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Superposition principle for the tensionless contact of a beam resting on a winkler or a pasternak foundation / Nobili, Andrea. - In: JOURNAL OF ENGINEERING MECHANICS. - ISSN 0733-9399. - STAMPA. - 139:10(2013), pp. 1470-1478. [10.1061/(ASCE)EM.1943-7889.0000555]

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18/12/2025 16:13

SUPERPOSITION PRINCIPLE FOR THE TENSIONLESS CONTACT OF A BEAM RESTING ON A WINKLER OR A PASTERNAK FOUNDATION

by Andrea Nobili¹

ABSTRACT

A Green function based approach is presented to address the nonlinear tensionless contact problem for beams resting on either a Winkler or a Pasternak two-parameter elastic foundation. Unlike the traditional solution procedure, this approach allows determining the contact locus position independently from the deflection curves. In so doing, a general nonlinear connection between the loading and the contact locus is found which enlightens the specific features of the loading that affect the position of the contact locus. It is then possible to build load classes sharing the property that their application leads to the same contact locus. Within such load classes, the problem is linear and a superposition principle holds. Several applications of the method are presented, including symmetric and non-symmetric contact layouts, which can be hardly tackled within the traditional solution procedure. Whenever possible, results are compared with the existing literature.

Keywords: Tensionless contact, Green function, two-parameter elastic foundation

INTRODUCTION

The contact problem for beams resting on elastic foundations has long attracted considerable attention, given its relevance in describing soil-structure interaction (Hetenyi 1946; Selvadurai 1979). In particular, a very extensive literature exists concerning beams resting on one, two and three-parameter elastic foundations (Kerr 1964). The existing literature is for the most part devoted to considering contact as a bilateral constraint, which fact limits the validity of the analysis to situations where lift-off plays a minor role. However, in so doing, the problem retains a valuable linear character and the superposition principle holds.

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When lift-off becomes an important feature, tensionless contact must be reverted to at the expense of the problem linearity. From a mathematical standpoint, tensionless contact determines a free-boundary problem (Kerr 1976; Nobili 2012).

Historically, interest in tensionless contact between a beam and a foundation arose in connection with railway systems. In this respect, Weitsman (1971), Lin and Adams (1987) and recently Chen and Chen (2011) considered detachment and stability for the problem of tensionless contact under a moving load. Besides, much research on tensionless structure-foundation contact is devoted to assessing its role in reducing the structural stress in a seismic event (Celep and Güler 1991; Psycharis 2008). Recently, Coskun (2003) studied forced harmonic vibrations of a finite beam supported by a tensionless Pasternak soil, while Zhang and Murphy (2004) studied a finite beam in tensionless contact in a non-symmetric contact scenario. Tensionless contact for an infinite beam in a multiple contact scenario was investigated by Ma et al. (2009a) and Ma et al. (2009b). An extensive body of literature exists regarding numerical strategies specifically devised to deal with tensionless contact. Recently, Sapountzakis and Kampitsis (2010) considered a boundary element method for beam-columns partly supported on a Winkler and, later (2011), a three-constant soil model.

The classic approach to solving a tensionless contact problem for a beam on an elastic foundation consists of integrating the deflection curves for the beam in contact, the beam in lift-off and the soil, and then matching the solutions at the yet unknown contact locus, that is the point where contact ceases and lift-off begins (Weitsman 1970; Kerr and Coffin 1991). This approach suffers from two major shortcomings. On the one hand, the procedure initially assumes a contact layout and then proceeds to determining the relevant quantities within such layout. It then remains to be checked that results are consistent with the assumptions. On the other hand, contact loci positions are determined through deflection curves integration. Since the general integrals of the governing equations depend on the loading, it appears that results are restricted to one particular loading.

In this paper, a Green function approach is adopted. Unlike the classic approach, this method consists of first determining the contact locus through a nonlinear equation and then solving the

linear problem for the deflection curves. In fact, only the first stage is here presented, the second being a classic problem. Although the method still requires some assumptions concerning the layout of the contact, nonetheless such assumptions are somewhat relaxed and a general connection between the contact locus and a family of loadings is obtained, so much so that a form of superposition is also retrieved. It is emphasized that this procedure differs from the integral approach of Tsai and Westmann (1967), which is still based on the Green function and yet it aims at determining the deflection curves and the contact locus in one stage.

THE FREE-BOUNDARY PROBLEM

The tensionless contact problem for a Euler–Bernoulli (E-B) beam resting on a tensionless elastic foundation is first stated in its simplest form, concerning a Winkler soil in a symmetric contact scenario (Fig.1). Let $[-X, X]$ denote the contact interval and $X > 0$ be the *contact locus*, i.e. the beam rests supported on the soil up to abscissa X and then it detaches from it. The beam detached from the soil is often addressed as lifting off the soil. The free soil extends beyond X to infinity. Here, the inverse of a reference length is introduced as the ratio between the soil modulus k and the beam flexural rigidity EI , i.e. $\beta^4 = k(4EI)^{-1}$. Then, the problem is cast in dimensionless form: $\Xi = \beta X$ is the dimensionless contact locus position and $u = \beta w$ denotes the beam dimensionless displacement. The beam displacement function, u , restricted to the contact interval $I^c = [0, \Xi]$ and to the lift-off interval $I^l = (\Xi, l]$, is denoted by u^c and u^l , respectively. $2l = 2\beta L$ is the beam dimensionless length and u^s is the soil dimensionless displacement in the unbounded region $I^s = [\Xi, +\infty)$, which is relevant for the Pasternak soil alone. Besides, $\sigma^c = \beta q^c/k$ and $\sigma^l = \beta q^l/k$ are the dimensionless loadings acting in I^c and I^l , respectively. In the contact interval I^c , the beam rests entirely supported on the soil and the governing equation reads

$$\frac{1}{4} (u^c)^{(iv)} + u^c = \sigma^c, \quad (1)$$

where superscripts within parenthesis denote the differentiation order with respect to ξ . To shorten notation, it is expedient to write the k -th derivative $(u^c)^{(k)}$ with respect to ξ as u_k^c . The problem

75 boundary conditions (BCs) due to symmetry are

$$u_1^c(0) = 0, \quad u_3^c(0) = 0, \quad (2)$$

76 while the BCs at the contact locus Ξ , enforce continuity for the beam of the bending moment and
77 of the shearing force

$$u_2^c(\Xi) = u_2^l(\Xi), \quad u_3^c(\Xi) = u_3^l(\Xi). \quad (3)$$

78 However, unlike an ordinary boundary value problem (BVP), here the contact locus is a problem
79 unknown, whence a further condition is demanded for its placing. This condition, named *con-*
80 *tact locus equation*, enforces displacement continuity with the Winkler foundation (which is here
81 assumed load free), i.e.

$$u^c(\Xi) = 0. \quad (4)$$

82 In more general terms, the problem may be rewritten formally as

$$D^c u^c = \sigma^c \quad (5)$$

83 where D^c denotes the differential operator embodying the dimensionless governing equation in the
84 contact region I^c , with its boundary conditions.

85 THE GREEN FUNCTION APPROACH

86 In this paper, a new solution procedure is introduced which takes advantage of the Green func-
87 tion to obtain an explicit connection between the loading and the contact locus position. Let the
88 adjoint problem for Eq.(5) be considered

$$\tilde{D}^c G(\xi, \zeta) = \delta(\xi, \zeta), \quad (6)$$

89 where $\delta(\xi, \zeta)$ is Dirac's delta function about $\xi = \zeta$ and \tilde{D}^c the adjoint operator. Let n indicate the
90 order of the operator D^c , i.e. $n = 4$ for both the Pasternak and the Winkler models. It is worth

recalling that the Green function G is determined assuming homogeneous boundary conditions at the boundary ∂I^c and it is thereby independent of the behavior in the lift-off region. The latter comes into play in the form of a boundary term $BT(\xi, \zeta)$. Furthermore, a over-determined system becomes an under-determined problem for the Green function. It is then possible to write the displacement at a point ζ in the contact region as

$$u^c(\zeta) = \int_{I^c} \sigma^c(\xi) G(\xi, \zeta) d\xi \quad (7)$$

and, accordingly, the condition setting the contact locus. For instance, for a Winkler foundation, it is

$$u^c(\Xi) = \lim_{\zeta \rightarrow \Xi} \int_{I^c} \sigma^c(\xi) G(\xi, \zeta) d\xi - [BT(\xi, \Xi)]_{\xi=0}^{\Xi} = 0. \quad (8)$$

Here, boundary terms are algebraic and have been gathered in $BT(\xi, \Xi)$. Eq.(8) sets an integral connection between the applied loading and the contact locus Ξ which has a three-fold purpose. First, it may be employed to test a given load distribution against the contact locus Ξ . Second, it may be employed to build the loading classes $\mathcal{Q}_{\mathcal{X}}$, whose elements share the property that their application produces the same set of contact loci $\mathcal{X} = \{\Xi_j\}$. Then, the nonlinear contact problem of a beam resting on a tensionless two-parameters elastic soil may be actually solved for any one representative of the load class, the solution for the other load members of that class being obtained by linear combination. The third purpose of the condition is to provide the contact locus without recurring to the actual integration of the deflection curves.

TENSIONLESS WINKLER-TYPE SOIL

Let us first consider the case of a E-B beam resting on a tensionless Winkler soil and acted upon by a line load σ^c (the resultant of which is indeed irrelevant owing to the homogeneous nature of the BC setting the contact locus) possibly extending up to (though vanishing at) the contact locus Ξ , in a symmetric continuous contact scenario. Here, the BCs (3) are homogeneous. The boundary term reads

$$BT = \frac{1}{4} [u_3^c G - u_2^c G' + u_1^c G'' - u^c G''']_0^{\Xi}. \quad (9)$$

Here, prime denotes differentiation with respect to ξ , while G is shorthand for $G(\xi, \zeta)$. It is easily seen that to warrant the vanishing of the boundary term, the Green function has to be subjected to symmetric conditions at $\xi = 0$

$$G'(0, \zeta) = G'''(0, \zeta) = 0 \quad (10)$$

and to the single condition

$$G''(\Xi, \zeta) = 0. \quad (11)$$

such that the beam slope $u_1^c(\Xi)$ drops out the boundary term. This result holds in general, even when the loading extends beyond the contact locus, which amounts to saying that the Green function is entirely independent of the lift-off part. The problem for the Green function is under-determined and it possesses one free integration parameter.

The ODE for the Green function is

$$\frac{1}{4}G^{(iv)}(\xi, \zeta) + G(\xi, \zeta) = \delta(\xi, \zeta), \quad (12)$$

whose general solution is written as

$$G(\xi, \zeta) = \left\{ \begin{array}{ll} a_i(\zeta, \Xi), & \xi < \zeta \\ b_i(\zeta, \Xi), & \xi > \zeta \end{array} \right\} \eta_i(\xi), \quad i = 1, \dots, n. \quad (13)$$

Here, $\{\eta_i(\xi)\}$ is the fundamental set and, for a Winkler soil,

$$\{\eta_i(\xi)\} = \{e^\xi \cos \xi, e^\xi \sin \xi, e^{-\xi} \cos \xi, e^{-\xi} \sin \xi\}. \quad (14)$$

Hereinafter, a summation convention is assumed for twice repeated subscripts, ranging from 1 to n . Let us further enforce the BC

$$G'''(\Xi, \zeta) = 0, \quad (15)$$

whence a self-adjoint formulation for G is set. Since the problem is self adjoint, the Green function

127 is symmetric as it allows exchanging the role of ξ and ζ . Through Eq.(13), the contact zone
 128 displacement is given by

$$u^c(\zeta) = a_i(\zeta, \Xi) \int_0^\zeta \sigma^c(\xi) \eta_i(\xi) d\xi + b_i(\zeta, \Xi) \int_\zeta^\Xi \sigma^c(\xi) \eta_i(\xi) d\xi, \quad \zeta \in [0, \Xi]. \quad (16)$$

129 In particular, letting $\zeta \rightarrow \Xi$, it is $u^c(\zeta) \rightarrow 0$ according to Eq.(4). Letting

$$F(\Xi) = \alpha_i(\Xi) A_i(\Xi), \quad (17)$$

130 where

$$A_i(\Xi) = a_i(\Xi, \Xi), \quad \alpha_i(\Xi) = \int_0^\Xi \sigma^c(\xi) \eta_i(\xi) d\xi, \quad (18)$$

131 it is $F(\Xi) = 0$. It is remarked that Eqs.(18) should be taken in a limiting sense as $\zeta \rightarrow \Xi$,
 132 although direct substitution is equally permitted for the Winkler foundation. In particular, explicit
 133 expressions are available for the functions A_i , namely

$$A_1 = A_3 = 2\Lambda^{-2} \cos(\Xi) \cosh(\Xi), \quad (19a)$$

$$A_2 = -A_4 = 2\Lambda^{-2} \sin(\Xi) \sinh(\Xi), \quad (19b)$$

134 having let the nonnegative quantity $\Lambda^2 = \sin(2\Xi) + \sinh(2\Xi)$. Eq.(17), with Eqs.(14) and (19),
 135 may be rewritten as

$$F(\Xi) = \alpha_+(\Xi) \cos(\Xi) \cosh(\Xi) + \alpha_-(\Xi) \sin(\Xi) \sinh(\Xi), \quad (20)$$

136 where $2\alpha_+(\Xi) = \alpha_2(\Xi) + \alpha_4(\Xi)$, $2\alpha_-(\Xi) = \alpha_1(\Xi) - \alpha_3(\Xi)$. The dependence from the loading is
 137 completely embedded in the functions $\alpha_+(\Xi)$, $\alpha_-(\Xi)$ and it is clear that different loadings giving
 138 the same functions are equivalent inasmuch as the contact locus is concerned. Eq.(20) acquires a
 139 particularly simple form when it exists $\rho^c < \Xi$ such that the loading vanishes outside the interval

140 $[0, \rho^c]$, for then

$$\tan(\Xi) \tanh(\Xi) = -\frac{\alpha_+(\rho^c)}{\alpha_-(\rho^c)} = r \quad (21)$$

141 and the RHS r is a constant with respect to Ξ . It is observed that for r positive the contact locus
 142 sits in the interval $(\pi/2, \pi)$ and, by solution continuity, for r negative in $(\pi, \frac{3}{2}\pi)$. In this situation,
 143 loadings are equivalent inasmuch as they exhibit the same ratio r . For instance, in the case of
 144 two symmetric pairs of concentrated forces, placed at Δ_1 and $\Delta_2 > \Delta_1$, it is

$$r = -\frac{\cos(\Delta_1) \cosh(\Delta_1) + \cos(\Delta_2) \cosh(\Delta_2)}{\sin(\Delta_1) \sinh(\Delta_1) + \sin(\Delta_2) \sinh(\Delta_2)} \quad (22)$$

145 such that solving the implicit equation $r = k$, k being a real constant, gives the set of pairs Δ_1, Δ_2
 146 yielding the same contact locus $\Xi(k)$. Fig.2 shows the curves $\Delta_2 - \Delta_1$ vs. Δ_1 for $k = 1, 5, 10$.
 147 The curves may be taken as a graphical representation of the sets \mathcal{Q}_k . Indeed, Fig.3 shows that
 148 for $k = 1$, the deformed beam profiles for the cases $\Delta_2 - \Delta_1 = 0.1$ and $\Delta_2 - \Delta_1 = 1$, to which
 149 it pertains respectively $\Delta_1 = 0.8857167949$ and $\Delta_1 = 0.2529526456$, exhibit the same contact
 150 locus position $\Xi(1) = 2.347045566$. Among such loadings the superposition principle does hold.
 151 Eq.(20) is generally nonlinear in Ξ owing to both the functions α_i and A_i .

152 Let us now investigate the contribution of the boundary term and consider the situation where
 153 the beam is loaded beyond the contact locus through the line load $\sigma^l(\xi), \Xi < \xi < l$. Then, a
 154 boundary term enters the function F . Exploiting the symmetry of the Green function and the
 155 continuity of its first derivative, Eq.(17) becomes

$$F(\Xi) = \left\{ \alpha_i(\Xi) - \frac{1}{4}u_3^c(\Xi)\eta_i(\Xi) + \frac{1}{4}u_2^c(\Xi)\eta_i'(\Xi) \right\} A_i(\Xi) \quad (23)$$

156 where, in analogy with the first of Eqs.(18), it is let $B_i(\Xi) = b_i(\Xi, \Xi)$. With a bit of work, Eq.(20)

157 is now

$$\alpha_+(\Xi) \cos(\Xi) \cosh(\Xi) + \alpha_-(\Xi) \sin(\Xi) \sinh(\Xi) = \frac{1}{8} u_3^c(\Xi) [\cosh(2\Xi) + \cos(2\Xi)] - \frac{1}{8} u_2^c(\Xi) [\sinh(2\Xi) - \sin(2\Xi)]. \quad (24)$$

158 Eq.(24) provides a nonlinear equation relating the loading and the contact locus, in a symmetric
 159 layout, which gathers all the nonlinear feature of the unilateral contact problem. It also provides a
 160 mean of determining whether the beam lifts off the foundation or, rather, rests entirely supported
 161 on it. To this aim, solutions of Eq.(24) are checked against the beam length l and when it is found
 162 that $\Xi > l$, then the beam rests entirely supported by the foundation.

163 **Applications for a Winkler soil**

164 *Symmetric case*

165 Let us consider the case of a beam loaded at midspan by a unit force. Then, it is $\alpha_+ = 1$,
 166 $\alpha_- = 0$ and Eq.(20) reduces to the simple relation

$$\cosh \Xi \cos \Xi = 0, \quad (25)$$

167 which corresponds to Eq.(7) of Weitsman (1970) and yields the well-known result $\Xi = \pi/2$. We
 168 are interested in adding an end force f^l and an end couple c^l such that the contact locus remains
 169 unchanged. To this aim, a relationship between $u_2^c(\Xi)$ and $u_3^c(\Xi)$ needs be sought in order that the
 170 boundary contribution drops out. Writing the latter as at the RHS of Eq.(24) and considering that

$$\frac{1}{4} u_2^c(\Xi) = c^l + f^l(l - \Xi), \quad \frac{1}{4} u_3^c(\Xi) = -f^l, \quad (26)$$

171 given that f^l is positive when downwards and c^l when clockwise, a connection is found between
 172 c^l and f^l as follows:

$$c^l = -f^l (l - \Xi + R^W(\Xi)), \quad (27)$$

173 where the positive function is let

$$R^W(\Xi) = \frac{\cosh(2\Xi) + \cos(2\Xi)}{\sinh(2\Xi) - \sin(2\Xi)}. \quad (28)$$

174 In particular, for $\Xi = \pi/2$, it is $R^W(\Xi) = 0.6536439910$.

175 As a second application, the case of a pair of concentrated forces, symmetric about $\xi = 0$ and
 176 placed at a distance $2\Delta > 0$ apart, is considered. Then, it is $\alpha_i = \eta_i(\Delta)$ and Eq.(21) gives a
 177 connection between the contact locus and the distance $\Delta < \Xi$, namely

$$\tan \Xi \tanh \Xi = -\frac{1}{\tan \Delta \tanh \Delta}. \quad (29)$$

178 It is immediate to see that the sign of both the left and the right hand side is given by the tangent
 179 terms: for $\Delta \in [0, \pi/2)$, the RHS is negative and solutions are to be found in the interval $\Xi \in$
 180 $[\pi/2, \pi)$. By the same token, for $\Delta \in [\pi/2, \pi)$, continuity of the solution suggests taking $\Xi \in$
 181 $[\pi, \frac{3}{2}\pi)$. It is further observed that the situation $\Delta = \Xi$ is not allowed. If the applied forces are far
 182 apart beyond a limiting spacing $2\tilde{\Delta}$, lift-off takes place in the neighborhood of the origin as well,
 183 in a discontinuous contact scenario. Such limiting spacing occurs when

$$u^c(0) = b_i(0, \Xi) \int_0^\Xi \sigma^c(\xi) \eta_i(\xi) d\xi = b_i(0, \Xi) \eta_i(\tilde{\Delta}) = 0 \quad (30)$$

184 and the grazing condition $u_1^c(0) = 0$ follows directly from the symmetry requirement. Here, it is

$$\begin{aligned} b_1(0, \Xi) &= \frac{1}{2\Lambda^2} [\cos(2\Xi) + \cosh(2\Xi) - \sinh(2\Xi) - \sin(2\Xi) + 2] \\ b_2(0, \Xi) &= \frac{1}{2\Lambda^2} [\cos(2\Xi) - \cosh(2\Xi) + \sinh(2\Xi) + \sin(2\Xi)] \end{aligned}$$

185 and $b_2(0, \Xi) + b_4(0, \Xi) = 1$, $b_1(0, \Xi) - b_3(0, \Xi) = -1$. For a general Δ , Eq.(30) with Eq.(29)
 186 yields

$$b_2(0, \Xi) - b_1(0, \Xi) \tan \Xi \tanh \Xi = f(\Delta), \quad (31)$$

187 being

$$f(\Delta) = -\frac{e^{-\Delta}}{2 \sinh \Delta} (1 + \cot \Delta). \quad (32)$$

188 Eq.(31) lends a connection between the contact locus and the spacing Δ . Since $\Delta > 0$ demands
 189 $\Xi > \pi/2$, the LHS of (31) is positive and to get a positive value for the RHS it must be $\Delta > \tilde{\Delta} =$
 190 2.356194490. Fig.4 shows the beam bending moment, shearing force and contact pressure in the
 191 contact interval. As on the verge of lifting-off, the latter vanishes at midspan.

192 As a third example, Eq.(24) is put to advantage for the case of a constant line loading q extend-
 193 ing up to the abscissa l_q and a concentrated force $2f_0$ at midspan. When $l_q = l$ the classic solution
 194 for a concentrated load $2f_0$ acting at midspan of a beam with weight per unit length q is obtained.
 195 This situation is generally more involved than the previous ones because, for l_q large enough, the
 196 contact locus sits within the loaded interval. Eq.(24) gives

$$2f_0 \cos \Xi \cosh \Xi + \frac{q}{2} [\sinh 2\Xi + \sin 2\Xi] = -q(l_q - \Xi) [\cosh 2\Xi + \cos 2\Xi] \\ - \frac{1}{2}q(l_q - \Xi)^2 [\sinh 2\Xi - \sin 2\Xi], \quad (33)$$

197 provided that $l_q > \Xi$. When $l_q < \Xi$ it is

$$2f_0 \cos \Xi \cosh \Xi + q [\cos l_q \sinh l_q + \sin l_q \cosh l_q] \cos \Xi \cosh \Xi \\ + q [-\cos l_q \sinh l_q + \sin l_q \cosh l_q] \sin \Xi \sinh \Xi = 0. \quad (34)$$

198 For $f_0 = 1$, Fig.5 plots both Eqs.(33,34) in their realms of validity, the boundary between them
 199 being represented by the bisector. It is seen that for q small ($q = 0.01$), the contact locus tends to
 200 the classic result $\pi/2$ in a wide range of l_q . At $q = 0.05$, it is observed that for a given l_q multiples
 201 solutions for Ξ are found and a maximum value for $l_q > \Xi$ appears. Beyond such maximum, a
 202 second branch of solution exists with $\Xi > l_q$. It rests to be seen whether the beam is long enough
 203 to warrant the admissibility of such solution. In order to discuss the multiplicity of solutions, Fig.6
 204 shows the beam profiles for $q = 0.05$ and $l_q = 3$, when the solution $\Xi < l_q$, curve (a), and $\Xi > l_q$,

curve (b), are considered. It is seen that the solution (b) leads to interpenetration and must be discarded. However, above the maximum value for l_q , solution (a) disappears and solution (b) becomes admissible.

Non-symmetric case

Let us now drop the symmetry assumption and deal with a general continuous contact scenario (Fig.7). Then, two contact loci, $\Xi_1 < \Xi_2$, are expected and Eq.(16) becomes

$$u^c(\zeta) = a_i(\zeta, \Xi_1, \Xi_2) \int_{\Xi_1}^{\zeta} \sigma^c(\xi) \eta_i(\xi) d\xi + b_i(\zeta, \Xi_1, \Xi_2) \int_{\zeta}^{\Xi_2} \sigma^c(\xi) \eta_i(\xi) d\xi. \quad (35)$$

Likewise, two limits are now considered

$$\lim_{\zeta \rightarrow \Xi_2} u^c(\zeta) = 0 \quad \Leftrightarrow \quad \alpha_i(\Xi_1, \Xi_2) A_i(\Xi_1, \Xi_2) = 0, \quad (36a)$$

$$\lim_{\zeta \rightarrow \Xi_1} u^c(\zeta) = 0 \quad \Leftrightarrow \quad \alpha_i(\Xi_1, \Xi_2) B_i(\Xi_1, \Xi_2) = 0, \quad (36b)$$

having let $A_i(\Xi_1, \Xi_2) = a_i(\Xi_2, \Xi_1, \Xi_2)$, $B_i(\Xi_1, \Xi_2) = b_i(\Xi_1, \Xi_1, \Xi_2)$ and $\alpha_i = \int_{\Xi_1}^{\Xi_2} \sigma^c(\xi) \eta_i(\xi) d\xi$.

Despite the fact that the analysis follows along the same path as in the symmetric situation, the increased mathematical complication suggests to limit the discussion to a single concentrated force.

Then, $\sigma^c = \delta(\xi, \Delta)$ and it is expedient to set the ξ -axis origin at $\xi = \Delta$ without loss of generality.

Eqs.(36) become

$$A_1(\Xi_1^*, \Xi_2^*) + A_3(\Xi_1^*, \Xi_2^*) = 0, \quad (37a)$$

$$B_1(\Xi_1^*, \Xi_2^*) + B_3(\Xi_1^*, \Xi_2^*) = 0, \quad (37b)$$

with the understanding that $\Xi_1^* = \Xi_1 - \Delta$ and likewise $\Xi_2^* = \Xi_2 - \Delta$. It is easy to show that for a symmetric disposition of the contact loci, i.e. $\Xi_1^* = -\Xi_2^*$, Eqs.(37) collapse into a single equation, which corresponds to Eq.(25). Indeed, every time a solution exists with $\Xi_1^* = -\Xi_2^*$ for either of the Eqs.(37), then it complies with both. It is natural to introduce $d = \Xi_1^* + \Xi_2^*$, the deviation

with respect to a symmetric condition (Fig.7). Fig.8 draws the solution curves d vs. Ξ_2^* for the first (dash curve) and the second (solid curve) of Eqs.(37). In this plot, each intersection point is a possible solution of the system. The shaded area, bounded from below by the dotted curve $d = 2\Xi_2^*$, is ruled out as it leads to a contact locus $\Xi_1^* > \Xi_2^*$. It is seen that a discrete number of solutions is available yet the ones with minimum Ξ_2^* and d are specially interesting. As long as $l_2 \geq \Delta + \pi/2$, which means that the solution points at $\Xi_2^* \geq \pi/2$ are admissible, the classic solution $d = 0$, corresponding to a symmetric layout, is retrieved (point A in Fig.8). When such condition no longer holds, one of the beam ends plunges into the foundation, say the right end, whence it is $\Xi_2 = l_2$ fixed. Then, only the second equation of (37) survives (solid curve) and it provides d vs. $\Xi_2^* = l_2 - \Delta$. Note that $\Xi_1^* = d - \Xi_2^*$ or, equivalently, $\Xi_1 = d - l_2 + 2\Delta$.

It is interesting to describe the system behavior as Δ increases and the loading is brought closer and closer to the beam end. Then, d is found moving along the solid curve from point A to point B and beyond, until the origin is reached. It is seen that d acquires decreasing (with Δ) negative values until the point B is reached, where the layout with maximum deviation from symmetry $|d|$ is found. Since, for the most part, the solid curve possesses unit slope, in the neighborhood of A it is $d \approx -\Delta$ and the left contact locus moves rightwards proportionally with Δ , i.e. $\Xi_1 \approx -l_2 + \Delta$. The contact imprint, however, is given by $l_c = \Xi_2 - \Xi_1$ and it shrinks as

$$l_c \approx 2l_2 - \Delta. \quad (38)$$

Fig.9 plots the position of the left contact locus against the loading offset Δ for a beam with $l_2 = \pi/2$, that is starting from point A . Since both the absolute position Ξ_1 and the relative position Ξ_1^* are given, the difference between the curves equals Δ , while the distance $l_2 - \Xi_1$ gives the contact imprint length l_c . The deviation from symmetry, d , is also shown as the difference from the dash-dot curve and $-l_2$. Dotted curves show the positive and negative unit slope, which confirm the behavior previously inferred for Ξ_1 , l_c and d . Beyond B , a substantial rotation of the beam occurs which leads to a very small contact imprint and an almost symmetric situation. Here,

246 d increases towards zero again.

247 It is easy to obtain the results numerically developed in Zhang and Murphy (2004) for a beam
248 of varying length l loaded symmetrically and non-symmetrically by a unit force. Three regimes
249 are considered:

- 250 1. when $\Xi_2 < l_2$ and $|\Xi_1| < l_1$,
- 251 2. when, say, $\Xi_2 > l_2$ and $|\Xi_1| < l_1$,
- 252 3. when $\Xi_2 > l_2$ and $|\Xi_1| > l_1$.

253 In regime 1, both left and right contact loci sit inside the beam, the symmetric solution $d = 0$ is
254 admitted and the contact imprint length $l_c = \Xi_2 - \Xi_1 = 2\Xi_2^*$ is constant. In regime 2, the beam right
255 length l_2 is too short to warrant that the right contact locus sits inside the beam. Conversely, the
256 left length l_1 accommodates the left contact locus. It is observed that this regime demands a non-
257 symmetric loading situation. Having let $l_2 = k_2 l$ and $\Delta = k_\Delta l$, where $k_2, k_\Delta < 1$, Eq.(38) shows
258 that the contact imprint length scales linearly with l with a proportionality coefficient $2k_2 - k_\Delta$.
259 Finally, regime 3 is such that both contact loci exceed the beam left and right length. The beam
260 rests entirely supported by the soil and the contact imprint length corresponds to the beam length.
261 In a symmetric layout, beam length scaling brings the system from regime 1 to regime 3 or vice
262 versa and the contact imprint length is either constant or equal to l , as numerically found in Zhang
263 and Murphy (2004). In a non-symmetric layout, the system undergoes all three regimes and, from
264 1 to 3, the contact imprint length is constant, decreases with coefficient $2k_2 - k_\Delta$ and finally equals
265 the beam length, i.e. coefficient 1.

266 PASTERNAK SOIL

267 Let us consider the case of a E-B beam resting on a tensionless Pasternak soil in a symmetric
268 continuous contact scenario. The governing ODE reads, in the contact interval,

$$\frac{1}{4}u_4^c - \alpha u_2^c + u^c = \sigma^c, \quad (39)$$

where the dimensionless quantity $\alpha = \beta^2 k_G / k$ is introduced and k_G is the shear modulus of the foundation (Selvadurai 1979). The BCs enforce symmetry at $\xi = 0$, as at Eqs.(2), and bending moment and shearing force continuity at the contact locus Ξ , as in (3). Furthermore, two BCs involve the soil profile through setting displacement and slope continuity (Kerr 1976), i.e.

$$u^c(\Xi) = u^s(\Xi), \quad u_1^c(\Xi) = u_1^s(\Xi). \quad (40)$$

The problem is only formally self-adjoint, as the Green function for the beam in the contact region is determined by symmetric BCs at $\xi = 0$, as at Eqs.(10), and homogeneous conditions at $\xi = \Xi$ as follows:

$$\frac{1}{4}G''(\Xi, \zeta) - \alpha G(\Xi, \zeta) = 0, \quad \frac{1}{4}G'''(\Xi, \zeta) - \alpha G'(\Xi, \zeta) = 0. \quad (41)$$

It is observed that, in the limit for $\alpha \rightarrow 0$, the BCs (11,15) for the Winkler soil are retrieved. The displacement in the contact region at $\zeta < \Xi$ is given by

$$u^c(\zeta) = \int_0^\Xi \sigma^c(\xi) G(\xi, \zeta) d\xi - \frac{1}{4} [u_3^c G - u_2^c G']_0^\Xi + \left[\left(-\frac{1}{4} G'' + \alpha G \right) u_1^c + \left(\frac{1}{4} G''' - \alpha G' \right) u^c \right]_0^\Xi, \quad (42)$$

where $G(\xi, \zeta)$ takes the shape (13) and, of course, the functions $a_i(\zeta, \Xi)$ and $b_i(\zeta, \Xi)$ differ from the case of the Winkler foundation. The fundamental set is taken in even/odd fashion

$$\{\eta_i(\xi)\} = \left\{ \cosh\left(\sqrt{\lambda_1}\xi\right), \sinh\left(\sqrt{\lambda_1}\xi\right), \cosh\left(\sqrt{\lambda_2}\xi\right), \sinh\left(\sqrt{\lambda_2}\xi\right) \right\}.$$

Here, for the sake of definiteness, it is assumed $\alpha > 1$ whence $\lambda_{1,2} = 2(\alpha \pm \sqrt{\alpha^2 - 1})$. The term in square brackets at RHS of (42) vanishes owing to the BCs (2,10,41).

In the absence of a soil loading σ^s , it is $u_1^s = -\alpha^{-1/2}u^s$, whence the BCs (40) yield the contact locus equation

$$u^c(\Xi) + \sqrt{\alpha}u_1^c(\Xi) = 0. \quad (43)$$

Eq.(43) sets the contact locus without recurring to the soil profile. It may be written as

$$F(\Xi) = \int_0^\Xi \sigma^c(\xi) K(\xi, \Xi) d\xi - \frac{1}{4} \lim_{\zeta \rightarrow \Xi} [u_3^c K(\Xi, \zeta) - u_2^c K'(\Xi, \zeta)], \quad (44)$$

wherein a new kernel function is defined in terms of the Green function G

$$K(\xi, \zeta) = G(\xi, \zeta) + \sqrt{\alpha} \frac{\partial G}{\partial \zeta}(\xi, \zeta). \quad (45)$$

Now the argument runs parallel to the treatment given for the Winkler soil. However, it is emphasized that neither the kernel G nor K is symmetric, for the problem for the Green function is no longer self-adjoint. When the beam lifting-off the soil is load-free, Eq.(44) gives an expression formally analogous to (17)

$$F(\Xi) = \mathcal{A}_i(\Xi) \alpha_i(\Xi), \quad (46)$$

being understood that $\mathcal{A}_i(\Xi) = A_i(\Xi) + \sqrt{\alpha} \bar{A}_i(\Xi)$ and

$$A_i(\Xi) = \lim_{\zeta \rightarrow \Xi} a_i(\zeta, \Xi), \quad \bar{A}_i(\Xi) = \lim_{\zeta \rightarrow \Xi} \frac{\partial a_i}{\partial \zeta}(\zeta, \Xi). \quad (47)$$

The symmetric layout accounts for the vanishing of the functions A_2, A_4 and likewise for \bar{A}_2, \bar{A}_4 . Besides, A_1 equals A_3 and \bar{A}_1 equals \bar{A}_3 provided that the role of λ_1 and λ_2 is exchanged. After some lengthy manipulations, it is found, omitting a common non-vanishing denominator,

$$\mathcal{A}_1 = \lambda_1(\lambda_1 - \lambda_2) \left[\cosh(\sqrt{\lambda_2} \Xi) + \sqrt{\alpha \lambda_2} \sinh(\sqrt{\lambda_2} \Xi) \right], \quad (48a)$$

$$\mathcal{A}_3 = \lambda_2(\lambda_2 - \lambda_1) \left[\cosh(\sqrt{\lambda_1} \Xi) + \sqrt{\alpha \lambda_1} \sinh(\sqrt{\lambda_1} \Xi) \right], \quad (48b)$$

whence A_1 and A_3 are easily retrieved letting $\alpha \rightarrow 0$. As expected, \mathcal{A}_3 equals \mathcal{A}_1 once the role of λ_1 and λ_2 is exchanged.

296 When accounting for the contribution from the lift-off interval, it is

$$F(\Xi) = \mathcal{A}_i(\Xi)\alpha_i(\Xi) - \frac{1}{4} [u_3^c(\Xi)\mathcal{A}_i(\Xi)\eta_i(\Xi) - u_2^c(\Xi)\mathcal{A}_i(\Xi)\eta_i'(\Xi)], \quad (49)$$

297 where use has been made of the continuity properties of the Green function G .

298 **Applications for a Pasternak soil**

299 Let us consider the classic situation of a beam resting on a tensionless Pasternak soil and
 300 loaded at midspan by a unit force. Then, it is $\sigma^c = \delta(\xi, 0)/2$ and Eq.(46), together with Eqs.(48)
 301 and divided through by $(\lambda_1 - \lambda_2)$, gives

$$F(\Xi) = \lambda_1 \left[\cosh \left(\sqrt{\lambda_2} \Xi \right) + \sqrt{\alpha \lambda_2} \sinh \left(\sqrt{\lambda_2} \Xi \right) \right] - \lambda_2 \left[\cosh \left(\sqrt{\lambda_1} \Xi \right) + \sqrt{\alpha \lambda_1} \sinh \left(\sqrt{\lambda_1} \Xi \right) \right]. \quad (50)$$

302 The first positive root of F gives, when $\beta = 2.5$, the result $\Xi = 0.8423946552$. We wish to
 303 determine the loading condition at the beam end such that the contact locus is preserved. Again,
 304 we need to vanquish the last term of Eq.(49), i.e.

$$c^l = f^l (l - \Xi + R^P(\Xi)), \quad R^P(\Xi) = \frac{\mathcal{A}_i(\Xi)\eta_i(\Xi)}{\mathcal{A}_i(\Xi)\eta_i'(\Xi)}.$$

305 It is observed that for $\alpha \rightarrow 0$ the Pasternak soil becomes a Winkler soil and indeed $R^P(\Xi) \rightarrow$
 306 $R^W(\Xi)$. In particular, for $\Xi = 0.8423946570$, it is $R^P(\Xi) = 0.2763085352$.

307 When two symmetrically placed unit forces are far apart enough, the beam stands on the verge
 308 of lifting off at the origin. Letting the force distance be 2Δ and making use of Eqs.(48), Eq.(46)
 309 specializes to

$$F(\Xi) = \lambda_1 \cosh(\sqrt{\lambda_1} \Delta) \left[\cosh \left(\sqrt{\lambda_2} \Xi \right) + \sqrt{\alpha \lambda_2} \sinh \left(\sqrt{\lambda_2} \Xi \right) \right] - \lambda_2 \cosh(\sqrt{\lambda_2} \Delta) \left[\cosh \left(\sqrt{\lambda_1} \Xi \right) + \sqrt{\alpha \lambda_1} \sinh \left(\sqrt{\lambda_1} \Xi \right) \right], \quad (51)$$

310 having omitted the common factor $\lambda_1 - \lambda_2$ and provided that $\alpha > 1$. Seeking the solution of
 311 $F(\Xi) = 0$ lends the curves Ξ vs. Δ . For a Pasternak foundation, the counterpart of Eq.(30)
 312 demands that the dimensionless contact pressure $-u_4^c/4$ vanishes at the origin. With Eq.(39), the
 313 requirement amounts to

$$\alpha_i(\Xi) \left(B_i(\Xi) - \alpha \bar{\bar{B}}_i(\Xi) \right) = 0, \quad (52)$$

314 being, in analogy with Eq.(47), $B_i(\Xi) = \lim_{\zeta \rightarrow \Xi} b_i(\zeta, \Xi)$, $\bar{\bar{B}}_i(\Xi) = \lim_{\zeta \rightarrow \Xi} \frac{\partial^2 b_i}{\partial \zeta^2}(\zeta, \Xi)$. Fig.10 plots
 315 the solution curves of Eq.(51) (dash) and Eq.(52) (solid curve) for $\alpha = 1.1, 5$ and 10 . The bisector
 316 is also plotted as a dotted line for solutions are admissible inasmuch as $\Xi > \Delta$. When the forces are
 317 brought farther apart, the contact locus position moves along the dash curve until the solid curve
 318 is met. At such limiting distance $2\tilde{\Delta}$, the continuous contact scenario breaks down and lift-off
 319 appears in the neighborhood of the origin.

320 CONCLUSIONS

321 In this paper, the free-boundary problem of tensionless contact for a beam resting on either
 322 a Winkler or a Pasternak two-parameter elastic foundation is addressed. The classic approach to
 323 the problem consists of integrating the deflection curves for the beam in contact with the soil, the
 324 beam lifting off it and the soil and then matching solutions at the contact locus, which is a problem
 325 unknown. When matching solutions, an extra condition exists that determines the contact locus.
 326 Conversely, in this paper, a Green function approach is put forward which aims at determining a
 327 direct (nonlinear) connection between the loading and the contact locus. Once the contact locus is
 328 set, the problem reduces to solving a classic linear BVP in the contact and lift-off regions. This
 329 way of approaching the problem lends considerable advantages over the classic one. First, the
 330 connection between the contact locus position and the loading is expressed as a general relation,
 331 which allows to determine what features of the loading affect the contact locus. This implies
 332 that it is possible to build the set of loadings whose application leads to the same contact locus.
 333 Among such loadings, the superposition principle holds. Second, solutions are obtained once some
 334 assumptions are made concerning the contact layout. Accordingly, results must be checked against

such assumptions at the end of the procedure. Although this part is common to both approaches, it is shown that here the required assumptions are weaker. For instance, the non-symmetric contact problem for a Winkler foundation is analyzed in general and two families of solution curves are obtained: one for the left and one for the right contact locus. When the beam length is insufficient to accommodate both contact loci, one curve is simply dropped in place of the constraint that fixes the contact at the beam end. Conversely, when deflection curves are integrated, whether lift-off exists needs be assumed from the start, given that the BCs depend on such assumption. Several applications are presented for both the cases of symmetric and non-symmetric contact. Furthermore, comparison with the existing literature is carried out.

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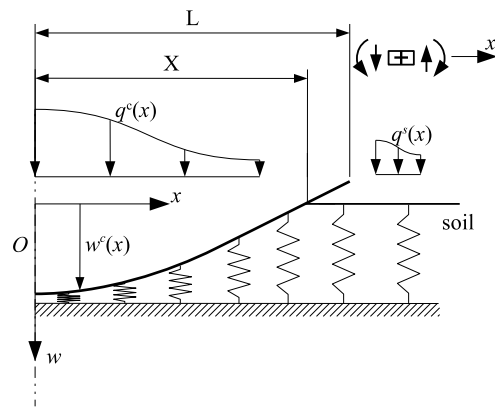


FIG. 1. Symmetric continuous contact scenario with lift-off for a beam on a tensionless Winkler foundation

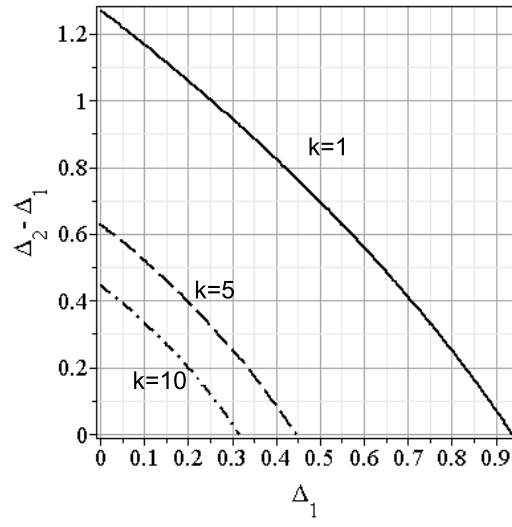


FIG. 2. $\Delta_1, \Delta_2 - \Delta_1$ pairs giving the same contact locus $\Xi(k)$

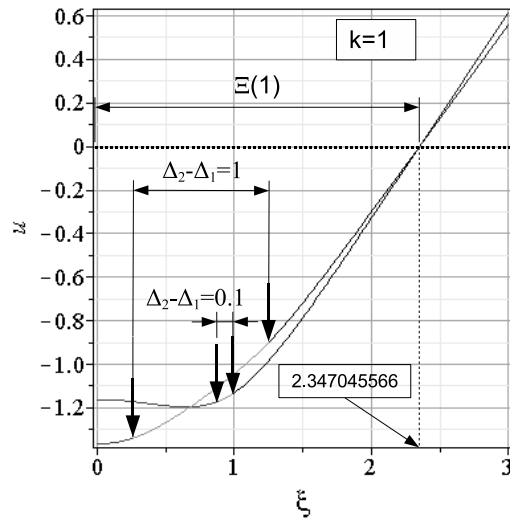


FIG. 3. Beam profile for $k = 1$ and either $\Delta_2 - \Delta_1 = 1$ or $\Delta_2 - \Delta_1 = 0.1$

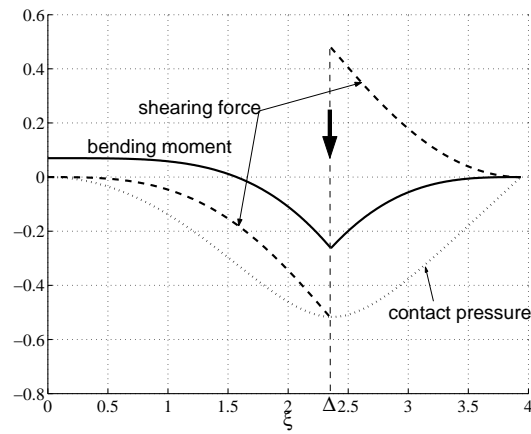


FIG. 4. Bending moment, shearing force and contact pressure for a beam on a Winkler foundation at the onset of midspan lift-off

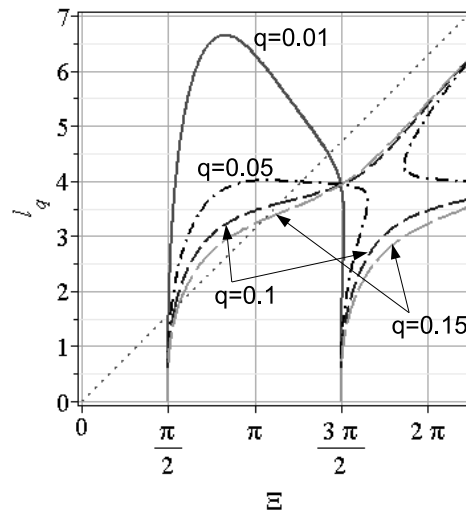


FIG. 5. l_q vs Ξ for $f_0 = 1$ as given by Eqs.(33,34)

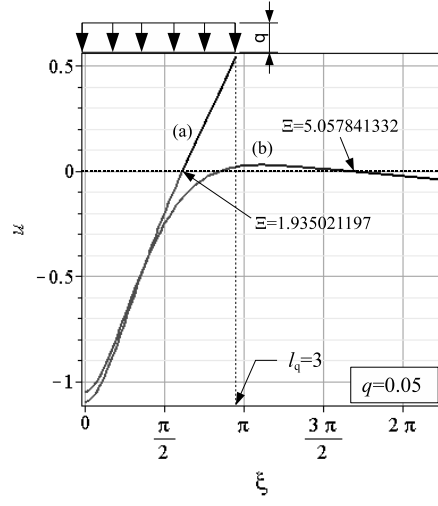


FIG. 6. Beam profiles for $\Xi < l_q$ (a) and $\Xi > l_q$ (b), for $q = 0.05$, $l_q = 3$, $f_0 = 1$

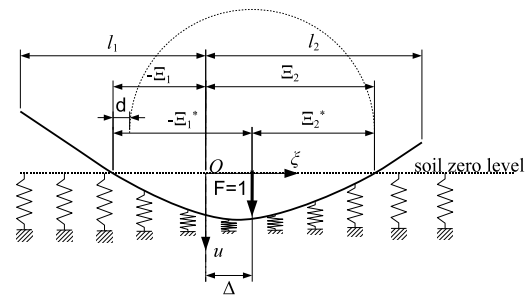


FIG. 7. Beam on a Winkler soil in a non-symmetric layout

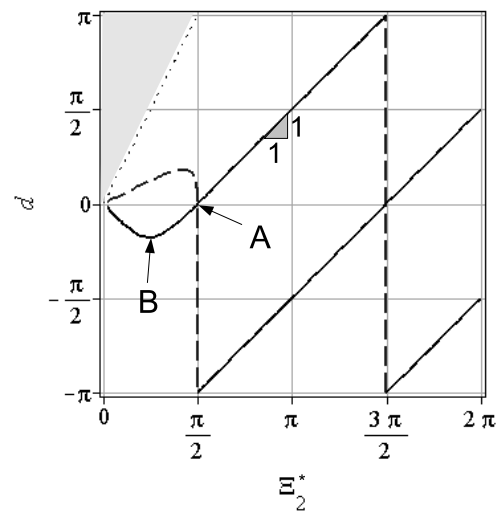


FIG. 8. Plots of the first (dash) and the second (solid) of Eqs.(37)

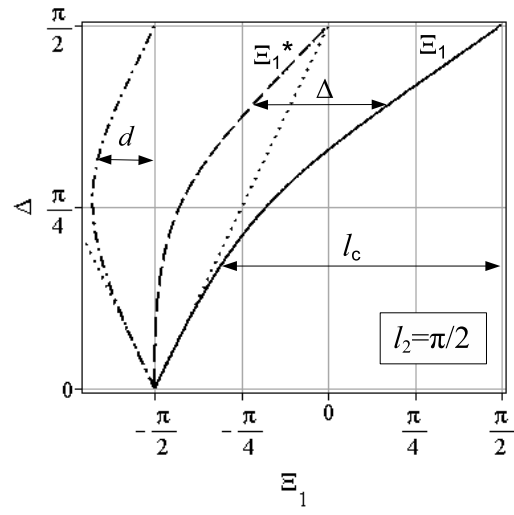


FIG. 9. Left contact locus absolute (solid) and relative (dash) position for a beam loaded by a concentrated force at $\xi = \Delta$.

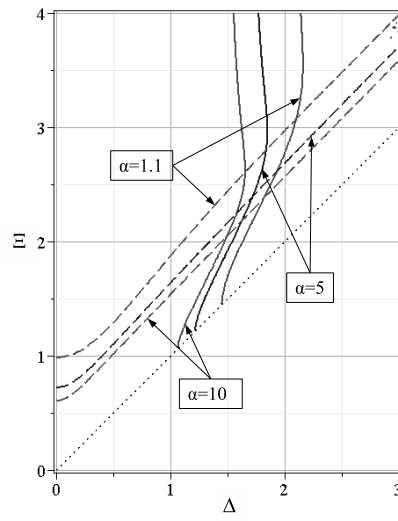


FIG. 10. Ξ vs. Δ curve (dash) for $\alpha = 1.1, 5$ and 10 , and limiting curve (solid) which marks the onset of a discontinuous contact scenario