International Journal of Engineering Science 225 (2026) 104548

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Full length article

Completing Eringen's nonlocal elasticity theory and its connection with surface elasticity

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ARTICLE INFO

Keywords:
Nonlocal elasticity
Surface elasticity
Kernel modification
Inhomogeneous boundary conditions
Elastic potential energy

ABSTRACT

Eringen's nonlocal elasticity theory is known to suffer from boundary-related inconsistencies, which arise from the presence of additional Boundary Conditions, commonly referred to as Constitutive (CBCs), which are embedded in the Green's function-type attenuation functions and supplement the problem BCs. To avoid over-determination, the method of kernel modification has been recently proposed, that enforces consistency between the CBCs and the prescribed BCs. In so doing, the kernel can no longer be of the difference type. Still, we prove that the influence of the system boundaries extends beyond the issue of over-determination. More specifically, we show that the differential and the integral formulation of nonlocal elasticity are equivalent provided that a boundary term is set to zero, that amounts to requiring that the motion equations are satisfied on the boundary. Indeed, this condition is necessary for the problem closure, because, once the BCs are incorporated into the kernels, they are automatically satisfied by *any* general solution of the associated differential formulation. Moreover, we show that Eringen's single-integral formulation fails to accommodate inhomogeneous BCs. Therefore, an extended integral formulation is introduced that matches the differential formulation in the presence of surface loads. This extended formulation admits a natural reinterpretation in terms of surface elasticity, thereby clarifying the role of boundary effects in nonlocal continua. As an application, the generalized Rayleigh problem is examined for a half-plane with an elastically constrained surface, which reveals the existence of localized surface waves that have no counterpart in classical elasticity.

1. Introduction

The theory of nonlocal elasticity has attracted sustained interest over the past decades as an effective framework for modeling size effects and long-range interactions in materials whose mechanical response cannot be adequately described by classical local continuum theories, see Eringen (1972, 1983), Eringen and Edelen (1972), Kröner (1967), Rogula (1982). By allowing the stress at a point to depend on the strain field over the entire spatial domain through the attenuation function (kernel) G, nonlocal models have been shown to capture experimentally observed behaviors in micro- and nano-structured solids, lattice materials, and heterogeneous media. However, when dealing with integral nonlocality, the equations of motions take the form of integro-differential equations, that are very difficult to solve. For this reason, Eringen (1983) proposed to adopt the integral kernel as the Green's function of

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https://doi.org/10.1016/j.ijengsci.2026.104548
Received 16 February 2026; Received in revised form 3 April 2026; Accepted 4 April 2026
Available online 8 April 2026
0020-7225/

a differential operator $\mathcal{L}$ with some specific properties. As a result, the problem may be entirely recast in differential form with considerable mathematical advantage, through the application of the differential operator to the original equations of motion. Although the integral form of nonlocal elasticity has been particularly influential, owing to its clear physical interpretation and its direct connection to interatomic or intermolecular interactions, the differential form is by far the simplest to deal with and for this reason it covers almost all the results currently available in the literature. As a noticeable exception, we refer to the paper by Martin (2026), which adopts the integral formulation.

Recently, and mostly in connection with nonlocal beam models, it was realized that the integral formulation carries along extra boundary conditions, named constitutive (CBCs), which come embedded in the Green's function kernel (Benvenuti & Simone, 2013; Mikhasev & Nobili, 2020; Romano et al., 2017). Since CBCs come on top of the problem BCs, the resulting formulation is over-determined and almost always ill-posed (Challamel & Wang, 2008). On the other hand, application of the differential operator $\mathcal{L}$ dispenses with the CBCs and, consequently, the differential formulation appears well-posed simply because the CBCs are neglected and, as a result, the problem differs from the original one under scrutiny. Very recently, Kaplunov et al. (2022, 2023) developed an asymptotic solution of the differential nonlocal problem for Rayleigh waves in a half-plane and found that it failed to satisfy the original equations of motion. Later, Nobili and Pramanik (2025a, 2025b), Pramanik and Nobili (2025) introduced the kernel modification approach, by which the CBCs are made to correspond with the BCs of the problem, thus removing ill-posedness. As a result, any general solution of the differential model, when plugged into the integral model, satisfies the BCs which can no longer be used to specify the general solution. Instead, this is achieved precisely by enforcing the original equations of motion, whence the apparent inconsistency revealed by Kaplunov et al. (2022) is, in fact, necessary for the problem closure.

This is the point of departure of this paper, that investigates the origin of this inconsistency by which the solution of the differential model fails to satisfy the original equations of motion from which the differential model was obtained. It is found that this outcome is strictly related to the presence of boundaries in the domain, which introduces boundary contributions in the differential model that have been neglected. Such boundary contributions underpin the fascinating connection between nonlocal and surface elasticity. Furthermore, we show that the integral formulation, that is mostly associated with nonlocal elasticity, namely $t_{ij} = \int G\sigma_{ij}$ where the kernel G possesses the reciprocity property, is, in fact, incapable of dealing with inhomogeneous BCs and it is incompatible with the differential constitutive formulation $\mathcal{L}t_{ij} = \sigma_{ij}$. Consequently, we introduce an improved integral formulation and show that this new framework successfully predicts the existence of generalized Rayleigh waves in an elastically constrained half-plane. On reducing this result to the case of classical elasticity, we find that generalized Rayleigh waves, now no longer dispersive, are still supported, provided that their speed is *lower* than the speed of shear waves. This constraint enforces an upper limit on the surface spring stiffness, whence the well-know result follows that, for local elasticity, Rayleigh waves are not supported by a rigid frictionless boundary.

2. Eringen's nonlocal theory

According to the *nonlocal* (linear) theory of elasticity, as introduced by Kröner (1967) and later expanded by Eringen and Edelen (1972), the nonlocal stress, t_{ij}, in the *isotropic* body $\mathbb{B}$, is obtained from the local stress, σ_{ij}, by integration through the *attenuation function (also influence function or kernel)*, G,

$$t_{ij}(\boldsymbol{x}) = \int_{\mathbb{B}} G(\boldsymbol{x},\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ denote position vectors in the domain $\mathbb{B}$ and $i, j \in \{1, 2, 3\}$. Consequently, nonlocal stress may be interpreted as a weighted average of the local stress throughout the body, with the close neighborhood of $\boldsymbol{x}$ bringing the largest contribution. For a discussion of the physical requirements that are imposed on the attenuation function G, the interested reader may see Eringen (1984) and, recently, Nobili and Pramanik (2025a). For our present purposes, we mention the reciprocity property, by which

$$G(\boldsymbol{x},\boldsymbol{\xi}) = G(\boldsymbol{\xi},\boldsymbol{x}), \tag{2}$$

that implies that the contribution at the point $\boldsymbol{x}$ induced by a source in $\boldsymbol{\xi}$ equals the action at $\boldsymbol{\xi}$ induced by a source in $\boldsymbol{x}$. In this paper, for the sake of clarity, we restrict attention to kernels which are the Green's function associated with a given operator $\mathcal{L}$. More specifically, we adopt the standard Helmholtz kernel that decays at infinity and it satisfies (Eringen, 1984, Eq.(7.28))

$$\mathcal{L}_{\xi} G(\boldsymbol{x},\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}), \tag{3}$$

where $\delta(\boldsymbol{x}) = \prod_{i=1}^{3} \delta(x_i)$ denotes Dirac's delta generalized function in three dimensions and we have defined the Helmholtz differential operator

$$\mathcal{L} \equiv 1 - \varepsilon^2 \,\triangle, \tag{4}$$

with $\triangle = \operatorname{div}\operatorname{grad}$ denoting the Laplace operator. When necessary, a subscript attached to an operator is used to specify the independent variable(s) on which it operates, e.g. $\mathcal{L}_{\xi}$ operates on ξ. Here, $\varepsilon > 0$ is a characteristic length which describes the length-scale of local stress attenuation, i.e. the contribution of the local to the nonlocal stress decays exponentially with the distance as measured in terms of ε.

Nonlocal stresses need to satisfy the equations of motion (in the absence of body forces)

$$t_{ij,j} = \rho \ddot{u}_i, \tag{5}$$

where $i, j \in \{1, 2, 3\}$ and twice repeated subscripts are summed over (Einstein's summation convention). Here, a subscript comma denotes differentiation with respect to the relevant coordinate, e.g. $u_{i,j} = \partial u_i/\partial x_j$ and a superscript dot indicates time differentiation, i.e. $\dot{u}_1 = \partial u_1/\partial t$. For notational simplicity, the mass density per unit volume, ρ, is assumed constant throughout the body. In light of (1), the motion. Eq. (5) constitute a set of integro-differential relations. In general, each nonlocal stress t_{ij} is associated with a specific attenuation function, α_{ij}. However, to simplify notation, we assume for the moment that a single attenuation function, G, applies to any set of subscripts i, j, which implies that all nonlocal stresses are obtained from Eringen's integral formula (1) using the same attenuation function G. We remark that this assumption is merely introduced to avoid a third pair of subscripts ij which would invalidate the summation convention and it may be easily dropped.

The usual solution procedure, already suggested by Eringen, exploits the fact that the attenuation function G is a Green's function. Consequently, one avoids dealing directly with the set of integro-differential Eq. (5) and, instead, is confronted with the simpler system of PDEs (Eringen, 1984, Eq.(8.8))

$$\sigma_{ij,j} = \rho \mathcal{L} \ddot{u}_i, \tag{6}$$

that is the so-called *differential nonlocal model*. In this paper, we shall show that this reduction from the integro-differential to the purely differential model, that is arrived at by an application of the operator $\mathcal{L}$ to the motion equations and by using the result (Eringen, 1984, Eq.(7.25))

$$\mathcal{L} t_{ij} = \sigma_{ij}, \tag{7}$$

is generally incorrect. As in local elasticity, the local stresses σ_{ij} are related to the displacement field u_i through some constitutive assumption and, for simplicity, we consider that for isotropic materials, as in (38). As usual, the constitutive response is expressed through the linear deformation tensor $E_{ij} = (u_{i,j} + u_{j,i})/2$. Consequently, the differential nonlocal model (6) may be entirely recast in terms of displacements as in Eq. (45), and it acquires a structure similar to that of the Navier equations of local elasticity, see Eq. (45) and Eringen (1984, Eq.(8.9)) and Eringen (2002, Eq.(6.9.15)).

In many cases, the system of PDEs (6) can be solved explicitly in terms of displacements pending a set of unknown constants $\{A_p\}$, $p = 1, 2, \ldots, N$, which, in analogy with local elasticity, are then set by enforcing the BCs of the problem.[1] However, this is only possible inasmuch as the BCs are also expressed in terms of displacement. Whenever stresses are called into question, they are nonlocal stresses (although, in the literature, there is ample evidence of the improper use of local stresses) and a set of Fredholm integral equations of the first kind is arrived at. This set of equations is generally very difficult, and, most often, impossible, to solve. Indeed, impossibility arises from the fact that, embedded in the attenuation functions, comes a set of boundary conditions, often named *constitutive* (CBCs), although the name is somewhat deceiving, which needs to be satisfied on top of the natural boundary conditions (BCs). Indeed, CBCs usually do not correspond to the BCs of the problem and, therefore, they set *extra* requirements which make for an over-determined formulation. As well known, over-determination most often leads to ill-posedness and lack of solutions, whence the impossibility. CBCs are implicitly present in any integral formulation (1) and failure to realize this fact has led to many misunderstandings, as discussed at large in Nobili and Pramanik (2025b). To further complicate matters, casting the integral (1) into the differential formulation, through the application of the differential operator $\mathcal{L}$, eliminates the CBCs embedded in the integral kernel, that are easily overlooked in the successive steps. Dropping the CBCs gets around the over-determination of the integral formulation and therefore produces the ill-conceived idea that the integral and the differential formulations lead to different outcomes[2] (which of course they do, if we impose different sets of BCs) and that only the latter is meaningful, see for example Kaplunov et al. (2022). This solution procedure is flawed under many aspects.

2.1. Difference-type kernels are admitted in the presence of decay conditions

We begin by assuming an unbounded domain $\mathbb{B}$ and decay conditions at infinity, which is the original setting where nonlocal elasticity was first conceived. Within this framework, we have that the attenuation function is a difference kernel

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = G(\boldsymbol{x} - \boldsymbol{\xi}), \tag{8}$$

whence, consequently,

$$\frac{\partial G}{\partial x_j}(\boldsymbol{x} - \boldsymbol{\xi}) = -\frac{\partial G}{\partial \xi_j}(\boldsymbol{x} - \boldsymbol{\xi}). \tag{9}$$

For the self-adjoint operator $\mathcal{L}_\xi$, we have the Green's (bilinear) identity (Lanczos, 1996, §5.6)

$$\int_{\mathbb{B}} \left(G \mathcal{L}_\xi t_{ij} - t_{ij} \mathcal{L}_\xi G \right) \mathrm{d}\boldsymbol{\xi} = -\varepsilon^2 \int_{\partial\mathbb{B}} \left(G \frac{\mathrm{d}t_{ij}}{\mathrm{d}\nu_\xi} - t_{ij} \frac{\mathrm{d}G}{\mathrm{d}\nu_\xi} \right) \mathrm{d}\boldsymbol{\xi}, \tag{10}$$

where we have included the term on the RHS to enable the extension of this argument to a finite domain. Here, $\boldsymbol{\nu}_\xi$ denotes the unit normal vector (pointing outwards from $\mathbb{B}$) evaluated at the point $\boldsymbol{\xi}$. The normal derivative of the attenuation function is named the *Poisson kernel* and it is denoted by $P(\boldsymbol{x}, \boldsymbol{\xi})$

$$P(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{\mathrm{d}G}{\mathrm{d}\nu_\xi}(\boldsymbol{x}, \boldsymbol{\xi}), \quad \boldsymbol{x} \in \mathbb{B}, \boldsymbol{\xi} \in \partial\mathbb{B}. \tag{11}$$

[1] $\{A_p\}$ generally form a set of *functions*, but this distinction is immaterial for the present argument.

[2] We cannot write "solutions", for the integral formulation often has none, see for example (Romano et al., 2017).

It is observed that

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} P(\boldsymbol{x},\boldsymbol{\xi}) = -\varepsilon^{-2}\delta_H(\|\boldsymbol{x}_0-\boldsymbol{\xi}\|), \quad \boldsymbol{x}_0,\boldsymbol{\xi}\in\partial\mathbb{B}, \tag{12}$$

where δ_H is Dirac's delta function on the surface $\partial\mathbb{B}$. Therefore, $P(\boldsymbol{x},\boldsymbol{\xi})$, just like $G(\boldsymbol{x},\boldsymbol{\xi})$, is a generalized function and diverges as $\boldsymbol{x}\to\boldsymbol{\xi}$. Next, from (1) and (9), we have

$$\begin{aligned} t_{ij,j}(\boldsymbol{x}) &= \int_{\mathbb{B}} \frac{\partial G}{\partial x_j}(\boldsymbol{x}-\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{B}} \frac{\partial G}{\partial \xi_j}(\boldsymbol{x}-\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = -\int_{\partial\mathbb{B}} G(\boldsymbol{x}-\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\nu_j\mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{B}} G(\boldsymbol{x}-\boldsymbol{\xi})\sigma_{ij,j}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \\ &= \rho\int_{\mathbb{B}} G(\boldsymbol{x}-\boldsymbol{\xi})\mathcal{L}_{\xi}\ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \end{aligned} \tag{13}$$

where the surface integral vanishes in light of the decay property of $G(\boldsymbol{x}-\boldsymbol{\xi})$ and, in the last equality, we have used (6). Besides, appealing to the Green's identity (10) to deal with the last equality in (13) and accounting for the decay at infinity of $\ddot{u}_i$, we finally obtain

$$t_{ij,j}(\boldsymbol{x}) = \rho\int_{\mathbb{B}} \mathcal{L}_{\xi}G(\boldsymbol{x}-\boldsymbol{\xi})\ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \rho\ddot{u}_i(\boldsymbol{x}),$$

that corresponds to the motion Eq. (5). It is therefore clear that the stipulated equivalence of the equations of motion (5) to the differential model (6), which is always assumed in the literature, really applies whenever *the attenuation function G is a difference kernel that decays at infinity, and so does the acceleration field* $\ddot{u}_i$. Indeed, when trying to reproduce the previous argument for a finite (or semi-infinite) domain, while keeping the difference form (8) for the attenuation function, boundary contributions appear, namely

$$t_{ij,j}(\boldsymbol{x}) = -\int_{\partial\mathbb{B}} G(\boldsymbol{x}-\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\nu_j\mathrm{d}\boldsymbol{\xi} - \rho\varepsilon^2\int_{\partial\mathbb{B}}\left[G(\boldsymbol{x}-\boldsymbol{\xi})\frac{\mathrm{d}\ddot{u}_i}{\mathrm{d}\nu_{\xi}}(\boldsymbol{\xi}) - P(\boldsymbol{x},\boldsymbol{\xi})\ddot{u}_i(\boldsymbol{\xi})\right]\mathrm{d}\boldsymbol{\xi} + \rho\ddot{u}_i(\boldsymbol{x}). \tag{14}$$

Therefore, in a bounded domain, after having solved the differential form of the equations of motion (6) expressed in terms of displacement, even if we could set the undetermined constants through enforcing the problem BCs, the resulting solution would not satisfy the original equations of motion (5), owing to the presence of the boundary terms. Failure to recognize this fact leads to the inconsistencies that were first identified by Kaplunov et al. (2022).

Interestingly, the appearance of the boundary integral in (14) was already observed by Eringen (1984, Eq.(8.3)) in connection with the motion equations. According to Eringen "the surface integral represents the contributions of the surface stresses (e.g. surface tension). Consequently, *nonlocal theory accounts for the surface physics as well*" (italics in original). However, Eq. (14) is obtained for a difference kernel, which is really admissible for unbounded domains only. Indeed, the need to abandon the difference form of the attenuation function in the presence of boundaries was already pointed out by Eringen (1984, §9), precisely in connection with surface waves traveling in a half-space. In fact, according to Eringen "the nonlocal moduli $G(|x'-x|)$ is appropriate only to homogeneous and isotropic solids. Half-space ceases to be homogeneous in the vicinity of the surface $x_2=0$, where in a boundary layer of a few atomic distances, the material is inhomogeneous and therefore a perturbation is necessary in $G(|x'-x|)$". The same point is also made by Kunin (1984). Eringen believed that the deviation from the homogeneous behavior due to the presence of the boundaries could be accounted for by simply incorporating a correction term in the attenuation function, whose leading order part remained of the difference type. However, we know that this approach is doomed to fail, owing to the implicit presence of the CBCs. Accordingly, we shall drop the assumption that the kernel is of the difference type in the presence of boundaries.

2.2. The original and the differential form of the motion equations are not equivalent for domains with boundaries

We now prove that, in the presence of boundaries, fulfillment of the differential form of the motion Eq. (6) is not enough to warrant that the original motion (5) are satisfied. The argument immediately follows from the Green's identity (10) applied to the vector

$$w_i = t_{ij,j} - \rho\ddot{u}_i,$$

that expresses unbalance. Given the importance of homogeneous boundary conditions, in the following we use the superscript "o" to denote quantities pertaining to the homogeneous problem, thus we write $G^o(\boldsymbol{x},\boldsymbol{\xi})$ meaning that

$$G^o(\boldsymbol{x},\boldsymbol{\xi}) \equiv 0, \quad \boldsymbol{x}\in\partial\mathbb{B}.$$

Thus we have, for the homogeneous attenuation function $G^o(\boldsymbol{x},\boldsymbol{\xi})$,

$$w_i(\boldsymbol{x}) = \int_{\mathbb{B}} G^o(\boldsymbol{x},\boldsymbol{\xi})\mathcal{L}_{\xi}w_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - \varepsilon^2\int_{\partial\mathbb{B}} P^o(\boldsymbol{x},\boldsymbol{\xi})w_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{15}$$

given that the first term in the volume integral of the Green's identity is reduced to the delta function in view of Eq. (3), i.e. G is a Green's function. It then clearly appears that, when the differential form of the motion (6) holds, only the first term at RHS in Eq. (15) disappears, while the boundary integral remains. Indeed, by using (6) and (7), it is $\mathcal{L}_{\xi}w_i\equiv 0$. Consequently, to obtain the original motion equations $w_i\equiv 0$ over $\mathbb{B}$, we need to enforce the further boundary conditions[3]

$$w_i(\boldsymbol{x}) \equiv 0, \quad \boldsymbol{x}\in\partial\mathbb{B}, \tag{16}$$

[3] Note that $\mathbb{B}$ is a closed set.

that amount to demanding that the original motion Eq. (5) stand *on the boundary*. It should be noticed that, in the literature, usually no check on the motion Eq. (5) is carried out after having solved the differential model (6) and, in the very few instances when this is done, as in Kaplunov et al. (2022), Nobili and Pramanik (2025a), Pham and Vu (2024), the original motion Eq. (5) are enforced over the whole domain, that is unnecessary. Eq. (15) shows that the integro-differential and the differential form of the equations of motion, that are assumed equivalent in the literature, in actual facts differ by a boundary term. It is further observed that this difference was first pointed out in Nobili and Pramanik (2025a), but it was there ascribed to mere differentiation and to the presence of arbitrary constants. We have now proved that this discrepancy is deeper than mere differentiation and it is due to the boundary conditions (16). Precisely this new set of BCs is required to determine the unknown constants $\{A_p\}$.

The result (15) shows that, when the equations of motion (5) do not hold on the boundary, they also fail over the whole body. In other words, we have produced a family of solutions which differ by the value that the equations of motion attain on the surface, i.e. for each set of constants $\{A_p\}$, we obtain the solution set with specific values of w_i over the boundary. A parallel may be drawn with the theory of surface elasticity, although different governing equations are used from those given in, for example, the surface elasticity theory developed by Gurtin and Murdoch (1978). In the language of surface elasticity, we may equivalently say that all solutions satisfy the motion equation *inside the body*, namely the differential equations of motion (6). Out of these solutions, we seek the one that accommodates for the motion Eq. (5) over the *surface*, and, as we have proved, this very solution satisfies the original motion Eq. (5) throughout. Indeed, within this surface elasticity viewpoint, the interior equation is the differential motion Eq. (6), while the surface equation is given by (16). Still, important differences exist with the classical surface elasticity theory. Concerning the surface equation, it is noted that the usual 3D divergence operator is adopted, as opposed to the *surface divergence* that appears in the classical surface theory. With regard to the interior equation, a modified inertia term is adopted. Besides, the connection between the surface stress t_{ij} and the interior stress σ_{ij} is no longer local, as in surface elasticity, but it takes place through the definition of the nonlocal stresses (1), i.e. in integral form. We are, however, still missing the boundary connection between the interior and the surface, and for this we need to expand the nonlocal theory as in Section 2.4.

2.3. *Kernel modification*

Very recently, in a series of papers (Nobili & Pramanik, 2025a, 2025b; Pramanik & Nobili, 2025), it was suggested to modify the attenuation functions of the integral formulation only in terms of the attached CBCs, by a procedure named *kernel modification*. Indeed, as thoroughly discussed in the literature, specification of the associated operator $\mathcal{L}$ is insufficient for the full determination of the attenuation functions, which also demand the incorporation of CBCs. Within kernel modification, CBCs are made to coincide with the BCs of the physical problem, which procedure gets away with CBCs of questionable physical interpretation and, at the same time, provides an even-determined and well-posed formulation. In essence, this procedure exploits the connection between the attenuation function α_{ij} and the Green's function of the operator $\mathcal{L}$ by incorporating the BCs coming from the physical problem. Within this approach, the BCs of the problem that involve the nonlocal stresses are *automatically* satisfied by *any* distribution of the local stresses σ_{ij}. These are most easily obtained through solving the differential formulation (6), although to arrive at the nonlocal stresses, it is still required to deal with the integrals (1). This is in fact the solution procedure followed in Nobili and Pramanik (2025a) with regard to Rayleigh waves in a nonlocal half-plane. Alternatively, Pham and Vu (2024) employed the differential connection (24) alongside homogeneous BCs to arrive at the same result.

It is important to emphasize that, within kernel modification, the BCs of the problem are automatically satisfied by *any* local stress distribution and therefore they may no longer be used to set the undetermined constants $\{A_p\}$ that arise from solving the differential problem (6) in terms of displacements. As it was already shown, the original motion equations on the boundary, Eq. (16), are used instead. Let us now apply kernel modification and write the new condition (16) for the Rayleigh problem in a half-plane. In particular, the Green's function can no longer be of the difference type (8), and in fact we assume that it is constructed as in Eq. (41) for the half-plane $\mathbb{B} = \{(\xi_1, \xi_2) : \xi_2 \leq 0\}$, namely

$$G^o(\boldsymbol{x}, \boldsymbol{\xi}) = G_1(x_1 - \xi_1, x_2 - \xi_2) + G_2(x_1 - \xi_1, x_2 + \xi_2), \tag{17}$$

where G_1 provides the source term located at (ξ_1, ξ_2) and G_2 is the sink placed at $(\xi_1, -\xi_2)$, that is outside the body $\mathbb{B}$. Thus,

$$\mathcal{L}_\xi G_1(\boldsymbol{x}, \boldsymbol{\xi}) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2), \quad \mathcal{L}_\xi G_2(\boldsymbol{x}, \boldsymbol{\xi}) = -\delta(x_1 - \xi_1)\delta(x_2 + \xi_2),$$

and, as in (41), $G_1(x_1 - \xi_1, x_2) + G_2(x_1 - \xi_1, x_2) = 0$ on the surface $\xi_2 = 0$, while decay stands at infinity. For the Helmholtz operator, it is

$$G_1(x_1, x_2, \xi_1, \xi_2) = -G_2(x_1, x_2, \xi_1, -\xi_2) = \frac{1}{2\pi\varepsilon^2} K_0\left(\varepsilon^{-1}\|\boldsymbol{x} - \boldsymbol{\xi}\|\right), \tag{18}$$

where $\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2}$ and K_0 is the zeroth order Bessel function of the second kind (Abramowitz & Stegun, 1965). Besides (no summation over j),

$$\frac{\partial G_1}{\partial x_j}(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{\partial G_1}{\partial \xi_j}(\boldsymbol{x}, \boldsymbol{\xi}), \quad \frac{\partial G_2}{\partial x_j}(\boldsymbol{x}, \boldsymbol{\xi}) = (-1)^{\delta_{1j}} \frac{\partial G_2}{\partial \xi_j}(\boldsymbol{x}, \boldsymbol{\xi}),$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Therefore (now k and j are summed over),

$$t_{ij,j}(\boldsymbol{x}) = \int_{\mathbb{B}} \frac{\partial G^o}{\partial x_j}(\boldsymbol{x}, \boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{B}} (-1)^{\delta_{2k}\delta_{2j}} \frac{\partial G_k}{\partial \xi_j}(\boldsymbol{x}, \boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$$

$$= -\int_{\partial\mathbb{B}} (-1)^{\delta_{2k}\delta_{2j}} G_k(\boldsymbol{x},\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\nu_j \mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{B}} (-1)^{\delta_{2k}\delta_{2j}} G_k(\boldsymbol{x},\boldsymbol{\xi})\sigma_{ij,j}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$$
$$= -\int_{\partial\mathbb{B}} (G_1 - G_2)(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2}(\boldsymbol{\xi})\nu_2 \mathrm{d}\boldsymbol{\xi} + \rho\int_{\mathbb{B}} G_1(\boldsymbol{x},\boldsymbol{\xi})\mathcal{L}_\xi \ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{B}} G_2(\boldsymbol{x},\boldsymbol{\xi})(\sigma_{i1,1} - \sigma_{i2,2})(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$$
$$= 2\int_{\partial\mathbb{B}} G_2(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + \rho\int_{\mathbb{B}} G^o(\boldsymbol{x},\boldsymbol{\xi})\mathcal{L}_\xi \ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - 2\int_{\mathbb{B}} G_2(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2,2}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

being $\nu_j = \delta_{2j}$. Application of the Gauss theorem to the last term cancels the first integral out,

$$t_{ij,j}(\boldsymbol{x}) = \rho\int_{\mathbb{B}} G^o(\boldsymbol{x},\boldsymbol{\xi})\mathcal{L}_\xi \ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + 2\int_{\mathbb{B}} \frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{19}$$

and, by the Green's identity applied to the first integral, one gets

$$w_i(\boldsymbol{x}) = \rho\varepsilon^2 \int_{-\infty}^{\infty} P^o(x_1 - \xi_1, x_2)\ddot{u}_i(\xi_1, 0)\mathrm{d}\xi_1 + 2\int_{\mathbb{B}} \frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{20}$$

where

$$P^o(x_1 - \xi_1, x_2) = \frac{\mathrm{d}G^o}{\mathrm{d}\xi_2}(x_1, x_2, \xi_1, 0).$$

In Appendix it is shown that Eq. (20) produces a set of homogeneous linear equations which differ from those obtained in Nobili and Pramanik (2025a) and yet the corresponding dispersion relations coincide.

2.4. Extending the nonlocal theory to inhomogeneous problems

So far we have been dealing with homogeneous nonlocal problems and we need to extend the theory to deal with the general inhomogeneous situation. We begin by showing that Eringen's integral formulation (1) and its differential counterpart (7) are generally incompatible in the presence of inhomogeneous BCs. Let us assume this homogeneous BCs hold in the portion of surface $S_0 \subset \partial\mathbb{B}$, while inhomogeneous conditions are set on $S_t = \partial\mathbb{B} - S_0$, namely (we use Dirichlet conditions here)

$$t_{ij}(\boldsymbol{x}) = t_{ij}^S(\boldsymbol{x}), \quad \boldsymbol{x} \in S_t, \tag{21}$$

and, obviously, $t_{ij}^S = t_{ji}^S$. It is emphasized that, usually, only the traction vector

$$t_{ij}^S \nu_j = p_i, \quad \boldsymbol{x} \in S_t, \tag{22}$$

is given on the boundary in a classical elasticity problem, whereas here the whole tensor t_{ij}^S is assumed to be prescribed on S_t. Besides, homogeneous BCs over $\partial\mathbb{B}$ are taken for G, whence we write G^o and for this kernel reciprocity holds. Then, the bilinear identity (10) gives

$$t_{ij}(\boldsymbol{x}) = \int_{\mathbb{B}} G^o(\boldsymbol{x},\boldsymbol{\xi})\mathcal{L}_\xi t_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - \varepsilon^2 \int_{S_t} P^o(\boldsymbol{x},\boldsymbol{\xi})t_{ij}^S(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{23}$$

which crucially generalizes Eringen's assumption (1). It is emphasized that Eringen's single integral formulation (1) may be considered still valid, provided that G^o is now replaced with the corresponding Green's function G_{ij}^S that incorporates the inhomogeneous data, namely

$$G_{ij}^S(\boldsymbol{x},\boldsymbol{\xi}) = \begin{cases} t_{ij}^S(\boldsymbol{x}), & \boldsymbol{x} \in S_t, \\ 0, & \boldsymbol{x} \in S_0, \end{cases}.$$

This step, however, comes at the sake of abandoning reciprocity, i.e. $G_{ij}^S(\boldsymbol{x},\boldsymbol{\xi}) \neq G_{ij}^S(\boldsymbol{\xi},\boldsymbol{x})$, and the simplification that comes from using one kernel for all stresses. Despite either form being equivalent, we prefer to stick with the two term notation (23). We now *assume*

$$\mathcal{L}_\xi t_{ij}(\boldsymbol{\xi}) = \sigma_{ij}(\boldsymbol{\xi}), \tag{24}$$

which, together with (3), becomes *the central hypothesis of our nonlocal theory*. Thus, we have

$$t_{ij}(\boldsymbol{x}) = \int_{\mathbb{B}} G^o(\boldsymbol{x},\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - \varepsilon^2 \int_{S_t} P^o(\boldsymbol{x},\boldsymbol{\xi})t_{ij}^S(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{25}$$

that extends Eq. (1). It is emphasized that, when ε is a small parameter, Eq. (25) indicates that the contribution of the inhomogeneous BCs is a perturbation of the homogeneous solution, somewhat in line with the above-mentioned Eringen's comment. However, this perturbation affects the homogeneous kernel G^o, which cannot be in the difference form on account of the presence of boundaries (and in the spirit of kernel modification). This means that the presence of boundaries is a leading order contribution, while the fact that boundaries express inhomogeneous BCs is merely a correction to the homogeneous case. The result (25) is hereinafter written in concise form as

$$t_{ij}(\boldsymbol{x}) = t_{ij}^V(\boldsymbol{x}) + t_{ij}^\Sigma(\boldsymbol{x}), \tag{26}$$

where $t^V_{ij}(\boldsymbol{x})$ is Eringen's classical term, while $t^\Sigma_{ij}(\boldsymbol{x})$ is the contribution coming from the inhomogeneous conditions. In particular, it is remarked that $t^S_{ij}(\boldsymbol{x})$ indicates the inhomogeneous data that are given on the surface, while $t^\Sigma_{ij}(\boldsymbol{x})$ denotes the surface integral at RHS in (25), the latter yielding the former when evaluated on the surface S_t.

We can now repeat the steps in Eqs. (19) and (20) to get

$$w_i(\boldsymbol{x}) = -\varepsilon^2 \int_{S_t} \frac{\partial P^o}{\partial x_j}(\boldsymbol{x},\xi) t^S_{ij}(\xi)\mathrm{d}\xi + 2\int_{\mathbb{B}} \frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\xi)\sigma_{i2}(\xi)\mathrm{d}\xi + \rho\varepsilon^2 \int_{\partial\mathbb{B}} P^o(\boldsymbol{x},\xi)\ddot{u}_i(\xi)\mathrm{d}\xi,$$

which, evaluated on the boundary S_t and on account of (12), yields

$$w_i(\boldsymbol{x}) = -\varepsilon^2 \int_{S_t} \frac{\partial P^o}{\partial x_j}(\boldsymbol{x},\xi) t^S_{ij}(\xi)\mathrm{d}\xi + 2\int_{\mathbb{B}} \frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\xi)\sigma_{i2}(\xi)\mathrm{d}\xi - \rho\ddot{u}_i(\boldsymbol{x}), \quad \boldsymbol{x} \in S_t. \tag{27}$$

This result provides the new BCs (16) for the inhomogeneous problem which are employed in Section 3 to find the dispersion relation for localized waves at the surface of a vertically constrained half-plane, where $r(\xi_1)$ is the reaction force of the constraint. In this scenario, we have

$$\varepsilon^2\delta_{2i} \int_{-\infty}^{\infty} \frac{\partial P^o}{\partial x_2}(x_1-\xi_1, 0) r(\xi_1)\mathrm{d}\xi_1 = 2\int_{\mathbb{B}} \frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\xi)\sigma_{i2}(\xi)\mathrm{d}\xi - \rho\ddot{u}_i(\boldsymbol{x}), \tag{28}$$

that is an integral equation for the reaction force density $r(\xi_1)$.

2.4.1. Elastic potential for inhomogeneous nonlocal problems

Incorporation of inhomogeneous BCs in the nonlocal theory calls for a new expression for the stored elastic potential (in the absence of body forces)

$$V = \tfrac{1}{2}\int_{\mathbb{B}} t^V_{ij} E_{ij}\mathrm{d}\boldsymbol{x} - \int_{\mathbb{B}} t^\Sigma_{ij,j} u_i \mathrm{d}\boldsymbol{x}, \tag{29}$$

where the first term at RHS matches the corresponding term in Polizzotto (2001, Eq.(42)), while the second term is similar to a body force and accounts for the inhomogeneous nature of the BCs. By reciprocity of G^o, it is

$$\delta\int_{\mathbb{B}} t^V_{ij} E_{ij}\mathrm{d}\boldsymbol{x} = \int_{\mathbb{B}} \delta t^V_{ij} E_{ij}\mathrm{d}\boldsymbol{x} + \int_{\mathbb{B}} t^V_{ij}\delta E_{ij}\mathrm{d}\boldsymbol{x} = 2\int_{\mathbb{B}} t^V_{ij}\delta u_{i,j}\mathrm{d}\boldsymbol{x}.$$

Then, accounting for the fact that $\delta t^\Sigma_{ij,j} = 0$,

$$\delta\int_{\mathbb{B}} \left(\tfrac{1}{2} t^V_{ij}E_{ij} - t^\Sigma_{ij,j}u_i\right)\mathrm{d}\boldsymbol{x} = \int_{\partial\mathbb{B}} t^V_{ij}\nu_j\delta u_i \mathrm{d}\boldsymbol{x} - \int_{\mathbb{B}} (t^V_{ij} + t^\Sigma_{ij})_{,j}\delta u_i\mathrm{d}\boldsymbol{x}, \tag{30}$$

where the surface integral vanishes because t^V_{ij} pertains to the homogeneous problem. Recalling the definition (26), one has

$$\delta V = 0 \quad \Leftrightarrow \quad t_{ij,j} = 0,$$

that are the equilibrium Eq. (5). Clearly, the potential (29) may be equally rewritten through the Gauss theorem as

$$V = \tfrac{1}{2}\int_{\mathbb{B}} t_{ij}E_{ij}\mathrm{d}\boldsymbol{x} + \tfrac{1}{2}\int_{\mathbb{B}} t^\Sigma_{ij}E_{ij}\mathrm{d}\boldsymbol{x} - \int_{S_t} p_i u_i \mathrm{d}\boldsymbol{x}, \tag{31}$$

where the first term is the classical elastic energy due to the nonlocal stress and the last term is the work done by the applied (surface) loads. In contrast, the middle term, V^Σ, is the elastic energy associated with the surface stresses, which reach out to the whole body due to nonlocality. As already anticipated, V^Σ is a small quantity in ε and, swapping the order of integration, it may be written as

$$V^\Sigma = \frac{\varepsilon^2}{2}\int_{S_t} t^S_{ij}e_{ij}\mathrm{d}\boldsymbol{x}, \tag{32}$$

where we have introduced the surface deformation field (the minus merely inverts the outward normal in the Poisson kernel)

$$e_{ij}(\boldsymbol{x}) = -\int_{\mathbb{B}} P^o(\xi,\boldsymbol{x})E_{ij}(\xi)\mathrm{d}\xi. \tag{33}$$

In this final form, the potential energy resembles that of surface elasticity, with a volume and a surface contribution, where the latter, however, requires the introduction of the surface strain field (33). It is most fitting that the strain field e_{ij} is obtained using the Poisson kernel, as opposed to the Green's function G, because, as it is well known, precisely this kernel relates the interior to the boundary.

2.4.2. Nonlocal elasticity as a surface elasticity problem

We can recast the inhomogeneous problem of nonlocal elasticity as a problem within surface elasticity. The interior equations read

$$\left.\begin{aligned} &\sigma_{ij,j} = \rho\mathcal{L}\ddot{u}_i,\\ &\sigma_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij},\\ &E_{ij} = \tfrac{1}{2}\left(u_{i,j}+u_{j,i}\right), \end{aligned}\right\} \quad \text{in } \mathbb{B}, \tag{34}$$

to be compared with Eq.(1) of Gurtin and Murdoch (1978). The surface equations read

$$\left.\begin{array}{l} t_{ij,j} = \rho\ddot{u}_i, \\ t_{ij} = \int_{\mathbb{B}} G^o \sigma_{ij} - \varepsilon^2 \int_{S_t} P^o t_{ij}^S, \end{array}\right\} \quad \text{on } S_t, \tag{35}$$

to be compared with Eq.(2) in Gurtin and Murdoch (1978). It is seen that the surface stress t_{ij} is already known on the surface S_t, namely $t_{ij} = t_{ij}^S$, and the governing Eq. (5) provides a restriction on $\ddot{u}_i$ and σ_{ij} that are evaluated from solving the interior problem. In particular, this restriction is given by $w_i \equiv 0$ on S, where, in the case of the half-plane, w_i is obtained from Eq. (27) in terms of t_{ij}^S, $\ddot{u}_i$ and σ_{ij}. Thus, the surface problem merely provides the closing boundary conditions for the interior problem, as already discussed. However, it is emphasized that the surface problem solution is also partly unknown, because t_{ij} is only given on the surface S and yet the diverge operation in the motion Eq. (5) involves the normal (to the surface) derivative, which requires the knowledge of t_{ij} in a neighborhood of the surface. It is worth mentioning that the asymptotic results presented in Chebakov et al. (2016) that highlight the importance of the near-surface behavior in connection with nonlocality acquire a suggestive surface elasticity interpretation.

3. Surface waves under inhomogeneous boundary conditions

We now apply the extended integral formulation (25) to investigate localized surface (Rayleigh) waves in a nonlocal half-plane under inhomogeneous BCs, that are given by vertical elastic springs uniformly distributed along the half-plane surface. As it is well-known, in the limit of infinite stiffness, this scenario does not support localized waves within local elasticity (Graff, 2012, §6.1.5). Let (x_1, x_2) define a rectangular coordinate system located on top of an isotropic elastic nonlocal half-plane, with the x_2 axis directed inwards such that the half-plane is defined as

$$\mathbb{B} = \{(x_1, x_2) : x_2 \geq 0\}. \tag{36}$$

Let u_1 and u_2 denote the displacement field inside the half-plane, from which we obtain the corresponding deformation field

$$\epsilon_{11} = u_{1,1}, \quad \epsilon_{12} = \epsilon_{21} = \tfrac{1}{2}(u_{1,2} + u_{2,1}), \quad \epsilon_{22} = u_{2,2}. \tag{37}$$

The corresponding local stresses immediately follow

$$\sigma_{11} = 2\mu\epsilon_{11} + \lambda(\epsilon_{11} + \epsilon_{22}), \quad \sigma_{12} = \sigma_{21} = 2\mu\epsilon_{12}, \quad \sigma_{22} = 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22}), \tag{38}$$

where μ and λ are the usual Lamé moduli of isotropic elasticity. From these, we immediately obtain the nonlocal stresses t_{ij} by integration according to (25), provided that we define the attenuation functions.

As well known, the natural BCs of the problem prescribe the surface vertical displacement

$$u_2(x_1, 0) = k^{-1} t_{22}(x_1, 0), \tag{39}$$

as well as the surface shear stress

$$t_{12}(x_1, 0) \equiv 0. \tag{40}$$

Eq. (39) generalizes the perfect support condition $u_2(x_1, 0) = 0$ by introducing the support elastic stiffness $k > 0$ (not to be confused with the wavenumber m, later to appear). Since the condition (40) is *homogeneous*, we may still apply the original one-integral formulation (1)

$$t_{12}(x_1, x_2) = \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \int_0^{+\infty} \alpha_{12}(x_1, x_2, \xi_1, \xi_2)\sigma_{12}(\xi_1, \xi_2)\mathrm{d}\xi_2,$$

where we have set the attenuation function

$$\alpha_{12}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2\pi\varepsilon^2}\left\{K_0\left(\frac{1}{\varepsilon}\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}\right) - K_0\left(\frac{1}{\varepsilon}\sqrt{(x_1 - \xi_1)^2 + (x_2 + \xi_2)^2}\right)\right\}, \tag{41}$$

by the method of images. We remark that the kernel (41) is not of the difference (or convolution) type and this feature is no accident, for it determines an inhomogeneous response, that changes in dependence of the vicinity of the boundary. In other words, as already observed by Kunin (1984) and by Eringen (1984), difference-type kernels are associated with *homogeneity* of the material, where the nonlocal stress is the same for any point at a given distance from the source, irrespectively of the presence of boundaries in the domain. In fact, the kernel (41) has the form (17).

From consideration of the BCs ((39),(40)), we see that t_{11} has no BC attached and therefore the CBCs in the corresponding attenuation function remains undetermined. For the sake of definiteness, we set

$$\alpha_{11}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2\pi\varepsilon^2} K_0\left(\frac{1}{\varepsilon}\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}\right), \tag{42}$$

that corresponds to prescribing *homogeneity*, i.e. symmetric decay in every direction. Consequently, this attenuation function is of the difference type.

We are then faced with the choice of the attenuation function α_{22}, in consideration of the inhomogeneous condition (39) at the surface. For this, we need the extended formulation (25)

$$t_{22}(\boldsymbol{x}) = \int_{\mathbb{B}} \alpha_{12}(\boldsymbol{x}, \boldsymbol{\xi})\sigma_{22}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - \varepsilon^2 \int_{\mathbb{R}} P^o(x_1, x_2, \xi_1, 0) t_{22}(\xi_1, 0)\mathrm{d}\xi_1, \tag{43}$$

where $P^o = -\frac{\partial \alpha_{12}}{\partial \xi_2}$ is the Poisson kernel. In Eq. (43), we have let $G^o = \alpha_{12}$ for it carries *homogeneous* boundary conditions, just as it is the case for t_{12}. It follows that it is the line integral (i.e. the second term at RHS) that accounts for the inhomogeneous form of the BC (39).

For convenience, we let the wavespeeds

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_S = \sqrt{\frac{\mu}{\rho}}, \tag{44}$$

respectively attached to longitudinal and transversal bulk waves, alongside their ratio $\kappa = c_L/c_S > 1$. We now write the differential model (6) in the form of the nonlocal Navier's equations

$$c_S^2 \triangle u_1 + (c_L^2 - c_S^2)(u_{1,1} + u_{2,2})_{,1} = \mathcal{L}\ddot{u}_1, \tag{45a}$$

$$c_S^2 \triangle u_2 + (c_L^2 - c_S^2)(u_{1,1} + u_{2,2})_{,2} = \mathcal{L}\ddot{u}_2, \tag{45b}$$

where it appears that, if we were to disregards the CBCs, the resulting problem would amount to the standard problem of classical isotropic elasticity with a modified inertia term. By introducing the dimensionless time $\tau = c_S L^{-1} t$, where L is a typical wavelength, we get

$$\triangle u_1 + (\kappa^2 - 1)(u_{1,1} + u_{2,2})_{,1} = L^{-2}\mathcal{L}u_{1,\tau\tau}, \tag{46a}$$

$$\triangle u_2 + (\kappa^2 - 1)(u_{1,1} + u_{2,2})_{,2} = L^{-2}\mathcal{L}u_{2,\tau\tau}. \tag{46b}$$

We now introduce the usual traveling wave assumption, such that

$$u_i(\boldsymbol{x}, \tau) = U_i(x_2) \exp\left[i\left(m x_1 - \Omega\tau\right)\right], \tag{47}$$

where $M = mL$ is the dimensionless wavenumber, $\Omega = \omega L c_S^{-1}$ the dimensionless angular frequency and $i^2 = -1$. The decaying traveling solution of (46) reads

$$U_1(x_2) = -im^{-1}\left[c_1 b_1^{-1} e^{-b_1 m x_2} + c_2 b_2 e^{-b_2 m x_2}\right], \tag{48a}$$

$$U_2(x_2) = m^{-1}\left[c_1 e^{-b_1 m x_2} + c_2 e^{-b_2 m x_2}\right], \tag{48b}$$

where c_1 and c_2 are unknown constants, while b_1 and b_2 are the decaying indices

$$b_1 = \sqrt{1 + \frac{v^2}{v^2 M^2 \varepsilon_1^2 - \kappa^2}}, \quad b_2 = \sqrt{1 + \frac{v^2}{v^2 M^2 \varepsilon_1^2 - 1}}. \tag{49}$$

Hereinafter, $v = \Omega/M$ is the dimensionless wavespeed, i.e. the dimensional wavespeed is $c = c_S v$, and $\varepsilon_1 = \varepsilon/L$ is the dimensionless microstructural parameter. We then obtain the local stresses

$$\sigma_{ij}(x_1, x_2, \tau) = \mu \Sigma_{ij}(x_2) \exp\left[i\left(m x_1 - \Omega\tau\right)\right], \tag{50}$$

with

$$\Sigma_{11}(x_2) = c_1(\kappa^2 b_1^{-1} - b_1(\kappa^2 - 2))e^{-b_1 m x_2} + 2b_2 c_2 e^{-b_2 m x_2}, \tag{51a}$$

$$\Sigma_{12}(x_2) = i\left[2c_1 e^{-b_1 m x_2} + c_2(1 + b_2^2)e^{-b_2 m x_2}\right], \tag{51b}$$

$$\Sigma_{22}(x_2) = -c_1(\kappa^2 b_1 - b_1^{-1}(\kappa^2 - 2))e^{-b_1 m x_2} - 2b_2 c_2 e^{-b_2 m x_2}. \tag{51c}$$

These results reduce to those obtained for local elasticity by letting

$$b_1 = b_{1l} = \sqrt{1 - \frac{v^2}{\kappa^2}}, \quad b_2 = b_{2l} = \sqrt{1 - v^2}. \tag{52}$$

In general, the local stress is obtained by superposition of two terms as

$$\Sigma_{ij}(x_2) = A_{ij} e^{-b_1 m x_2} + B_{ij} e^{-b_2 m x_2}, \tag{53}$$

whence, to obtain the nonlocal stresses, it suffices to integrate the relevant attenuation function with an exponentially decaying term. To this effect, we use the integral representation of the zeroth-order Bessel function of the second kind given in Nobili and Pramanik (2025a)

$$K_0\left(\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}\right) = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{-i\varepsilon^{-1} s_1 (x_1 - \xi_1)}}{\sqrt{1 + s_1^2}} e^{-\varepsilon^{-1}\sqrt{1+s_1^2}|x_2 - \xi_2|} \mathrm{d}s_1, \tag{54}$$

that immediately gives $\alpha_{11}(x_1, x_2, \xi_1, \xi_2)$. Similarly, for t_{12}, we use the attenuation function

$$\alpha_{12}(x_1, x_2, \xi_1, \xi_2) =$$

$$\frac{1}{2\pi\varepsilon^2}\int_{\mathbb{R}}\frac{e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)}}{\sqrt{1+s_1^2}}\left\{\begin{array}{ll} e^{-\varepsilon^{-1}\sqrt{1+s_1^2}x_2}\sinh\left[\varepsilon^{-1}\sqrt{1+s_1^2}\xi_2\right], & x_2>\xi_2>0,\\ e^{-\varepsilon^{-1}\sqrt{1+s_1^2}\xi_2}\sinh\left[\varepsilon^{-1}\sqrt{1+s_1^2}x_2\right], & \xi_2>x_2>0,\end{array}\right. \mathrm{d}s_1,$$

which clearly enjoys the reciprocity property, i.e. $\alpha_{12}(\boldsymbol{x},\boldsymbol{\xi})=\alpha_{12}(\boldsymbol{\xi},\boldsymbol{x})$. This may be rewritten as

$$\alpha_{12}(x_1,x_2,\xi_1,\xi_2)=\frac{1}{4\pi\varepsilon^2}\int_{\mathbb{R}}\frac{e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)}}{\sqrt{1+s_1^2}}\left(e^{-\varepsilon^{-1}\sqrt{1+s_1^2}|x_2-\xi_2|}-e^{-\varepsilon^{-1}\sqrt{1+s_1^2}(x_2+\xi_2)}\right)\mathrm{d}s_1, \tag{55}$$

From this, we easily obtain the Poisson kernel

$$\frac{\partial\alpha_{12}}{\partial\xi_2}(x_1,x_2,\xi_1,\xi_2)=\frac{1}{2\pi\varepsilon^3}\int_{\mathbb{R}}e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)}\left\{\begin{array}{ll} e^{-\varepsilon^{-1}\sqrt{1+s_1^2}x_2}\cosh\left[\varepsilon^{-1}\sqrt{1+s_1^2}\xi_2\right], & x_2>\xi_2>0\\ -e^{-\varepsilon^{-1}\sqrt{1+s_1^2}\xi_2}\sinh\left[\varepsilon^{-1}\sqrt{1+s_1^2}x_2\right], & \xi_2>x_2>0\end{array}\right. \mathrm{d}s_1, \tag{56}$$

and, in particular, on the surface $\xi_2=0$, one finds

$$\frac{\partial\alpha_{12}}{\partial\xi_2}(x_1,x_2,\xi_1,0)=\frac{1}{2\pi\varepsilon^3}\int_{\mathbb{R}}e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)-\varepsilon^{-1}\sqrt{1+s_1^2}x_2}\mathrm{d}s_1. \tag{57}$$

In general, for the nonlocal stresses we have

$$t_{ij}(x_1,x_2,\tau)=\mu T_{ij}(x_1,x_2)\exp\left[-i\Omega\tau\right], \tag{58}$$

and, starting from T_{11}, one gets

$$T_{11}(x_1,x_2)=\frac{A_{11}}{4\pi\varepsilon^2}\int_0^\infty\mathrm{d}\xi_2\int_{-\infty}^\infty\mathrm{d}\xi_1\int_{-\infty}^\infty\frac{e^{i(m+\varepsilon^{-1}s_1)\xi_1}}{\sqrt{1+s_1^2}}e^{-\varepsilon^{-1}\sqrt{1+s_1^2}|x_2-\xi_2|-bm\xi_2}e^{-i\varepsilon^{-1}s_1x_1}\mathrm{d}s_1, \tag{59}$$

having accounted only for one out of the two terms in (53). Hence, integrating first along ξ_1 and then along s_1, one gets

$$T_{11}(x_1,x_2)=\frac{A_{11}}{2\varepsilon}\frac{e^{imx_1}}{\sqrt{1+m^2\varepsilon^2}}\int_0^\infty e^{-\varepsilon^{-1}\sqrt{1+m^2\varepsilon^2}|x_2-\xi_2|-bm\xi_2}\mathrm{d}\xi_2, \tag{60}$$

and further integrating along ξ_2, we finally find

$$T_{11}(x_1,x_2)=A_{11}e^{imx_1}\frac{e^{-mbx_2}-\frac{b_3+m\varepsilon b}{2b_3}e^{-\varepsilon^{-1}b_3x_2}}{1+(1-b^2)m^2\varepsilon^2}, \tag{61}$$

where $b_3=\sqrt{1+m^2\varepsilon^2}$. Thus, the complete expression is obtained accounting for both terms in Eq. (53), namely

$$\begin{aligned}T_{11}(x_1,x_2)&=e^{imx_1}\left\{c_1\frac{\kappa^2b_1^{-1}-b_1(\kappa^2-2)}{1+(1-b_1^2)m^2\varepsilon^2}\left[e^{-mb_1x_2}-\frac{b_3+m\varepsilon b_1}{2b_3}e^{-\varepsilon^{-1}b_3x_2}\right]\right.\\&\left.+c_2\frac{2b_2}{1+(1-b_2^2)m^2\varepsilon^2}\left[e^{-mb_2x_2}-\frac{b_3+m\varepsilon b_2}{2b_3}e^{-\varepsilon^{-1}b_3x_2}\right]\right\},\end{aligned} \tag{62}$$

that features three exponentially decaying terms.

Proceeding in similar fashion for T_{12} and using the kernel α_{12}, we find

$$T_{12}(x_1,x_2)=ie^{imx_1}\left\{c_1\frac{2}{1+(1-b_1^2)m^2\varepsilon^2}\left[e^{-mb_1x_2}-e^{-\varepsilon^{-1}b_3x_2}\right]+c_2\frac{1+b_2^2}{1+(1-b_2^2)m^2\varepsilon^2}\left[e^{-mb_2x_2}-e^{-\varepsilon^{-1}b_3x_2}\right]\right\}. \tag{63}$$

Finally, for T_{22}, one gets

$$\begin{aligned}T_{22}(x_1,x_2)&=e^{imx_1}\left\{c_1\frac{b_1^{-1}(\kappa^2-2)-\kappa^2b_1}{1+(1-b_1^2)m^2\varepsilon^2}\left[e^{-mb_1x_2}-e^{-\varepsilon^{-1}b_3x_2}\right]\right.\\&\left.-c_2\frac{2b_2}{1+(1-b_2^2)m^2\varepsilon^2}\left[e^{-mb_2x_2}-e^{-\varepsilon^{-1}b_3x_2}\right]+(c_1+c_2)k_1e^{-\varepsilon^{-1}b_3x_2}\right\},\end{aligned} \tag{64}$$

where the last term, proportional to c_1+c_2, comes from the boundary. In fact, it features the dimensionless elastic stiffness $k_1=(m\mu)^{-1}k$ and the surface vertical displacement c_1+c_2. It can be easily verified that the nonlocal stresses (62)–(64) indeed satisfy the natural boundary conditions (39) and (40) as well as the decaying property as $x_2\to\infty$, for any set of values of the arbitrary constants c_1 and c_2.

We now turn our attention to the determination of the arbitrary constants c_1 and c_2, which are no longer set by the natural BCs of the problem as in local elasticity, for these are already implied in the choice of the attenuation functions (41) and (43). Instead, we use (16) whereby the nonlocal stresses (62), (63), and (64) satisfy the equation of motion in their original form (5) on the boundary. This gives the homogeneous linear system

$$R_{11}c_1+R_{12}c_2=0, \tag{65a}$$

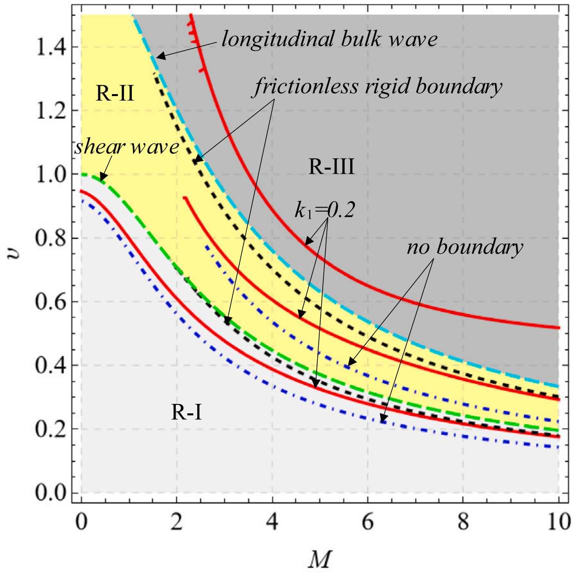

Fig. 1. Dispersion diagram (i.e. dimensionless speed υ vs. dimensionless wavenumber M) of nonlocal Rayleigh waves for $\kappa = 1.7$, $\epsilon_1 = 0.5$. The curve for the dimensionless surface stiffness, $k_1 = 0.2$ (red, solid) is plotted together with the limiting cases of a free surface, $k_1 \to 0$ (blue, dot-dashed) and a frictionless rigid boundary, $k_1 \to \infty$ (black, dashed). The bulk shear and longitudinal wave speeds, $\upsilon = 1/b_3$ (green, long-dashed) and $\upsilon = \kappa/b_3$ (cyan, long-dashed), bound three regions of space denoted as R-I, R-II, and R-III.

$$(k_1 + R_{21})c_1 + (k_1 + R_{22})c_2 = 0, \tag{65b}$$

where

$$\left.\begin{aligned} R_{11} &= \frac{4b_3^2 - M\epsilon_1(b_3 + M\epsilon_1 b_1)\left(\kappa^2 b_1^{-1} - b_1(\kappa^2-2)\right)}{1+(1-b_1^2)M^2\epsilon_1^2}, \\ R_{12} &= \frac{2(1+b_2^2)b_3^2 - 2M\epsilon_1 b_2(b_3 + M\epsilon_1 b_2)}{1+(1-b_2^2)M^2\epsilon_1^2}, \\ R_{21} &= \frac{\kappa^2 b_1 - b_1^{-1}(\kappa^2-2) - 2M\epsilon_1 b_3^{-1}}{1+(1-b_1^2)M^2\epsilon_1^2}, \\ R_{22} &= \frac{2b_2 - M\epsilon_1 b_3^{-1}(1+b_2^2)}{1+(1-b_1^2)M^2\epsilon_1^2}. \end{aligned}\right\} \tag{66}$$

A non-trivial solution of this homogeneous system exists if and only if the determine of the coefficient matrix vanishes

$$(R_{11}R_{22} - R_{12}R_{21}) + k_1(R_{11} - R_{12}) = 0, \tag{67}$$

and the corresponding dispersion diagram in plotted in Fig. 1. Within the framework of local elasticity, this dispersion Eq. (67) reduces to

$$\upsilon^2 + k_1^{-1}\left(4b_{2l} - b_{1l}^{-1}\left(1 + b_{2l}^2\right)^2\right) = 0, \tag{68}$$

which admits real solutions up to a limiting value of the surface stiffness, in correspondence of which the nondispersive Rayleigh wave speed turns into the bulk shear wavespeed c_S, see Fig. 2.

4. Rayleigh waves in the case of a frictionless rigid boundary

When the half-plane surface operates as a rigid frictionless boundary, the BC (39) is replaced by

$$u_2(x_1, 0) = 0, \tag{69}$$

that comes as a particular case of the elastically reacting boundary condition in the limit as the elastic stiffness goes to infinity, i.e., $k \to \infty$. This case is interesting and it is here developed because the boundary conditions (16) are no longer coupled. Indeed, in this case, the normal reaction force at the surface can be specified as

$$t_{22}(x_1, 0) = re^{imx_1}. \tag{70}$$

Using the displacement boundary condition in Eq. (69) together with the traveling wave assumption (47), and from Eq. (48b), we obtain a relationship between the unknown constants as

$$c_2 = -c_1. \tag{71}$$

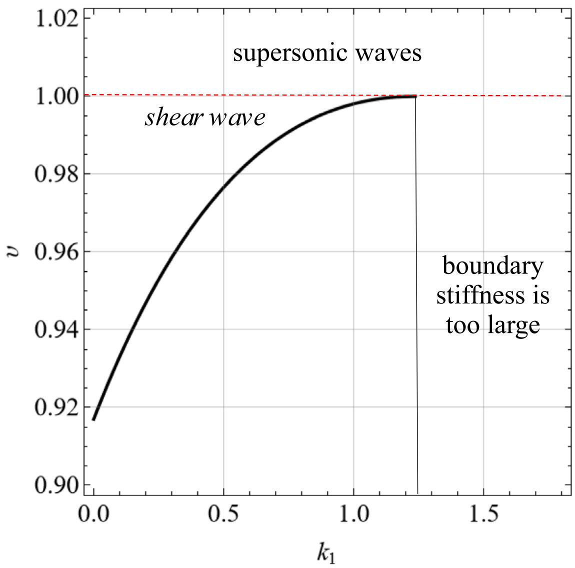

Fig. 2. Dimensionless wave speed (that is independent of the wavenumber) vs. dimensionless surface stiffness k_1 for local (classical) Rayleigh waves, having set $\kappa = 1.7$. The curve stops when the wave speed reaches the dimensionless shear wave speed $\upsilon = 1$.

With this condition, the stress components t_{11} and t_{12} remain the same as those given in Eqs. (62) and (63), respectively, while the normal stress component t_{22}, after omitting the time-dependent factor, is given by

$$\begin{aligned} T_{22}(x_1, x_2) &= -c_1 e^{imx_1} \left\{ \frac{\kappa^2 b_1 - b_1^{-1}(\kappa^2 - 2)}{1 + (1 - b_1^2) m^2 \varepsilon^2} \left[e^{-mb_1 x_2} - e^{-\varepsilon^{-1} b_3 x_2} \right] \right. \\ &\quad \left. - \frac{2b_2}{1 + (1 - b_2^2) m^2 \varepsilon^2} \left[e^{-mb_2 x_2} - e^{-\varepsilon^{-1} b_3 x_2} \right] \right\} + r_1 e^{-\varepsilon^{-1} b_3 x_2} e^{imx_1}, \end{aligned} \tag{72}$$

where $r_1 = r/\mu$ is the dimensionless reaction force. Now, with the BC (16) yields the homogeneous linear system

$$c_1(R_{11} - R_{12}) = 0, \tag{73a}$$

$$R_1 + c_1(R_{21} - R_{22}) = 0, \tag{73b}$$

where R_{ij}, $i, j \in \{1, 2\}$ are defined in Eq. (66). For non-trivial solutions ($c_1 \neq 0$), Eq. (73a) immediately provides the dispersion relation

$$R_{11} - R_{12} = 0, \tag{74}$$

while Eq. (73b) lends the boundary reaction force

$$r_1 = -c_1(R_{21} - R_{22}), \tag{75}$$

which cannot be directly derived from the general elastic support case. In particular, for local elasticity, the dispersion relation (74) and the reaction force defined in Eq. (75) reduce to

$$\upsilon = 0, \quad r_1 = 0, \tag{76}$$

that, as expected, no longer supports Rayleigh-type surface waves.

5. Conclusions

We revisit the theory of nonlocal elasticity, as developed by Kröner (1967) and Eringen (1984), to address its inconsistencies related to the presence of boundaries. In particular, we show that

1. Convolution (or difference-type) attenuation functions are only admissible for unbounded domains, because they imply material homogeneity, as already observed by Kunin (1984) and Eringen (1984). For such unbounded domains, the original theory is consistent and the integral and the differential formulations are immediately equivalent.

2. For bounded domains, the integral and the differential formulations are only equivalent provided that a boundary term is set to zero. This term expresses the original equilibrium equations at the boundary and provides the missing conditions that are required to specify the general solution of the differential problem, given that the natural BCs have already been incorporated in the attenuation functions to avoid over-determination (kernel modification).
3. Indeed, expressing this boundary term for the Rayleigh problem provides a new set of conditions that lend the same dispersion curves already obtained in Nobili and Pramanik (2025a), with the added advantage that only local stresses need to be evaluated at the boundary.
4. When considering inhomogeneous BCs, Eringen's one-integral definition of the nonlocal stress is no longer adequate, assuming the attenuation function satisfies reciprocity (that is necessary to obtain a quadratic energy functional). We provide a generalized formulation that contains a boundary integral which brings in the inhomogeneous character of the BCs, in the lack of which the differential constitutive relation connecting local and nonlocal stresses is no longer equivalent to the integral formulation. This formulation admits a corresponding new expression for the elastic energy which differs, in several respects, from that given, for example, in Polizzotto (2001).
5. The above results may be interpreted in the light of surface elasticity, where the differential formulation is enforced in the interior of the body for the local stress, while the boundary term may be seen as the equilibrium equation on the surface for the nonlocal stress. The surface problem admits an integral solution (and it is therefore nonlocal) that gives the nonlocal stress through the local stress field in the interior of the body. Similarly, the elastic energy may be rewritten to account for this surface elasticity interpretation, by introducing a nonlocal strain field on the surface.

Finally, the extended theory is applied to solve the generalized Rayleigh problem, where the half-plane surface is vertically restrained by elastic springs. The corresponding dispersion relation is obtained and waves are supported also in the case of a rigid frictionless boundary. In the limit of classical elasticity, Rayleigh waves can be found up to a limiting stiffness, in correspondence of which Rayleigh waves move at the shear wave speed. Consequently, the well-known result is retrieved that Rayleigh waves are not admitted by a rigid frictionless boundary.

CRediT authorship contribution statement

Andrea Nobili: Writing – review & editing, Writing – original draft, Supervision, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Dipendu Pramanik:** Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Andrea Nobili reports financial support was provided by Italian government Ministry of University and Research (MUR). If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We acknowledge financial support under the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.1, Call for tender No. 1409 published on 14.9.2022 by the Italian Ministry of University and Research (MUR), funded by the European Union NextGenerationEU, Project Title: Sustainable CompositE materials for the construction iNdusTry - CUP P2022P3Y2T - Grant Assignment Decree No. 1381 adopted on 1.9.2023 by the Italian Ministry of University and Research (MUR).

Appendix. Rayleigh wave dispersion for traction-free half-space

Let us derive the dispersion equation for Rayleigh waves propagating in a traction-free half-plane by using the theory developed in Section 2.2 and show that results corresponds to those reported in Nobili and Pramanik (2025a). However, it can be noted that, if we use only the single Green's function for the nonlocal stress component $t_{11}(\boldsymbol{x})$, such that

$$t_{11}(\boldsymbol{x}) = \int_{\mathbb{B}} \alpha_{11}(\boldsymbol{x},\boldsymbol{\xi})\sigma_{11}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{A.1}$$

then the Eq. (20) becomes

$$t_{ij,j}(\boldsymbol{x}) - \rho\ddot{u}_i(\boldsymbol{x}) = \int_{\mathbb{B}} \left(\delta_{i1}\frac{\mathrm{d}G_2}{\mathrm{d}\xi_1}(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i1}(\boldsymbol{\xi}) + 2\frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\boldsymbol{\xi})\sigma_{i2}(\boldsymbol{\xi})\right)\mathrm{d}\boldsymbol{\xi} + \rho\varepsilon^2\int_{\partial\mathbb{B}}\frac{\mathrm{d}G^o}{\mathrm{d}\nu}(\boldsymbol{x},\boldsymbol{\xi})\ddot{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{A.2}$$

which can be further specialized as

$$\begin{aligned} &t_{ij,j}(\boldsymbol{x}) - \rho\ddot{u}_i(\boldsymbol{x}) \\ &= \int_0^\infty \mathrm{d}\xi_2\int_{-\infty}^\infty \left(\delta_{i1}\frac{\mathrm{d}G_2}{\mathrm{d}\xi_1}(\boldsymbol{x},\xi_1,\xi_2)\sigma_{i1}(\xi_1,\xi_2) + 2\frac{\mathrm{d}G_2}{\mathrm{d}\xi_2}(\boldsymbol{x},\xi_1,\xi_2)\sigma_{i2}(\xi_1,\xi_2)\right)\mathrm{d}\xi_1 \end{aligned}$$

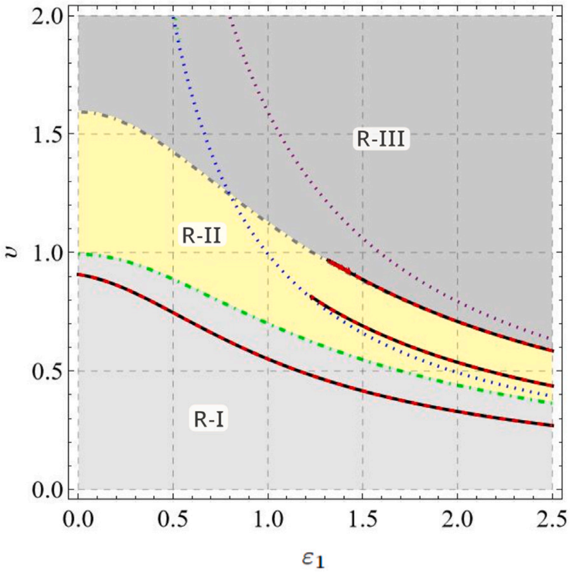

Fig. A.3. Dispersion diagram for nonlocal Rayleigh waves computed according to Nobili and Pramanik (2025a, Eq.(29)) (solid, black) and according to (A.8) (dashed, red): the two families of curves perfectly overlap (here $\kappa = 1.6$ and $\varepsilon_1 = m\varepsilon$, where m is the wavenumber and ε the nonlocal parameter, is used for consistency with (Nobili & Pramanik, 2025a)). Three regions of space, named R-I, R-II and R-III, are bounded by the bulk wavespeed for shear $\upsilon = 1/b_3$ (green, dot-dashed) and longitudinal waves $\upsilon = \kappa/b_3$ (gray, dot-dashed). The dispersion curves begin/end at the limit speeds $\upsilon = \varepsilon_1^{-1}$ (blue, dotted) and $\upsilon = \kappa\varepsilon_1^{-1}$ (purple, dotted).

$$
\begin{aligned}
&-\rho\varepsilon^2\int_{-\infty}^{\infty}\frac{\mathrm{d}G^o}{\mathrm{d}\xi_2}(\boldsymbol{x},\xi_1,0)\ddot{u}_i(\xi_1,0)\mathrm{d}\xi_1,\\
&=\frac{\mu}{4\pi\varepsilon^3}\int_0^{\infty}\mathrm{d}\xi_2\int_{-\infty}^{\infty}\mathrm{d}\xi_1\int_{-\infty}^{\infty}\frac{-\delta_{i1}is_1\Sigma_{i1}(\xi_2)+2\sqrt{1+s_1^2}\,\Sigma_{i2}(\xi_2)}{\sqrt{1+s_1^2}}\,\frac{e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)}}{e^{\varepsilon^{-1}\sqrt{1+s_1^2}(x_2+\xi_2)}}e^{i(m\xi_1-\Omega\tau)}\mathrm{d}s_1\\
&\quad+\frac{\rho\omega^2U_i(0)}{2\pi\varepsilon}\int_{-\infty}^{\infty}\mathrm{d}s_1\int_{-\infty}^{\infty}e^{-i\varepsilon^{-1}s_1(x_1-\xi_1)-\varepsilon^{-1}\sqrt{1+s_1^2}x_2}e^{i(m\xi_1-\Omega\tau)}\mathrm{d}\xi_1,\\
&=\frac{\mu}{2b_3\varepsilon^2}e^{i(mx_1-\Omega\tau)}\int_0^{\infty}\left[\delta_{i1}im\varepsilon\Sigma_{i1}(\xi_2)+2b_3\Sigma_{i2}(\xi_2)\right]e^{-\varepsilon^{-1}b_3(x_2+\xi_2)}\mathrm{d}\xi_2\\
&\quad+\frac{\rho\omega^2U_i(0)}{\varepsilon}\int_{-\infty}^{\infty}e^{-\varepsilon^{-1}\sqrt{1+s_1^2}x_2}e^{-i(\varepsilon^{-1}s_1x_1+\Omega\tau)}\delta(m+\varepsilon^{-1}s_1)\mathrm{d}s_1,
\end{aligned}
\tag{A.3}
$$

where the integral representation of the Green's functions $G_2 = -\alpha_{11}(x_1-\xi_1, x_2+\xi_2)$, $G^o = \alpha_{12}(\boldsymbol{x},\boldsymbol{\xi})$ defined in Eqs. (54), (55), together with the traveling wave assumption (47), (50) are employed. Then, using the local stress defined in Eq. (53), we have

$$
\begin{aligned}
&t_{ij,j}(\boldsymbol{x})-\rho\ddot{u}_i(\boldsymbol{x})\\
&=\frac{\mu}{\varepsilon}\left[\frac{1}{2b_3}\left(\frac{im\varepsilon\delta_{i1}A_{i1}+2b_3A_{i2}}{b_3+m\varepsilon b_1}+\frac{im\varepsilon\delta_{i1}B_{i1}+2b_3B_{i2}}{b_3+m\varepsilon b_2}\right)+\varepsilon m^2\upsilon^2U_i(0)\right]e^{-\varepsilon^{-1}b_3x_2}e^{i(mx_1-\Omega\tau)}.
\end{aligned}
\tag{A.4}
$$

Thus, the nonlocal stresses satisfy the equations of motion (5) provided that

$$
\frac{1}{2b_3}\left(\frac{im\varepsilon\delta_{i1}A_{i1}+2b_3A_{i2}}{b_3+m\varepsilon b_1}+\frac{im\varepsilon\delta_{i1}B_{i1}+2b_3B_{i2}}{b_3+m\varepsilon b_2}\right)+\varepsilon m^2\upsilon^2U_i(0)=0,\quad i=1,2,
\tag{A.5}
$$

which can be further simplified and decomposed into a homogeneous linear system for the arbitrary constants c_1, c_2 as

$$
R^*_{11}c_1+R^*_{12}c_2=0,
\tag{A.6a}
$$

$$
R^*_{21}c_1+R^*_{22}c_2=0,
\tag{A.6b}
$$

where,

$$
\left.
\begin{aligned}
R^*_{11}&=\frac{1}{2b_3}\frac{im\varepsilon P_1+2b_3P_2}{b_3+m\varepsilon b_1}-im\varepsilon\upsilon^2b_1^{-1}, & R^*_{21}&=\frac{P_3}{b_3+m\varepsilon b_1}+m\varepsilon\upsilon^2,\\
R^*_{12}&=\frac{1}{2b_3}\frac{im\varepsilon Q_1+2b_3Q_2}{b_3+m\varepsilon b_2}-im\varepsilon\upsilon^2b_2, & R^*_{22}&=\frac{Q_3}{b_3+m\varepsilon b_2}+m\varepsilon\upsilon^2,
\end{aligned}
\right\}
\tag{A.7}
$$

the quantities P_i, Q_i follows from (51) in the forms $A_{ij} = P_{i+j-1}c_1$, $B_{ij} = Q_{i+j-1}c_2$. Non-trivial solutions of (A.6) require the determinant of the coefficient matrix to vanish, yielding the dispersion equation for the Rayleigh wave in a traction-free half-space as

$$R^*_{11}R^*_{22} - R^*_{12}R^*_{21} = 0. \tag{A.8}$$

A comparison between the dispersion Eq. (A.8), obtained from the boundary condition (20), and the dispersion equation presented in Nobili and Pramanik (2025a, Eq.(29)) for the same problem, this time derived by enforcing the equations of motion, is presented in Fig. A.3. Remarkably, the dispersion curves perfectly coincide, despite having being derived through two different procedures: on the one side the expressions of the nonlocal stress components (Nobili & Pramanik, 2025a, Eq.(29)) is demanded, on the other only the local stress components are required, whence this approach gets away from the need to compute the nonlocal stresses.

Data availability

No data was used for the research described in the article.

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