whence the BCs (40) yield the contact locus equation

$$u^c(\Xi) + \sqrt{\alpha}u_1^c(\Xi) = 0. \tag{43}$$

Eq.(43) sets the contact locus without recurring to the soil profile. It may be written as

$$F(\Xi) = \int_0^{\Xi} \sigma^c(\xi) K(\xi, \Xi) d\xi - \frac{1}{4} \lim_{\zeta \to \Xi} \left[u_3^c K(\Xi, \zeta) - u_2^c K'(\Xi, \zeta) \right], \tag{44}$$

wherein a new kernel function is defined in terms of the Green function G

$$K(\xi, \zeta) = G(\xi, \zeta) + \sqrt{\alpha} \frac{\partial G}{\partial \zeta}(\xi, \zeta). \tag{45}$$

Now the argument runs parallel to the treatment given for the Winkler soil. However, it is emphasized that neither the kernel G nor K is symmetric, for the problem for the Green function is no longer self-adjoint. When the beam lifting-off the soil is load-free, Eq.(44) gives an expression formally analogous to (17)

$$F(\Xi) = \mathcal{A}_i(\Xi) \alpha_i(\Xi), \tag{46}$$

being understood that $\mathcal{A}_i(\Xi) = A_i(\Xi) + \sqrt{\alpha} \bar{A}_i(\Xi)$ and

$$A_i(\Xi) = \lim_{\zeta \to \Xi} a_i(\zeta, \Xi), \qquad \bar{A}_i(\Xi) = \lim_{\zeta \to \Xi} \frac{\partial a_i}{\partial \zeta}(\zeta, \Xi). \tag{47}$$

The symmetric layout accounts for the vanishing of the functions A_2, A_4 and likewise for $\bar{A}_2, \bar{A}_4$. Besides, A_1 equals A_3 and $\bar{A}_1$ equals $\bar{A}_3$ provided that the role of λ_1 and λ_2 is exchanged. After some lengthy manipulations, it is found, omitting a common non-vanishing denominator,

$$\mathcal{A}_1 = \lambda_1(\lambda_1 - \lambda_2) \left[\cosh(\sqrt{\lambda_2}\Xi) + \sqrt{\alpha \lambda_2} \sinh(\sqrt{\lambda_2}\Xi) \right], \tag{48a}$$

$$\mathcal{A}_3 = \lambda_2(\lambda_2 - \lambda_1) \left[\cosh(\sqrt{\lambda_1}\Xi) + \sqrt{\alpha \lambda_1} \sinh(\sqrt{\lambda_1}\Xi) \right], \tag{48b}$$

whence A_1 and A_3 are easily retrieved letting $\alpha \to 0$. As expected, $\mathcal{A}_3$ equals $\mathcal{A}_1$ once the role of λ_1 and λ_2 is exchanged.

When accounting for the contribution from the lift-off interval, it is

$$F(\Xi) = \mathcal{A}_i(\Xi)\alpha_i(\Xi) - \frac{1}{4}\left[u_3^c(\Xi)\mathcal{A}_i(\Xi)\eta_i(\Xi) - u_2^c(\Xi)\mathcal{A}_i(\Xi)\eta_i'(\Xi)\right], \tag{49}$$

where use has been made of the continuity properties of the Green function G.

Applications for a Pasternak soil

Let us consider the classic situation of a beam resting on a tensionless Pasternak soil and loaded at midspan by a unit force. Then, it is $\sigma^c = \delta(\xi, 0)/2$ and Eq.(46), together with Eqs.(48) and divided through by $(\lambda_1 - \lambda_2)$, gives

$$\begin{aligned} F(\Xi) = \lambda_1 &\left[\cosh\left(\sqrt{\lambda_2}\Xi\right) + \sqrt{\alpha\lambda_2}\sinh\left(\sqrt{\lambda_2}\Xi\right)\right] \\ &- \lambda_2\left[\cosh\left(\sqrt{\lambda_1}\Xi\right) + \sqrt{\alpha\lambda_1}\sinh\left(\sqrt{\lambda_1}\Xi\right)\right]. \end{aligned} \tag{50}$$

The first positive root of F gives, when $\beta = 2.5$, the result $\Xi = 0.8423946552$. We wish to determine the loading condition at the beam end such that the contact locus is preserved. Again, we need to vanquish the last term of Eq.(49), i.e.

$$c^l = f^l\left(l - \Xi + R^P(\Xi)\right), \quad R^P(\Xi) = \frac{\mathcal{A}_i(\Xi)\eta_i(\Xi)}{\mathcal{A}_i(\Xi)\eta_i'(\Xi)}.$$

It is observed that for $\alpha \to 0$ the Pasternak soil becomes a Winkler soil and indeed $R^P(\Xi) \to R^W(\Xi)$. In particular, for $\Xi = 0.8423946570$, it is $R^P(\Xi) = 0.2763085352$.

When two symmetrically placed unit forces are far apart enough, the beam stands on the verge of lifting off at the origin. Letting the force distance be 2Δ and making use of Eqs.(48), Eq.(46) specializes to

$$\begin{aligned} F(\Xi) = \lambda_1 \cosh(\sqrt{\lambda_1}\Delta)&\left[\cosh\left(\sqrt{\lambda_2}\Xi\right) + \sqrt{\alpha\lambda_2}\sinh\left(\sqrt{\lambda_2}\Xi\right)\right] \\ &- \lambda_2\cosh(\sqrt{\lambda_2}\Delta)\left[\cosh\left(\sqrt{\lambda_1}\Xi\right) + \sqrt{\alpha\lambda_1}\sinh\left(\sqrt{\lambda_1}\Xi\right)\right], \end{aligned} \tag{51}$$

having omitted the common factor $\lambda_1 - \lambda_2$ and provided that $\alpha > 1$. Seeking the solution of $F(\Xi) = 0$ lends the curves Ξ vs. Δ. For a Pasternak foundation, the counterpart of Eq.(30) demands that the dimensionless contact pressure $-u_4^c/4$ vanishes at the origin. With Eq.(39), the requirement amounts to

$$\alpha_i(\Xi)\left(B_i(\Xi) - \alpha\bar{\bar{B}}_i(\Xi)\right) = 0, \tag{52}$$

being, in analogy with Eq.(47), $B_i(\Xi) = \lim_{\zeta\to\Xi} b_i(\zeta,\Xi)$, $\bar{\bar{B}}_i(\Xi) = \lim_{\zeta\to\Xi} \frac{\partial^2 b_i}{\partial\zeta^2}(\zeta,\Xi)$. Fig.10 plots the solution curves of Eq.(51) (dash) and Eq.(52) (solid curve) for $\alpha = 1.1$, 5 and 10. The bisector is also plotted as a dotted line for solutions are admissible inasmuch as $\Xi > \Delta$. When the forces are brought farther apart, the contact locus position moves along the dash curve until the solid curve is met. At such limiting distance $2\tilde{\Delta}$, the continuous contact scenario breaks down and lift-off appears in the neighborhood of the origin.

CONCLUSIONS

In this paper, the free-boundary problem of tensionless contact for a beam resting on either a Winkler or a Pasternak two-parameter elastic foundation is addressed. The classic approach to the problem consists of integrating the deflection curves for the beam in contact with the soil, the beam lifting off it and the soil and then matching solutions at the contact locus, which is a problem unknown. When matching solutions, an extra condition exists that determines the contact locus. Conversely, in this paper, a Green function approach is put forward which aims at determining a direct (nonlinear) connection between the loading and the contact locus. Once the contact locus is set, the problem reduces to solving a classic linear BVP in the contact and lift-off regions. This way of approaching the problem lends considerable advantages over the classic one. First, the connection between the contact locus position and the loading is expressed as a general relation, which allows to determine what features of the loading affect the contact locus. This implies that it is possible to build the set of loadings whose application leads to the same contact locus. Among such loadings, the superposition principle holds. Second, solutions are obtained once some assumptions are made concerning the contact layout. Accordingly, results must be checked against such assumptions at the end of the procedure. Although this part is common to both approaches, it is shown that here the required assumptions are weaker. For instance, the non-symmetric contact problem for a Winkler foundation is analyzed in general and two families of solution curves are obtained: one for the left and one for the right contact locus. When the beam length is insufficient to accommodate both contact loci, one curve is simply dropped in place of the constraint that fixes the contact at the beam end. Conversely, when deflection curves are integrated, whether lift-off exists needs be assumed from the start, given that the BCs depend on such assumption. Several applications are presented for both the cases of symmetric and non-symmetric contact. Furthermore, comparison with the existing literature is carried out.

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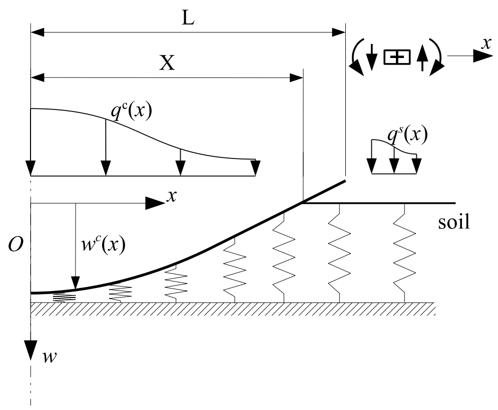

FIG. 1. Symmetric continuous contact scenario with lift-off for a beam on a tensionless Winkler foundation

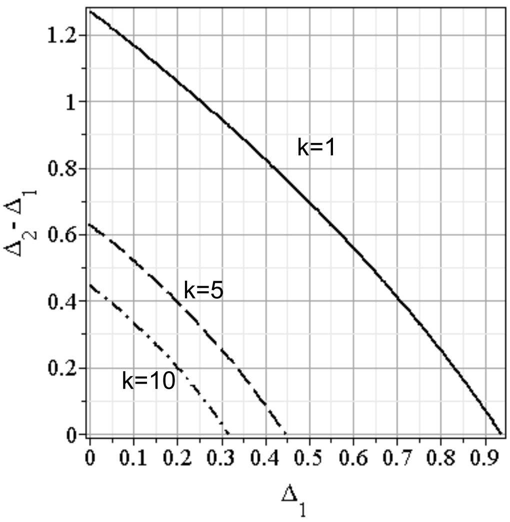

FIG. 2. $\Delta_1, \Delta_2 - \Delta_1$ **pairs giving the same contact locus** $\Xi(k)$

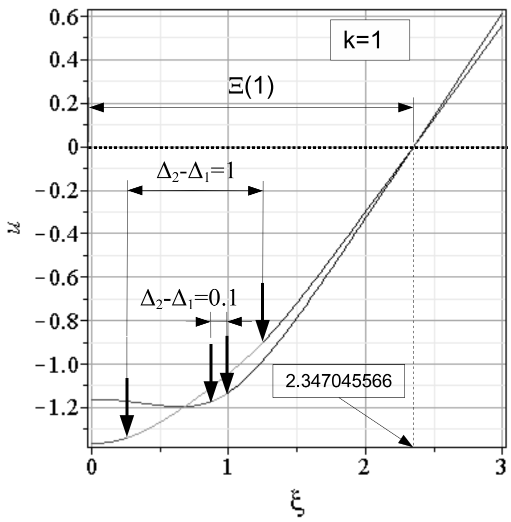

FIG. 3. Beam profile for $k = 1$ and either $\Delta_2 - \Delta_1 = 1$ or $\Delta_2 - \Delta_1 = 0.1$

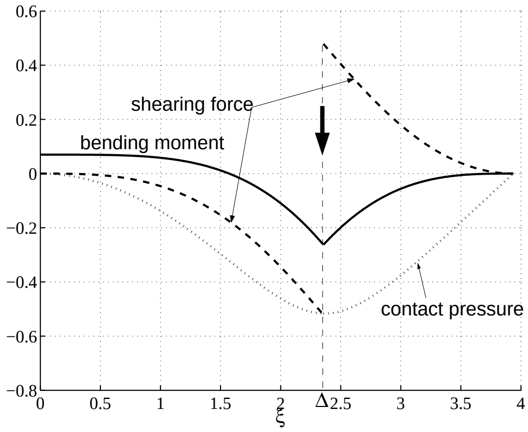

FIG. 4. Bending moment, shearing force and contact pressure for a beam on a Winkler foundation at the onset of midspan lift-off

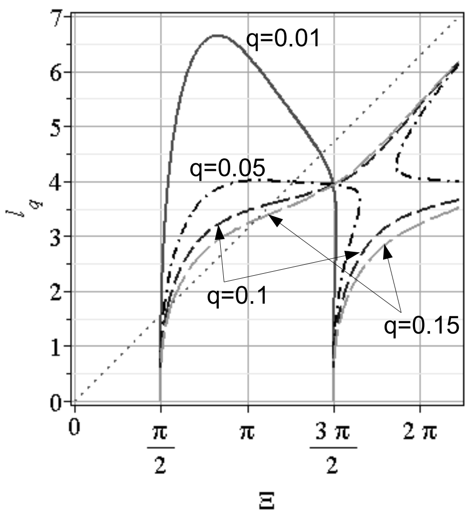

FIG. 5. l_q **vs** Ξ **for** $f_0 = 1$ **as given by Eqs.(33,34)**

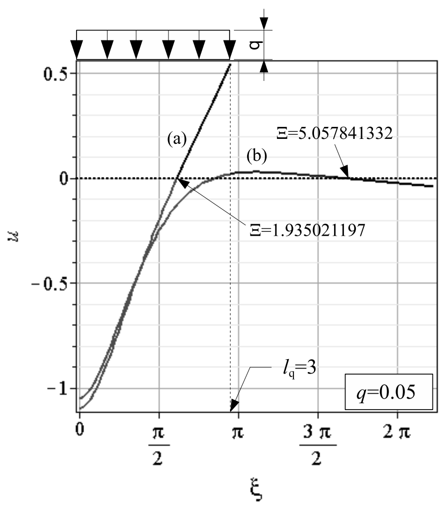

FIG. 6. Beam profiles for $\Xi < l_q$ **(a) and** $\Xi > l_q$ **(b), for** $q = 0.05$**,** $l_q = 3$**,** $f_0 = 1$

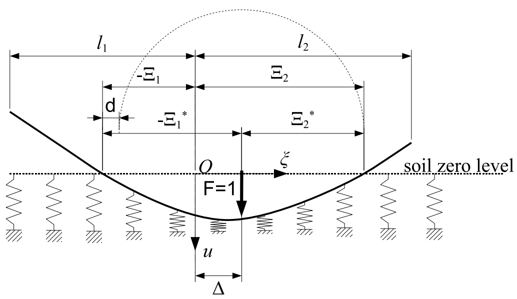

FIG. 7. Beam on a Winkler soil in a non-symmetric layout

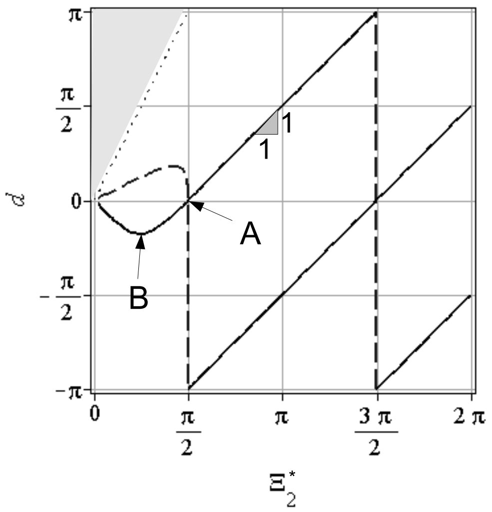

FIG. 8. Plots of the fist (dash) and the second (solid) of Eqs.(37)

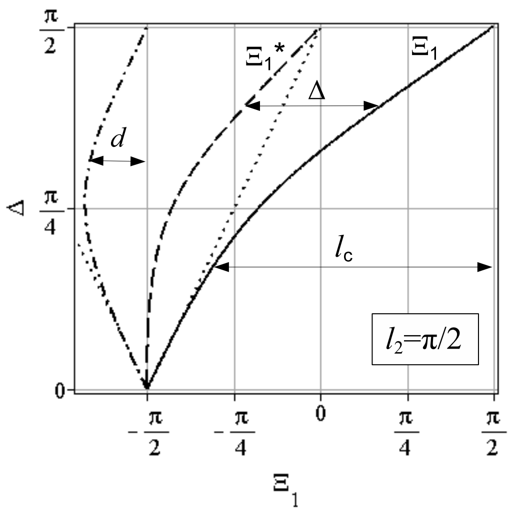

FIG. 9. Left contact locus absolute (solid) and relative (dash) position for a beam loaded by a concentrated force at $\xi = \Delta$.

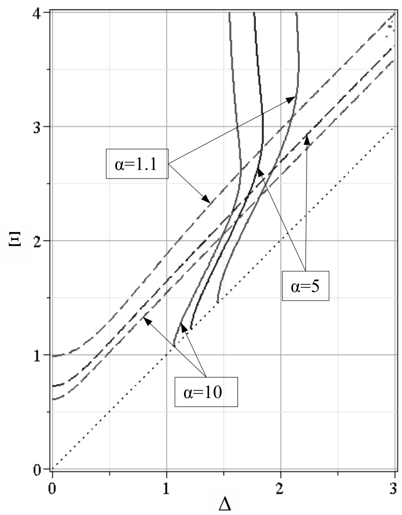

FIG. 10. Ξ **vs.** $\triangle$ **curve (dash) for** $\alpha = 1.1$**,** 5 **and** 10**, and limiting curve (solid) which marks the onset of a discontinuous contact scenario**