Mediterr. J. Math. (2025) 22:27
https://doi.org/10.1007/s00009-024-02790-2

Mediterranean Journal of Mathematics

Trisections of PL 4-Manifolds Arising from Colored Triangulations

Maria Rita Casali and Paola Cristofori

Abstract. The purpose of the present paper is twofold: first to extend to non-orientable compact 4-manifolds the notion of *gem-induced trisection*, directly obtained from colored triangulations (or, equivalently, from colored graphs encoding them, called *gems*); second to prove that, both in the orientable and non-orientable case, if the boundary is homeomorphic to a connected sum of sphere bundles over $\mathbb{S}^1$, gem-induced trisections naturally give rise to trisections of the corresponding closed 4-manifold. As a consequence, an estimation of the trisection genus of any closed orientable 4-manifold is obtained via colored triangulations, in terms of the combinatorial properties of a Kirby diagram representing it.

Mathematics Subject Classification. 57Q15, 57K40, 57M15.

Keywords. Trisection, compact 4-manifold, colored triangulation.

1. Introduction

In 2018 (see [2]), Bell, Hass, Rubinstein and Tillmann introduced the use of (singular) triangulations to study *trisections* of closed orientable 4-manifolds, i.e., decompositions into three 4-dimensional handlebodies, mutually intersecting in 3-dimensional handlebodies and globally intersecting in a closed surface (according to the definition by Gay and Kirby [17]). The same combinatorial approach has been applied in [27] to the special case of *colored triangulations* of (closed) PL 4-manifolds with exactly ten edges, to compute the *trisection genus* (i.e., the minimum genus of the intersecting surface) of connected sums of $\mathbb{CP}^2$—possibly with reversed orientation—$\mathbb{S}^2 \times \mathbb{S}^2$ and the $K3$-surface.

The above ideas have been generalized in [8], where the assumption on the number of edges of the colored triangulation is relaxed and orientable

This work was supported by GNSAGA of INDAM and by the University of Modena and Reggio Emilia, project: *"Discrete Methods in Combinatorial Geometry and Geometric Topology"*.

Birkhäuser

Published online: 16 January 2025

4-manifolds with connected boundary are included, too: the resulting decompositions are called *gem-induced trisections*, since they are directly obtained from *gems* (i.e., *Graphs Encoding Manifolds*).

While in the closed setting a gem-induced trisection is a particular kind of trisection, when the boundary is non-empty the decomposition is different from that introduced in [13], which is commonly intended as a trisection in the boundary case: in fact, it does not involve compression bodies and open book decompositions, but only handlebodies and Heegaard splittings, according to a suggestion by Rubinstein and Tillmann in [26, Section 4.5].

On the other hand, in [24], the classical definition of trisection has been broadened to non-orientable manifolds. It is then natural to address the problem of extending to the non-orientable case the notion of gem-induced trisection of a compact 4-manifold.

Section 3 of the present paper describes how such a decomposition can be obtained from a colored triangulation of the associated singular manifold, or, equivalently, from its dual graph, which is called a *gem* of the compact manifold itself (see Sect. 2 for a brief overview of this combinatorial representation for compact PL manifolds).

Actually, gem-induced trisections turn out to fit the definition of trisection—hence allowing a direct estimation of the trisection genus—only for a suitable class of closed orientable 4-manifolds (possibly comprehending all simply-connected ones, according to Kirby problem n. 50): see Proposition 3.5.

Nevertheless, the particular type of extension to the boundary case performed by gem-induced trisections allows also an "indirect" approach to trisections of closed (orientable and non-orientable) 4-manifolds. In fact, in Sect. 4 we prove that, in case of a compact 4-manifold M whose boundary is homeomorphic to a connected sum of sphere bundles over $\mathbb{S}^1$, any gem-induced trisection of M naturally gives rise to a trisection of the associated closed 4-manifold $\bar{M}$ with the same intersecting surface; therefore, an upper bound for the trisection genus of $\bar{M}$ is obtained (see Theorem 4.1).

In particular, trisections arising from colored triangulations turn out to minimize the trisection genus for a wide class of (orientable and non-orientable) 4-manifolds: see Proposition 4.4.

Moreover, combining the above results with those contained in [6] and [8], we prove the existence, for each closed orientable 4-manifold $\bar{M}$, of trisections arising from (suitable) gems of the compact 4-manifold M, that consists of the 0-, 1- and 2-handles in a handle-decomposition of $\bar{M}$. Hence, an estimation of the trisection genus $g_T(\bar{M})$ is obtained, in terms of the combinatorial properties of a Kirby diagram representing $\bar{M}$ (see Sect. 4 for details):

Theorem 1.1.

(i) *For each closed orientable* 4*-manifold* $\bar{M}$*,*

$$g_T(\bar{M}) \leq s + 1,$$

s being the crossing number of a (connected and with dotted components in "good position") Kirby diagram representing $\bar{M}$*.*

(ii) *Furthermore, if $\bar{M}$ admits a handle decomposition lacking in 1-handles, then*

$$g_T(\bar{M}) \leq m_\alpha,$$

m_α being the number of α-colored regions in a chess-board coloration of a (connected and with no dotted component) Kirby diagram representing $\bar{M}$.

2. Colored Triangulations and Gems of Compact PL Manifolds

Throughout this paper we will work in the PL category, therefore manifolds and maps under consideration will always be PL.

By a *singular n-manifold* we mean a closed connected n-dimensional polyhedron admitting a simplicial triangulation where the link of vertices are closed connected $(n-1)$-manifolds while the links of the h-simplices, with $h > 0$, are $(n-h-1)$-spheres. Vertices whose links are not spheres will be called *singular*.

A *colored triangulation* of a singular n-manifold N is a (pseudo) triangulation K of N endowed with a labeling of its vertices by the set of integers $\Delta_n = \{0, \ldots, n\}$ which is injective on each simplex.

Such a colored triangulation is usually combinatorially visualized by the 1-skeleton $\Gamma(K)$ of its dual complex, endowed with the edge-coloration inherited from the labeling of K: an edge e of $\Gamma(K)$ has color $c \in \Delta_n$ iff no vertex of the $(n-1)$-simplex of K dual to e is labeled c. $\Gamma(K)$ is an $(n+1)$-*colored graph* (i.e., a regular multigraph with degree $n+1$, such that its edge-coloration by Δ_n is injective on adjacent edges[1]), which is said to *represent* the singular manifold N.

Remark 2.1. The duality between K and $\Gamma(K)$ establishes a bijective correspondence between the $(n-h)$-simplices of K whose vertices are labeled by $\Delta_n - \{c_1, \ldots, c_h\}$ and the connected components of the subgraph $\Gamma_{\{c_1,\ldots,c_h\}}$ obtained from $\Gamma(K)$ by considering only edges colored by $\{c_1, \ldots, c_h\}$, which are called $\{c_1, \ldots, c_h\}$*-residues*. In particular, the connected components obtained from $\Gamma(K)$ by deleting all c-colored edges ($c \in \Delta_n$), called $\hat{c}$*-residues* of $\Gamma(K)$, are n-colored graphs representing the disjoint links[2] of the c-labeled vertices of K (which are closed $(n-1)$-manifolds, since K triangulates a singular n-manifold).

---

[1]According to a well-established literature (see, for example, [9] and references within), an n-dimensional pseudocomplex $K(\Gamma)$ may be associated to any $(n+1)$-colored graph Γ, by considering an n-simplex, with vertices labeled by the elements of Δ_n, for each vertex of Γ and by gluing two n-simplices along their $(n-1)$-dimensional faces opposite to c-labeled vertices, whenever the corresponding vertices of Γ are c-adjacent ($c \in \Delta_n$). Note that, if $|K(\Gamma)|$ is a singular manifold, both $\Gamma(K(\Gamma)) = \Gamma$ and $K(\Gamma(K)) = K$ hold, so colored triangulations of singular manifolds and colored graphs representing them can be associated unambiguously.

[2]Given an h-simplex σ^h of K, the *disjoint star* of σ^h in K is the pseudocomplex obtained by taking all n-simplices of K having σ^h as a face and re-identifying only their faces that contain σ^h. The *disjoint link* of σ^h in K is the subcomplex of the disjoint star formed by those simplices that do not intersect σ^h.

As a consequence, by setting from now on $\Gamma = \Gamma(K)$, the following characterization holds:

$|K|$ *is a closed* n*-manifold iff, for each color* $c \in \Delta_n$,
all $\hat{c}$*-residues of* Γ *represent the* $(n-1)$*-sphere.*

A $\hat{c}$*-residue* of Γ will be called *singular* if it corresponds to a singular vertex of $|K|$, i.e., if it does not represent the sphere. A color c is said to be *singular* if at least one $\hat{c}$-residue is singular.

Moreover, the number of $\{c_1, \dots, c_h\}$-residues will be denoted by $g_{c_1,\dots,c_h}$ (or, for short, by $g_{\hat{c}}$ if $h = n$ and $\{c\} = \Delta_n - \{c_1, \dots, c_n\}$).

If M is a compact n-manifold, then a singular n-manifold $\widehat{M}$ can be constructed by capping off each component of ∂M by a cone over it. Therefore, $(n+1)$–colored graphs may be used to represent compact PL n-manifolds, as well:

Definition 2.2. An $(n+1)$-colored graph *represents* a compact n-manifold M (or, equivalently, it is a *gem* of M, where gem means *Graph Encoding Manifold*) if and only if the dual pseudocomplex is a colored triangulation of the singular manifold $\widehat{M}$.

The so called *gem theory*, or *crystallization theory*, is based on the following existence result (which extends the one originally stated in [25] for closed manifolds):

Theorem 2.3 [10]. *Any compact orientable (resp. non orientable)* n*-manifold* M *admits a bipartite (resp. non-bipartite)* $(n+1)$*-colored graph* Γ *representing it.*
In particular, if M *has empty or connected boundary:*

- Γ *may be assumed to have color* n *as its unique possible singular color, and exactly one* $\hat{n}$*-residue (in this case we will say that* Γ *belongs to the class* $G_s^{(n)}$*)*[3]
- Γ *may be assumed to have exactly one* $\hat{c}$*-residue,* $\forall c \in \Delta_n$ *(in this case* Γ *is called a* crystallization *of* M*).*[4]

In this paper only manifolds with empty or connected boundary will be considered. Note that, according to Definition 2.2, the same $(n+1)$-colored graph can represent both a closed n-manifold and the compact n-manifold obtained by deleting from it the interior of an n-ball. To avoid this ambiguity, from now on we will also exclude manifolds with spherical boundary. However, all definitions and results of this paper regarding closed manifolds (for which, obviously, $M = \widehat{M}$ holds) can be easily translated to fit the case of spherical boundary.

[3]Equivalently, the associated colored triangulation K of $\widehat{M}$ has exactly one vertex labeled n, which is the only possible singular vertex of K (in this case we will say that K belongs to the class $\mathcal{K}^{(n)};_s$). Note that, obviously, the vertex of K labeled n (or, equivalently, the $\hat{n}$-residue of Γ) is singular if and only if ∂M is non-empty and non-spherical.

[4]Equivalently, the associated colored triangulation of $\widehat{M}$ has exactly $n+1$ vertices.

Let us recall [16] that, given a connected bipartite (resp. non-bipartite) $(n+1)$-colored graph Γ of order $2p$, then for each cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ of Δ_n, up to inverse, there exists a particular type of cellular embedding, called *regular*,[5] of Γ into an orientable (resp. non-orientable) closed surface. Moreover, the genus (resp. half the genus) of this surface, denoted by $\rho_\varepsilon(\Gamma)$, satisfies

$$2 - 2\rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_{n+1}} g_{\varepsilon_j, \varepsilon_{j+1}} + (1-n)p. \quad (1)$$

The *regular genus* of Γ is defined as

$$\rho(\Gamma) = min\{\rho_\varepsilon(\Gamma) \mid \varepsilon \text{ is a cyclic permutation of } \Delta_n\}.$$

The *regular genus* of a compact n-manifold M is a PL invariant extending to higher dimension the classical genus of a surface and the Heegaard genus of a 3-manifold: it is defined as

$$\mathcal{G}(M) = min\{\rho(\Gamma) \mid \Gamma \text{ is a gem of } M\}.$$

It was proved in [15] that regular genus zero characterizes spheres in any dimension; moreover, other classification results via regular genus are available, especially in dimension 4 and 5 (see [4,7,12] and their references). Sections 3 and 4 will show that, in dimension 4, the regular genus is also strongly involved in the estimation of the trisection genus via colored triangulations.

3. Gem-induced Trisections of Non-orientable 4-Manifolds

As already recalled in Sect. 1, gem-induced trisections of compact orientable 4-manifolds with empty or connected boundary have been introduced in [8], as an effective tool for the study of trisections in the PL setting. They have been also used in [23] to investigate connections between trisections of closed orientable 4-manifolds and *colored tensor models* [11].

In this section we will show how the concept of gem-induced trisection can be extended so as to include also the non-orientable case.

Let M be a compact 4-manifold with empty or connected boundary and let K be a colored triangulation of the singular manifold $\widehat{M}$. Note that, by Theorem 2.3, it is always possible to suppose $K \in \mathcal{K}_s^{(4)}$, i.e., to assume that K has only one possible singular vertex, labeled by color 4. Furthermore, let us fix a cyclic permutation ε of Δ_4 and suppose, for simplicity, $\varepsilon_4 = 4$.

Then—by generalizing an idea of [2,27]—we can construct a particular decomposition of the singular manifold $\widehat{M}$ associated to K and ε in the following way:

- let σ be the standard 2-simplex, whose vertices are denoted by v_0, v_1, v_2 and let $\mu : K(\Gamma) \to \sigma$ be the simplicial map defined by $\mu(v) = v_1$ (resp. $\mu(v) = v_2$) iff v is a vertex of K labeled by ε_0 or ε_2 (resp. ε_1 or

[5] More precisely, a regular embedding is a cellular embedding whose regions are bounded by the images of the $\{\varepsilon_j, \varepsilon_{j+1}\}$-colored cycles of Γ, for each $j \in \mathbb{Z}_{n+1}$.

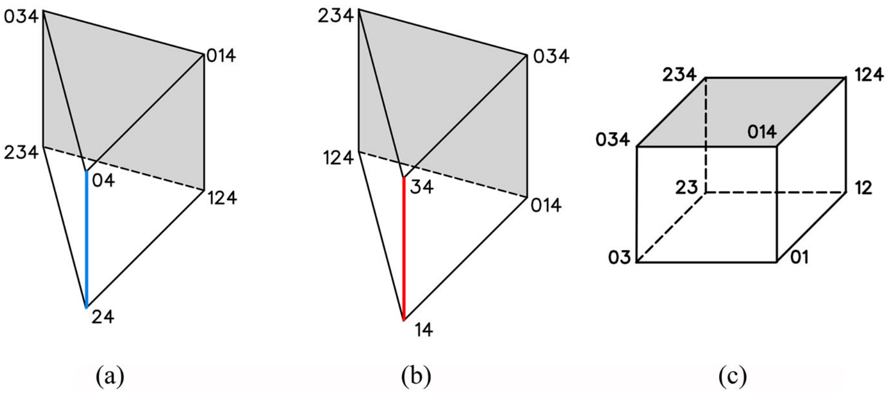

Figure 1. The intersections of a 4-simplex of $K(\Gamma)$ with H_{01}, H_{02} and H_{12} respectively

ε_3); finally let us set $\mu(w) = v_0$, where w is the unique ε_4-labeled vertex of K.

- The preimage $\widehat{H}_0$ of the star of v_0 in the first barycentric subdivision σ' of σ is the cone over the disjoint link of w, while the preimage H_1 (resp. H_2) of the star of v_1 (resp. v_2) in σ' is a regular neighborhood of the 1-subcomplex of K generated by the ε_0- and ε_2-labeled (resp. ε_1- and ε_3-labeled) vertices. Therefore both H_1 and H_2 are 4-dimensional handlebodies.
- The 3-dimensional subcomplex $H_{01} = \widehat{H}_0 \cap H_1$ (resp. $H_{02} = \widehat{H}_0 \cap H_2$) is a 3-dimensional handlebody. In fact, its intersection with each 4-simplex of K is a triangular prism as the one in Fig. 1(a) (resp. Fig. 1(b), whose vertices are barycenters of a 1- or 2-simplex, which we indicate by the labels of its spanning vertices. The quadrangular face of the prism opposite to the blue (resp. red) edge is free; so, H_{01} (resp. H_{02}) collapses to a graph.
- Note that $\Sigma = \widehat{H}_0 \cap H_1 \cap H_2$ is formed by the free faces of the above prisms (see Fig. 1, where $H_{12} = H_1 \cap H_2$), one for each 4-simplex of K; since each edge of such quadrangular faces corresponds to a 3-simplex of K, it is shared by exactly two faces and hence Σ turns out to be a closed connected surface. Moreover, its Euler characteristic is $\chi(\Sigma) = g_{0,1} + g_{1,2} + g_{2,3} + g_{0,3} - 4p + 2p = 2 - 2\rho_{\varepsilon_{\hat{4}}}(\Gamma(K)_{\hat{4}})$, where $\varepsilon_{\hat{4}} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

As a consequence of the above construction, the following result can be stated, which extends to the non-orientable case the analogous one in [8].

Proposition 3.1. *Let $\widehat{M}$ be a singular 4-manifold with one singular vertex at most. For each colored triangulation $K \in \mathcal{K}_s^{(4)}$ of $\widehat{M}$ and for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 4)$ of Δ_4, the triple $(\widehat{H}_0, H_1, H_2)$ satisfies the following properties:*

(i) *$\widehat{M} = \widehat{H}_0 \cup H_1 \cup H_2$ and the interiors of $\widehat{H}_0$, H_1, H_2 are pairwise disjoint;*

(ii) *both H_1 and H_2 are 4-dimensional handlebodies of genus $g_{\varepsilon_1,\varepsilon_3,\varepsilon_4} - g_{\widehat{\varepsilon_0}} - g_{\widehat{\varepsilon_2}} + 1$ and $g_{\varepsilon_0,\varepsilon_2,\varepsilon_4} - g_{\widehat{\varepsilon_1}} - g_{\widehat{\varepsilon_3}} + 1$ respectively;*
(iii) *$\widehat{H}_0$ is homeomorphic to the cone over the disjoint link of the singular vertex of $\widehat{M}$ (or to a 4-ball, if $\widehat{M}$ is a 4-manifold);*
(iv) *$H_{01} = \widehat{H}_0 \cap H_1$ and $H_{02} = \widehat{H}_0 \cap H_2$ are 3-dimensional handlebodies;*
(v) *$\Sigma = \widehat{H}_0 \cap H_1 \cap H_2$ is a closed connected surface.*

Moreover, if $H_{12} = H_1 \cap H_2$ is a 3-dimensional handlebody, too, then all the above handlebodies, as well as the surface Σ, are orientable or not according to the orientability of $\widehat{M}$. Therefore, in the first case Σ has genus $\rho_{\varepsilon_{\hat{4}}}(\Gamma(K)_{\hat{4}})$, while in the second one it has genus $2\rho_{\varepsilon_{\hat{4}}}(\Gamma(K)_{\hat{4}})$.

Proof. Statements (i), (iii), (iv) and (v) directly follow by construction, as well as the fact that H_1 and H_2 are handlebodies. To complete the proof of statement (ii), note that the 1-dimensional subcomplex $K_{\varepsilon_0\varepsilon_2}$ (resp. $K_{\varepsilon_1\varepsilon_3}$) of K generated by the $\{\varepsilon_0, \varepsilon_2\}$-colored (resp. $\{\varepsilon_1, \varepsilon_3\}$-colored) vertices, has exactly $g_{\varepsilon_1,\varepsilon_3,4}$ (resp. $g_{\varepsilon_0,\varepsilon_2,4}$) edges and $g_{\widehat{\varepsilon_0}} + g_{\widehat{\varepsilon_2}}$ (resp. $g_{\widehat{\varepsilon_1}} + g_{\widehat{\varepsilon_3}}$) vertices. Since K is a pseudomanifold, $K_{\varepsilon_0\varepsilon_2}$ (resp. $K_{\varepsilon_1\varepsilon_3}$) is connected; hence H_1 (resp. H_2) has genus $g_{\varepsilon_1,\varepsilon_3,4} - g_{\widehat{\varepsilon_0}} - g_{\widehat{\varepsilon_2}} + 1$ (resp. $g_{\varepsilon_0,\varepsilon_2,4} - g_{\widehat{\varepsilon_1}} - g_{\widehat{\varepsilon_3}} + 1$).

Let us now observe that (H_{01}, H_{02}, Σ) is a Heegaard splitting of $\partial\widehat{H}_0$, which is the disjoint link of the (possibly singular) vertex w of K; moreover, if H_{01}, H_{02}, H_{12} are all handlebodies, then for each $i \in \{1, 2\}$, (H_{ij}, H_{ik}, Σ), with $\{j, k\} = \{0, 1, 2\} - \{i\}$, is a Heegaard splitting of the 3-manifold ∂H_i, which is a connected sum of copies of the $\mathbb{S}^2$-bundle over $\mathbb{S}^1$, orientable or non-orientable according to H_i. Therefore, it is obvious that, if $\widehat{M}$ is orientable, all 4-dimensional "pieces" $\widehat{H}_0$, H_1 and H_2 are orientable and the same holds for the 3-dimensional handlebodies and the surface Σ.

On the other hand, if $\widehat{M}$ is non-orientable, then all the handlebodies and the surface Σ must be non-orientable. In fact, the existence of the above Heegaard splittings allows easily to check that, if one of the 4-dimensional "pieces" were orientable, then the Heegaard splitting of its boundary would be formed by orientable elements and, as a consequence, both the third 3-dimensional handlebody and the other two 4-dimensional "pieces" would be orientable, too.

In the same way, if one of the 3-dimensional handlebodies, say H_{ij}, (resp. if the surface Σ) were orientable, the existence of the above Heegaard splittings would imply the orientability of the other 3-dimensional handlebodies, as well as of the boundaries of both the 4-dimensional "pieces" intersecting in H_{ij} (resp. of all 3-dimensional handlebodies, as well as of the boundaries of all 4-dimensional "pieces"), and therefore of all 4-dimensional "pieces", too.

Moreover, the final statement regarding the genus of Σ is a trivial consequence of the computation $\chi(\Sigma) = 2 - 2\rho_{\varepsilon_{\hat{4}}}(\Gamma(K)_{\hat{4}})$. □

It is easy to check that, if $\widehat{M}$ is the singular manifold associated to a compact 4-manifold M, the above triple $(\widehat{H}_0, H_1, H_2)$ naturally induces a decomposition $\mathcal{T}(\Gamma(K), \varepsilon) = (H_0, H_1, H_2)$ of M, where H_0 is the collar on ∂M obtained by deleting from $\widehat{H}_0$ a suitable neighborhood of the singular vertex of K (or $H_0 = \widehat{H}_0$ is a 4-ball, if $\partial M = \emptyset$).

In full analogy with what was already introduced in the orientable case, we can give the following definition:

Definition 3.2. Let M be a compact 4-manifold with empty or connected boundary. A *gem-induced trisection* of M is a decomposition $\mathcal{T}(\Gamma(K),\varepsilon) = (H_0, H_1, H_2)$ of M such that H_{12} is a 3-dimensional handlebody, $K \in \mathcal{K}_s^{(4)}$ being a colored triangulation of $\widehat{M}$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 4)$ a cyclic permutation of Δ_4.

In this case, Σ is called the *central surface* of the gem-induced trisection, and we refer to the common genus of all 3-dimensional handlebodies as the *genus* of the gem-induced trisection.

The term "gem-induced" refers to the fact that the obtained decomposition can be thought as induced both by the triangulation K of $\widehat{M}$ and by its dual 5-colored graph $\Gamma(K) \in G_s^{(4)}$, which in turn is also a gem of the manifold M. Moreover, in the following we will often refer explicitly only to the graph, by using the notations $\mathcal{T}(\Gamma,\varepsilon)$ for the gem-induced trisection and *genus*$(\mathcal{T}(\Gamma,\varepsilon))$ for its genus (which is equal to $\rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}})$).

Remark 3.3. Note that, in the case of a gem-induced trisection $\mathcal{T}(\Gamma,\varepsilon) = (H_0, H_1, H_2)$ of a compact 4-manifold M, then (H_{01}, H_{02}, Σ) is a Heegaard splitting of $\partial\widehat{H}_0 = \partial H_0 = \mathbb{S}^3$ if M is closed, while, if $\partial M \neq \emptyset$, it is a Heegaard splitting of $\partial\widehat{H}_0 = \partial M$.
Moreover, if M (and so $\widehat{M}$) is non-orientable, $\partial\widehat{H}_0$ must be non-orientable; as a consequence no closed non-orientable 4-manifold can admit a gem-induced trisection and the same happens when M is non-orientable with orientable boundary.

Remark 3.4. Let $\mathcal{T}(\Gamma,\varepsilon) = (H_0, H_1, H_2)$ be a gem-induced trisection of a compact 4-manifold M. The existence of the Heegaard splittings of ∂H_1 and ∂H_2 given by the 3-dimensional handlebodies of the decomposition, together with suitable applications of Van–Kampen's theorem, implies that $\pi_1(\widehat{H}_0)$ surjects to $\pi_1(\widehat{M})$ (details can be found in [8], since the arguments do not depend on orientability). Since $\widehat{H}_0$ is contractible, the simply-connectedness of $\widehat{M}$ is a necessary condition for the existence of gem-induced trisections of M.

By construction and by Definition 3.2, a gem-induced trisection of a closed 4-manifold is a particular type of trisection (in the sense of [17]; see also Sect. 4), where one of the 4-dimensional handlebodies is a 4-ball. On the other hand, in the boundary case, gem-induced trisections provide a decomposition which differs in substance from the extended notion of trisection introduced in [13] (see [8, Remark 16]). Therefore, as a consequence of Remark 3.3 and [8, Proposition 25], which characterizes closed orientable 4-manifolds admitting gem-induced trisections, we can state that gem-induced trisections fit the usual definition of trisection only for a particular class of closed orientable 4-manifolds; however, it is worthwhile to note that this class possibly comprehends all simply-connected ones, according to Kirby problem n. 50.

Proposition 3.5. *Let M be a compact PL 4-manifold with empty or connected boundary. A gem-induced trisection of M is a trisection if and only if M is closed and orientable.*

Moreover, a closed orientable 4-manifold admits a gem-induced trisection if and only if it admits a handle decomposition lacking in 3-handles (or in 1-handles). □

In the orientable setting, a combinatorial condition ensuring $\mathcal{T}(\Gamma, \varepsilon)$ to be a gem-induced trisection of M was already presented in [8]; the same condition works as well also in the non-orientable case:

Proposition 3.6. *Let M be a compact 4-manifold with empty or connected boundary and $\Gamma \in G_s^{(4)}$ a gem of M of order $2p$; if there exists an ordering $(e_1, \ldots, e_p)$ of the 4-colored edges of Γ such that for each $j \in \{1, \ldots, p\}$:*
(*) there exists $i \in \Delta_3$ such that all 4-colored edges of the $\{4, i\}$-colored cycle containing e_j belong to the set $\{e_1, \ldots, e_j\}$,
then $\mathcal{T}(\Gamma, \varepsilon)$ is a gem-induced trisection of M, for each cyclic permutation ε of Δ_4.

Proof. The involved arguments are exactly the same as in the orientable case: see the proof of [8, Proposition 20], together with the final part of the proof of [8, Proposition 17]. □

Definition 3.7. Let M be a compact 4-manifold with empty or connected boundary, that admits gem-induced trisections. The *G-trisection genus* of M is defined as:

$$g_{GT}(M) = \min\{genus(\mathcal{T}(\Gamma, \varepsilon)) \mid \mathcal{T}(\Gamma, \varepsilon) \text{ is a gem-induced trisection of } M\}.$$

The properties of the G-trisection genus which were already proved in [8] for orientable 4-manifolds can be easily extended to the non-orientable case, since the arguments do not depend on orientability.

Proposition 3.8. *Let M be a compact 4-manifold with empty or connected boundary. Then:*

(i) $g_{GT}(M) \leq \rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}})$, *for any gem Γ of M so that $\mathcal{T}(\Gamma, \varepsilon)$ is a gem-induced trisection.*
(ii) $g_{GT}(M) = 0 \quad \Leftrightarrow \quad M \cong \mathbb{S}^4$.
(iii) *If M has non-empty boundary, then $g_{GT}(M) \geq \mathcal{H}(\partial M)$, where $\mathcal{H}(\partial M)$ denotes the Heegaard genus of the boundary.*
(iv) *The G-trisection genus is subadditive with respect to both the (internal) connected sum and the boundary connected sum.*

□

Example 3.1. Figure 2 shows a gem $\Gamma \in G_s^{(4)}$ of the genus one non-orientable handlebody $\tilde{\mathbb{Y}}_1^4$. It is easy to see that $\mathcal{T}(\Gamma, \varepsilon)$ is a gem-induced trisection of $\tilde{\mathbb{Y}}_1^4$ for each cyclic permutation ε of Δ_4, since Γ satisfies condition (*) of Proposition 3.6; moreover, $\rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}}) = 1$. Therefore, by statements (i) and (ii) of Proposition 3.8, $g_{GT}(\tilde{\mathbb{Y}}_1^4) = 1$.

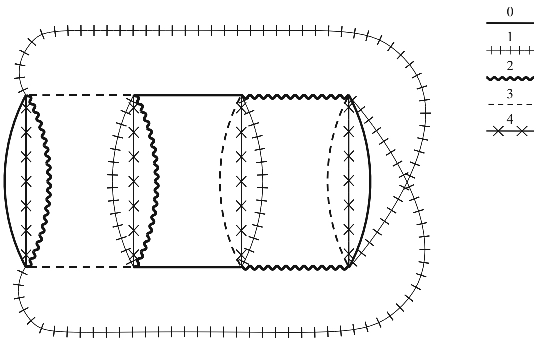

Figure 2. A gem of the genus one non-orientable handlebody $\tilde{\mathbb{Y}}^4_1$

Furthermore, if $\tilde{\mathbb{Y}}^4_m$ denotes the genus m non-orientable handlebody ($m \geq 1$), $g_{GT}(\tilde{\mathbb{Y}}^4_m) = m$ directly follows by statements (iii) and (iv) of Proposition 3.8. The same value was already known from [8] to be the G-trisection genus of the genus m orientable handlebody $\mathbb{Y}^4_m$.

4. Trisections Arising from Gem-induced Trisections

In the present section, we will show that gem-induced trisections of 4-manifolds with boundary allow an "indirect" approach to trisections of closed 4-manifolds, both in the orientable and non-orientable case.

Let us first recall that, according to [17,24], a *trisection* of or a closed 4-manifold $\bar{M}$ is a decomposition of $\bar{M}$ into three 4-dimensional handlebodies with disjoint interiors, whose pairwise intersections are 3-dimensional handlebodies and all intersecting into a closed connected surface (called the *central surface* of the trisection). Moreover, the seven "pieces" of the trisection (three in dimension 4, three in dimension 3, plus the central surface) are all orientable if and only if $\bar{M}$ is orientable and all non-orientable if and only if $\bar{M}$ is non-orientable. It is also a direct consequence of the definition that all the 3-dimensional handlebodies have the same genus, which is called the *genus* of the trisection.

The *trisection genus* $g_T(\bar{M})$ of any closed 4-manifold $\bar{M}$ is defined as the minimum genus among all trisections of $\bar{M}$.

Throughout this section, the following lower bound for $g_T(\bar{M})$ will turn out to be useful:

$$g_T(\bar{M}) \geq \beta_1(\bar{M};\mathbb{Z}_2) + \beta_2(\bar{M};\mathbb{Z}_2) \qquad (2)$$

where $\beta_i(\bar{M};\mathbb{Z}_2)$ denotes the i-th Betti number of $\bar{M}$ with coefficient in $\mathbb{Z}_2$.

The proof of inequality (2) is due to [27], where—however—the trisection genus is defined as the minimal genus of a central surface, i.e., it coincides with $g_T(\bar{M})$ (resp. $2g_T(\bar{M})$) if $\bar{M}$ is orientable (resp. non-orientable).

The following theorem, by proving how trisections of closed 4-manifolds may arise from gem-induced trisections of bounded 4-manifolds, also yields an upper bound for the trisection genus in terms of the combinatorial invariant G-trisection genus defined in Sect. 3.

Theorem 4.1. *Let M be a compact orientable (resp. non-orientable) 4-manifold with boundary $\partial M \cong \#_m(\mathbb{S}^2 \times \mathbb{S}^1)$ (resp. $\partial M \cong \#_m(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1)$), $m > 0$. If M admits a gem-induced trisection, then the closed 4-manifold $\bar{M}$, uniquely obtained by gluing a 4-dimensional handlebody along ∂M (i.e., $\bar{M} \cong M \cup \mathbb{Y}^4_m$ or $\bar{M} \cong M \cup \tilde{\mathbb{Y}}^4_m$, according to M being orientable or not) admits a trisection with the same central surface.*

As a consequence,

$$g_T(\bar{M}) \leq g_{GT}(M).$$

Proof. By hypothesis, a colored triangulation $K \in \mathcal{K}_s^{(4)}$ of $\widehat{M}$ (or, equivalently, a gem $\Gamma \in G_s^{(4)}$ of M) and a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 4)$ of Δ_4 exist, so that $\mathcal{T}(\Gamma, \varepsilon) = (H_0, H_1, H_2)$ is a gem-induced trisection of M, where H_1 and H_2 are 4-dimensional handlebodies and H_0 is a collar on ∂M; moreover, all pairwise intersections $H_0 \cap H_1, H_1 \cap H_2, H_2 \cap H_0$ are 3-dimensional handlebodies of the same genus. In virtue of a celebrated theorem in [21], together with its non-orientable version in [24], a 4-dimensional handlebody of genus m $\overset{(\sim)}{\mathbb{Y}}{}^4_m$ (orientable or non-orientable, according to the orientability of M) may be glued in a unique way to the "free" boundary of H_0, so as to obtain $\bar{H}_0 = H_0 \cup \overset{(\sim)}{\mathbb{Y}}{}^4_m \cong \overset{(\sim)}{\mathbb{Y}}{}^4_m$. It is now easy to check that the triple $(\bar{H}_0, H_1, H_2)$ actually constitutes a trisection of the closed 4-manifold $\bar{M} = M \cup \overset{(\sim)}{\mathbb{Y}}{}^4_m$, with the same pairwise intersections as $\mathcal{T}(\Gamma, \varepsilon)$, and hence with the same intersecting surface.

The second part of the statement directly follows. □

In the following, a trisection of a closed (orientable or non-orientable) 4-manifold $\bar{M}$ will be said to *arise from a colored triangulation* if it is either a gem-induced trisection of $\bar{M}$ or it is obtained from a gem-induced trisection of M (such that $\bar{M} \cong M \cup \overset{(\sim)}{\mathbb{Y}}{}^4_m$) according to Theorem 4.1.

Moreover, for sake of conciseness, we will denote by $\mathbb{S}^{n-1} \otimes \mathbb{S}^1$ either $\mathbb{S}^{n-1} \times \mathbb{S}^1$ or $\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1$, i.e., the orientable or non-orientable $\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$.

Example 4.1. In [8], $\mathbb{Y}^4_1$ has been proved to admit a gem-induced trisection of genus 1, through its well-known order eight crystallization; on the other hand, Example 3.1 proves that $\tilde{\mathbb{Y}}^4_1$ admits a gem-induced trisection of genus 1, too, through the (order eight) crystallization of Fig. 2. Hence, $g_T(\mathbb{S}^3 \otimes \mathbb{S}^1) \leq 1$ follows from Theorem 4.1. Actually, the equality holds via the well-known characterization of $\mathbb{S}^4$ as the only closed 4-manifold with trisection genus equal to zero (see also Proposition 3.8(ii)). Thus, both $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ are proved to admit a trisection with minimal genus arising from a colored triangulation.

We want now to describe a possible way to pass from a gem of the closed 4-manifold $\bar{M}$ to a gem of a compact 4-manifold M so that $\bar{M} \cong M \cup \overset{(\sim)}{\mathbb{Y}}{}^4_m$; to this aim, the notion of ρ-pair[6] turns out to be very useful.

Definition 4.2. A *ρ_h-pair* $(1 \le h \le n)$ of color $i \in \Delta_n$ in an $(n+1)$-colored graph Γ is a pair of i-colored edges (e, f) sharing the same $\{i, c\}$-colored cycle for each $c \in \{c_1, \ldots, c_h\} \subseteq \Delta_n$. Colors $c_1, \ldots, c_h$ are said to be *involved*, while the other $n-h$ colors are said to be *not involved* in the ρ_h-pair.

The *switching* of (e, f) consists in canceling e and f and establishing new i-colored edges between their endpoints; the reversed operation is obviously the switching of a ρ_{n-h}-pair. Although, in general, the switching may be performed in two different ways, it is uniquely determined if $\Gamma \in G_s^{(n)}$, $h \in \{n-1, n\}$ and the bipartition of each non-singular $\hat{c}$-residue is preserved.

The topological effects of the switching of ρ_{n-1}- and ρ_n-pairs have been completely determined in the case of closed n-manifolds: see [1], where it is proved that a ρ_{n-1}-pair (resp. ρ_n-pair) switching does not affect the represented n-manifold (resp. either induces the splitting into two connected summands, or the "loss" of a $\mathbb{S}^{n-1} \otimes \mathbb{S}^1$ summand in the represented n-manifold).

Proposition 4.3. *Let $\bar{M}$ be a closed 4-manifold and Γ a crystallization of $\bar{M}$. If Γ contains m ρ_1-pairs of color i $(i \in \Delta_3)$ and involving color 4, so that the sequence of m switchings yields a 5-colored graph $\Gamma' \in G_s^{(4)}$ admitting a gem-induced trisection $\mathcal{T}(\Gamma', \varepsilon)$ (for a suitable cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 4)$ of Δ_4), then*

$$g_T(\bar{M}) \le \rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}}) + m.$$

Proof. First, let us consider the case $m = 1$: let $\Gamma' \in G_s^{(4)}$ be the 5-colored graph obtained from Γ (representing the closed 4-manifold $\bar{M}$) by switching a ρ_1-pair of color i $(i \in \Delta_3)$ and involving color 4. It is easy to check that Γ' represents a compact 4-manifold M whose boundary is PL-homeomorphic to $\mathbb{S}^2 \otimes \mathbb{S}^1$: in fact, the $\hat{4}$-residue $\Gamma'_{\hat{4}}$ represents the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ (according to Γ' being bipartite or not), since it contains a ρ_3-pair whose switching yields the $\hat{4}$-residue $\Gamma_{\hat{4}}$ representing $\mathbb{S}^3$, while all other $\hat{c}$-residues $\Gamma'_{\hat{c}}$ $(c \in \Delta_3)$ represent $\mathbb{S}^3$, since they give rise to $\Gamma_{\hat{c}}$, representing $\mathbb{S}^3$, by switching a ρ_2-pair. Moreover, $\bar{M}$ is obtained from M by attaching a 3-handle to its boundary, and this 3-handle is orientable or non-orientable according to the orientability of ∂M. In fact, as shown in Fig. 3, the switching of the ρ_3-pair in Γ' can be factorized by inserting a 4-colored edge (whose end-points belong to different bipartition classes in $\Gamma'_{\hat{4}}$ if and only if Γ is bipartite) and subsequently canceling a 3-*dipole*, i.e., a subgraph consisting of two vertices joined by three colored edges, so that the vertices belong to different bicolored cycles involving the remaining colors. It is not difficult to check that the insertion of the 4-colored edge corresponds to "breaking" a

[6] ρ-pairs and their switching were introduced by Lins [22] and subsequently studied in [1,5,14].

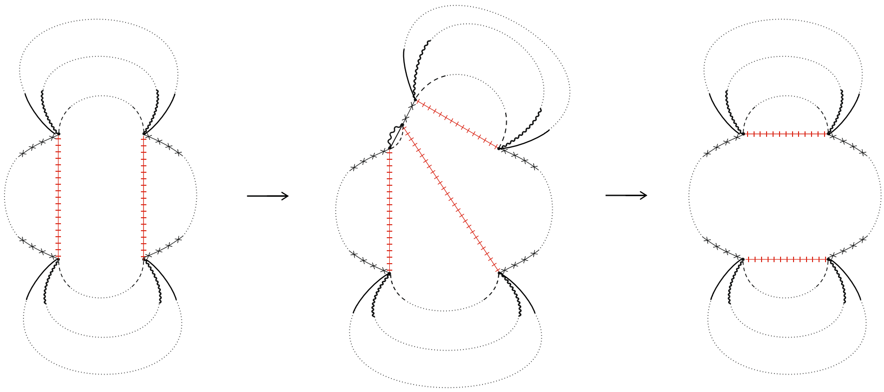

Figure 3. Factorization of a ρ_3-pair switching into an edge insertion and a 3-dipole cancelation

tetrahedral boundary face of M and inserting a new pair of 4-simplices sharing the same 3-dimensional face opposite to the 4-labeled vertex, so that the boundary is transformed into a 3-sphere; hence, this operation may be seen as the attachment of a polyhedron homeomorphic to $\mathbb{D}^3 \times \mathbb{D}^1$ to the boundary of M, without affecting its interior.[7] On the other hand, the elimination of the 3-dipole does not affect the represented manifold, since it corresponds to a re-triangulation of a subcomplex of $|K(\Gamma)|$ homeomorphic to a 4-ball (see [7] and references within).

The case $m > 1$ may be proved by induction via similar arguments, by taking into account that, at the k-th step ($2 \leq k \leq m$), the switching of the considered ρ_3-pair transforms the boundary, represented by the $\hat{4}$-residue, from $\#_k(\mathbb{S}^2 \otimes \mathbb{S}^1)$ into $\#_{k-1}(\mathbb{S}^2 \otimes \mathbb{S}^1)$. Hence, Γ' represents a compact 4-manifold M such that $\bar{M} \cong M \cup \overset{(\sim)}{\mathbb{Y}}{}^4_m$.

Now, if $\Gamma' \in G_s^{(4)}$ admits a gem-induced trisection $\mathcal{T}(\Gamma', \varepsilon)$ (for a suitable cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 4)$ of Δ_4), $g_{GT}(M) \leq \rho_{\varepsilon_{\hat{4}}}(\Gamma'_{\hat{4}})$ holds by Proposition 3.8(i). On the other hand, it is easy to check that, by switching a single ρ_1-pair of color i involving color 4, the number of $\{i, j\}$-cycles, with $j \in \Delta_3$, is decreased by one, while the number of $\{r, s\}$-cycles, with $r, s \in \Delta_3 - \{i\}$, is unaffected; as a consequence, by formula (1) in Section 2, $\rho_{\varepsilon_{\hat{4}}}(\Gamma'_{\hat{4}}) = \rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}}) + m$ holds.

Hence, the thesis follows from Theorem 4.1:

$$g_T(\bar{M}) \leq g_{GT}(M) \leq \rho_{\varepsilon_{\hat{4}}}(\Gamma_{\hat{4}}) + m.$$

□

Example 4.2. $g_T(\mathbb{S}^3 \otimes \mathbb{S}^1) = 1$ may be re-obtained also via Proposition 4.3, starting from the crystallizations of $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ depicted in Fig. 4(a)

[7] An analogous argument has already been used in [6, Proposition 11(ii)] to prove the correspondence between the switching of ρ_3-pairs in a graph representing $[\#_m(\mathbb{S}^2 \times \mathbb{S}^1)] \times I$ and the attachment of 3-handles.

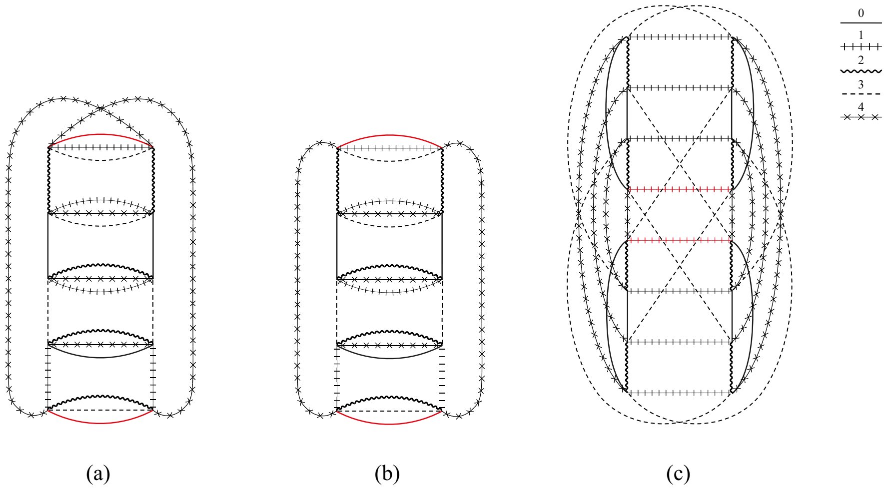

Figure 4. minimal crystallizations of $\mathbb{S}^3 \times \mathbb{S}^1$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ and $\mathbb{RP}^4$, with ρ_1-pairs in red

and (b) respectively: in fact, it is easy to check that each of them admits a ρ_1-pair of color 0 involving color 4, whose switching yields a gem of $\overset{(\sim)}{\mathbb{Y}}{}^4_1$ satisfying the condition of Proposition 3.6. Since both starting crystallizations have $\rho_{\varepsilon_{\hat{i}}}(\Gamma_{\hat{i}}) = 0$ for each $i \in \Delta_4$, $g_T(\mathbb{S}^3 \otimes \mathbb{S}^1) \leq 0 + 1 = 1$ follows from Proposition 4.3. As already pointed out, the equality actually holds, since trisection genus zero characterizes $\mathbb{S}^4$ among all closed 4-manifolds.

Example 4.3. $g_T(\mathbb{RP}^4) = 2$ is realized by a trisection arising from a colored triangulation, too. In fact, it is easy to check that the crystallization of $\mathbb{RP}^4$ depicted in Fig. 4(c) admits a ρ_1-pair of color 1 involving color 4, whose switching yields a gem satisfying the condition of Proposition 3.6. Since the starting crystallization has $\rho_{\varepsilon_{\hat{i}}}(\Gamma_{\hat{i}}) = 1$ for each $i \in \Delta_4$, $g_T(\mathbb{RP}^4) \leq 1 + 1 = 2$ follows from Proposition 4.3; then the above claim is a direct consequence of inequality (2).

Example 4.4. $g_T(\mathbb{S}^2 \times \mathbb{RP}^2) = 3$ is realized by a trisection arising from a colored triangulation, too. In fact, it is easy to check that the (order 24) crystallization of $\mathbb{S}^2 \times \mathbb{RP}^2$ depicted in [3, Figure 3] admits a ρ_1-pair of color 1 and involving color 0; by a suitable permutation of colors, the ρ_1-pair of color 1 involves color 4, and its switching yields a gem satisfying the condition of Proposition 3.6. Since the above crystallization of $\mathbb{S}^2 \times \mathbb{RP}^2$ has $\rho_{\varepsilon_{\hat{i}}}(\Gamma_{\hat{i}}) = 2$ for each $i \in \Delta_4$, $g_T(\mathbb{S}^2 \times \mathbb{RP}^2) \leq 2 + 1 = 3$ follows from Proposition 4.3; then the above claim is a direct consequence of inequality (2).

As already mentioned in the Introduction, the existence of trisections of minimal genus arising from colored triangulations can now be proved for a large class of closed 4-manifolds:

Proposition 4.4. *Let* $\bar{M} \cong_{PL} (\#_p \mathbb{CP}^2) \# (\#_{p'}(-\mathbb{CP}^2)) \# (\#_q(\mathbb{S}^2 \times \mathbb{S}^2)) \# (\#_r K3) \# (\#_s(\mathbb{S}^3 \otimes \mathbb{S}^1)) \# (\#_t \mathbb{RP}^4) \# (\#_u(\mathbb{S}^2 \times \mathbb{RP}^2))$, *with* $p, p', q, r, s, t, u \geq 0$. *Then, its trisection genus* $g_T(\bar{M}) = (p + p' + 2q + 22r) + s + 2t + 3u$ *is realized by a trisection arising from a colored triangulation.*

Proof. For each summand, the existence of a trisection of minimal genus arising from a colored triangulation (or, equivalently, from a suitable gem) is ensured by Examples 4.1 (or 4.2), 4.3 and 4.4, together with results in [27] concerning minimal gem-induced trisections of $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and the K3-surface. Note that all the involved gems represent closed 4-manifolds if and only if $s + t + u = 0$.

Now, the so called *graph connected sum* may be performed on the above gems, yielding a gem of the compact 4-manifold, with empty or connected boundary, obtained by connected sum (or by boundary connected sum, if $s + t + u \geq 2$) of the represented manifolds. Indeed, according to gem theory, given two $(n+1)$-colored graphs Γ_1 and Γ_2, their graph connected sum with respect to vertices v_1 and v_2 (v_i in Γ_i, $\forall i \in \{1, 2\}$) is the graph $\Gamma_1 \#_{v_1,v_2} \Gamma_2$ obtained by deleting v_1 and v_2 and by welding the hanging edges of the same color; if Γ_1 (resp. Γ_2) represents the n-manifold M_1 (resp. M_2), then $\Gamma_1 \#_{v_1,v_2} \Gamma_2$ is known to represent $M_1 \# M_2$ in case at least one of M_1 and M_2 is closed, and $M_1 \natural M_2$ in case both M_1 and M_2 have connected non-empty boundary (see [19, Section 7] for details).

Moreover, it is not difficult to check that, by suitably performing the above sequence of graph connected sums, the obtained 5-colored graph yields a gem-induced trisection with genus equal to the sum of the genera of the trisections (assumed to be of minimal genus) of each summand: see Proposition 3.8(iv) and the proof of [8, Proposition 26(ii)].

The existence of the required trisection of $\bar{M}$ follows by making use of Theorem 4.1, in case $s + t + u > 0$.

On the other hand, the trisection genus of $\bar{M}$ exactly coincides with the sum of the genera, in virtue of inequality (2). □

Let us now restrict the attention to the orientable case, where trisections arising from colored triangulations can be proved to exist for each closed 4-manifold, via Kirby diagrams and associated colored graphs.

In fact, if $\bar{M}$ is a closed orientable 4-manifold, it is well-known that—in virtue of the already cited result by Laudenbach and Poenaru [21]—a Kirby diagram (L, d) of $\bar{M}$ describes only the attachments of the 1-handles (via dotted components) and 2-handles (via framed components) of a handle decomposition of $\bar{M}$. Hence, (L, d) actually coincides with a Kirby diagram of the compact 4-manifold M consisting only of the h-handles of $\bar{M}$, with $h \leq 2$. Obviously, if no 3-handle appears, M can be identified with $\bar{M}$, up to capping off its spherical boundary.

On the other hand, by results in [6] and [8, Section 4], a gem-induced trisection of M can be algorithmically constructed starting from a Kirby diagram. Thus, Theorem 4.1 directly yields the required trisection of $\bar{M}$ arising from a colored triangulation.

Let us specify that, as in [6,8], the considered Kirby diagrams are *connected*, in the sense that their associated planar projections are supposed to be connected; moreover, the (possible) dotted components of (L, d) are assumed to be in *good position*, i.e., they are unknotted, unlinked and with overcrossings and undercrossings never alternating along them. Note that—without loss of generality—any closed orientable 4-manifold can be represented by a Kirby diagram satisfying these conditions (see [18]).

According to Theorem 1.1 stated in the Introduction, we are now able to extend to the whole class of closed orientable 4-manifolds the estimation of the trisection genus via Kirby diagrams that was already obtained in [8] under the assumption of the existence of a handle decomposition lacking in 3-handles.

Proof of Theorem 1.1. Let $\bar{M}$ be a closed orientable 4-manifold represented by a (connected) Kirby diagram (L, d) whose dotted components, if any, are in good position.

Further, let us suppose that the handle decomposition of $\bar{M}$ associated to (L, d) contains $q > 0$ 3-handles (if $q = 0$, both estimations were already proved in [8, Corollary 4]) and let M be the compact 4-manifold consisting only of the h-handles, with $h \leq 2$, of the above handle decomposition.

In the general case when (L, d) contains both dotted and framed components (i.e. the handle decomposition of M contains both 1-handles and 2-handles: case (i) of the statement), [6, Theorem 12] and [8, Theorem 3] allow to construct a gem of M, so that it admits a genus $s + 1$ gem-induced trisection (s being the crossing number of (L, d)). On the other hand, if (L, d) contains no dotted components (i.e. the handle decomposition of M contains only 2-handles: case (ii) of the statement), [6, Theorem 7(ii)] and [8, Theorem 3] yield again a gem of M, so that it admits a genus m_α gem-induced trisection (m_α being the number of α-colored regions in a chess-board coloration of (L, d), where α is the color of the unbounded region).

In both cases, the thesis follows from Theorem 4.1, since $\bar{M} \cong M \cup \mathbb{Y}^4_q$: in case (i) (resp. (ii)), we have

$$g_T(\bar{M}) \leq g_{GT}(M) \leq s + 1 \quad \text{(resp. } g_T(\bar{M}) \leq g_{GT}(M) \leq m_\alpha\text{)}.$$

□

Example 4.5. $g_T(\mathbb{S}^3 \times \mathbb{S}^1) = 1$ may be re-obtained also via Theorem 1.1(i), starting from the Kirby diagram of $\mathbb{S}^3 \times \mathbb{S}^1$ consisting only of one dotted component: in fact, since there are no crossings, $g_T(\mathbb{S}^3 \times \mathbb{S}^1) \leq 0 + 1 = 1$ trivially follows.

Remark 4.5. We point out that an estimation of the trisection genus similar to the one given in Theorem 1.1(i) has been obtained in [20] via trisection diagrams.

Note also that, when there are no dotted components, the inequality in Theorem 1.1(ii) should be preferred to the one in Theorem 1.1(i), since it obviously gives a better estimation. Therefore, for example, if $\bar{M}$ is a closed orientable 4-manifold admitting a handle decomposition with no 3-handles, it could be more convenient to apply statement (ii) to the Kirby diagram representing the dual handle decomposition.

We conclude the paper by pointing out that Theorem 1.1 can be easily generalized to the case of disconnected Kirby diagrams. In fact, it is well-known that, in this case, the represented compact manifold is the boundary connected sum of the compact manifolds represented by each connected component; hence, the estimation of the trisection genus directly follows from multiple application of Theorem 1.1, together with subadditivity (see Proposition 3.8(iv) and the proof of Proposition 4.4):

Corollary 4.6. *Let $\bar{M}$ be a closed orientable 4-manifold and (L, d) a Kirby diagram of $\bar{M}$ with c connected components and whose dotted components—if any—are in good position. Then:*

(i)
$$g_T(\bar{M}) \leq s + c,$$
s being the crossing number of (L, d).

(ii) *Furthermore, if (L, d) has no dotted components, then*
$$g_T(\bar{M}) \leq m_\alpha + c - 1,$$
m_α being the number of α-colored regions in a chess-board coloration of (L, d).

□

Author contributions All authors contributed equally to the work.

Funding Open access funding provided by Università degli Studi di Modena e Reggio Emilia within the CRUI-CARE Agreement.

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Bandieri, P., Gagliardi, C.: Rigid gems in dimension n. Bol. Soc. Mat. Mex. **18**(3), 55–67 (2012)

[2] Bell, M., Hass, J., Rubinstein, J.H., Tillmann, S.: Computing trisections of 4-manifolds. Proc. Nat. Acad. Sci. USA **115**(43), 10901–10907 (2018)

[3] Basak, B., Casali, M.R.: Lower bounds for regular genus and gem-complexity of PL 4-manifolds. Forum Math. **29**(4), 761–773 (2017)

[4] Casali, M.R.: On the regular genus of 5-manifolds with free fundamental group. Forum Math. **15**, 465–475 (2003)

[5] Casali, M.R., Cristofori, P.: Cataloguing PL 4-manifolds by gem-complexity. Electron. J. Combin. **22**(4), P4.25 (2015)

[6] Casali, M.R., Cristofori, P.: Kirby diagrams and 5-colored graphs representing compact 4-manifolds. Rev. Mat. Complut. **36**, 899–931 (2023)

[7] Casali, M.R., Cristofori, P.: Classifying compact 4-manifolds via generalized regular genus and G-degree. Ann. Inst. Henri Poincaré Comb. Phys. Interact. **10**(1), 121–158 (2023)

[8] Casali, M.R., Cristofori, P.: Gem-induced trisections of compact PL 4-manifolds. Forum Math. **36**(1), 87–109 (2024)

[9] Casali, M.R., Cristofori, P., Gagliardi, C.: Crystallizations of compact 4-manifolds minimizing combinatorially defined PL-invariants. Rend. Istit. Mat. Univ. Trieste **52**, 431–458 (2020)

[10] Casali, M.R., Cristofori, P., Grasselli, L.: G-degree for singular manifolds. RACSAM **112**(3), 693–704 (2018)

[11] Casali, M.R., Cristofori, P., Dartois, S., Grasselli, L.: Topology in colored tensor models via crystallization theory. J. Geom. Phys. **129**, 142–167 (2018)

[12] Casali, M.R., Gagliardi, C.: Classifying PL 5-manifolds up to regular genus seven. Proc. Am. Math. Soc. **120**, 275–283 (1994)

[13] Castro, N.A., Gay, D.T., Pinzon-Caicedo, J.: Trisections of 4-manifolds with boundary. Proc. Nat. Acad. Sci. USA **115**(43), 10861–10868 (2018)

[14] Cristofori, P., Fominykh, E., Mulazzani, M., Tarkaev, V.: 4-colored graphs and knot/link complements. Results Math. **72**, 471–490 (2017)

[15] Ferri, M., Gagliardi, C.: The only genus zero n-manifold is $\mathbb{S}^n$. Proc. Am. Math. Soc. **85**, 638–642 (1982)

[16] Gagliardi, C.: Extending the concept of genus to dimension n. Proc. Am. Math. Soc. **81**, 473–481 (1981)

[17] Gay, D., Kirby, R.: Trisecting 4-manifolds. Geom. Topol. **20**, 3097–3132 (2016)

[18] Gompf, R.E., Stipsicz, A.I.: 4-Manifolds and Kirby Calculus, vol. 20. American Mathematical Society, Providence (1999)

[19] Grasselli, L., Mulazzani, M.: Compact n-manifolds via $(n+1)$-colored graphs: a new approach. Algebra Colloq. **27**(1), 95–120 (2020)

[20] Kepplinger, W.: An algorithm taking Kirby diagrams to trisection diagrams. Pac. J. Math. **318**(1), 109–126 (2022)

[21] Laudenbach, F., Poenaru, V.: A note on 4-dimensional handlebodies. Bull. Soc. Math. Fr. **100**, 337–344 (1972)

[22] Lins, S.: Gems, Computers and Attractors for 3-Manifolds. Knots and Everything, vol. 5. World Scientific, River Edge (1995)

[23] Martini, R., Toriumi, R.: Trisections in colored tensor models. Ann. Inst. Henri Poincaré Comb. Phys. Interact. **11**(3), 453–502 (2024)

[24] Miller, M., Naylor, P.: Trisections of non-orientable 4-manifolds. Mich. Math. J. **74**(2), 403–447 (2024)

[25] Pezzana, M.: Sulla struttura topologica delle varietà compatte. Atti Semin. Mat. Fis. Univ. Modena **23**, 269–277 (1974)

[26] Rubinstein, J.H., Tillmann, S.: Multisections of piecewise linear manifolds. Indiana Univ. Math. J. **69**(6), 2209–2239 (2020)

[27] Spreer, J., Tillmann, S.: Determining the trisection genus of orientable and non-orientable PL 4-manifolds through triangulations. Exp. Math. **31**(3), 897–907 (2022)

Maria Rita Casali and Paola Cristofori
Department of Physics, Informations and Mathematics
University of Modena and Reggio Emilia
Via Campi 213 B
41125 Modena
Italy
e-mail: paola.cristofori@unimore.it

Maria Rita Casali
e-mail: casali@unimore.it

Received: August 30, 2024.
Revised: November 29, 2024.
Accepted: December 15, 2024.