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# Joint Complete Records occurrence over FGM sequences

Amir Khorrami Chokami[1] and Silvia Montagna[2]

[1] ESOMAS Department, Università di Torino and Collegio Carlo Alberto, Torino, Italy
amir.khorramichokami@unito.it,
[2] Dipartimento di Economia "Marco Biagi", Università degli Studi di Modena e Reggio Emilia, via Berengario 51, Modena 41121, Italy

**Abstract.** A record over a sequence of iid absolutely continuous random variables is defined as a random variable surpassing all preceding ones. The extension to the multivariate case is not obvious. In this project, we focus on the so-called complete records. We examine the probability of appearance of two complete records at two different times, whenever the within sequences of random vectors where the dependence among components follows the Farlie-Gumbe1-Morgenstern copula.

**Keywords:** complete records, time dependence, standard max stable distribution, FGM copula.

## 1 Introduction

In recent times, there has been a growing interest on the occurrence of extreme events. The study of maxima in a multivariate context presents the challenge of tackling the dependence among extreme events, in connection with the parent copula of the underlying sequence of random vectors (see [9], [8] among others). Let us consider now a sequence of iid random variables $X_1, X_2, \ldots$. The rv $X_m$ is called a *record* if $X_m > \max(X_1, \ldots, X_{m-1})$, $m \geq 2$ (obviously, $X_1$ is the first record because it is the first vriable appearing in the sequence). In this simple case, various results concerning the time of appearance of a record, the asymptotic distribution of the $m$-th record as $m \to \infty$, increments between records, are available (see [1], [4, Sections 6.2 and 6.3], [2] and [3], among others). In particular, if we call $I_m^{\mathrm{R}} = \mathbb{1}\left(X_m > \max(X_1, \ldots, X_{m-1})\right)$, $m \in \mathbb{N}$, then

$$\mathsf{P}\big(I_j^{\mathrm{R}} = 1, I_k^{\mathrm{R}} = 1\big) = \mathsf{P}\big(I_j^{\mathrm{R}} = 1\big)\,\mathsf{P}\big(I_k^{\mathrm{R}} = 1\big) = j^{-1}k^{-1}, \quad j < k, \tag{1}$$

indicating independence between the appearance of records at two different times.

Th extension to the multivariate case is not an easy task. The first issue we have to face, regards the non-unique definition of a multivariate record. In this work, we consider the so-called complete records, defined as follows. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots,$ be iid random vectors (rv) in $\mathbb{R}^d$. Put for integers $j < k$

$$\boldsymbol{M}_j := \max_{1 \leq i \leq j} \boldsymbol{X}_i, \quad \boldsymbol{M}_j^k := \max_{j \leq i \leq k} \boldsymbol{X}_i,$$

where the maximum is taken componentwise. a rv $\boldsymbol{X}_m$ is a *complete record* if each component is a univariate record itself, i.e., $\boldsymbol{X}_m > \boldsymbol{M}_{m-1}$. The event indicating the appearance of a complete record at time $m$ is then $I_m^{\mathrm{CR}} := \mathbb{1}(\boldsymbol{X}_m > \boldsymbol{M}_{m-1})$. Some findings regarding complete records across sequences of random vectors with independent components and with Marshall-Olkin copula are available in [2] and [7], respectively.

This paper studies the left-hand side of Equation (1) in a bivariate context, precisely when the copula of the underlying sequence of random vectors is of the Farlie-Gumbe1-Morgenstern (FGM) type:

$$F(x,y) = \mathrm{e}^{(x+y)} \left(1 + \lambda \left(1 - \mathrm{e}^{x}\right)\left(1 - \mathrm{e}^{y}\right)\right), \quad x, y \leq 0 \text{ and } |\lambda| \leq 1. \tag{2}$$

with bivariate density given by

$$f(x,y) = \mathrm{e}^{x+y} \left(1 + \lambda \left(2\mathrm{e}^{x} - 1\right)\left(2\mathrm{e}^{y} - 1\right)\right). \tag{3}$$

The challenge in studying records arises from the immediate complexity of computations required when transitioning from independent cases to those where components are dependent. The FGM copula, albeit with difficulties, enables the derivation of exact formulas for complete record occurrences. This copula is valuable due to its simple structure and straightforward interpretation of the parameter $\lambda$, which captures dependence characteristics. These qualities have made the FGM distribution a popular choice for various applications, including capital allocation and the so-called collective risk models. For a thorough understanding of this copula in an extreme-value context, refer to [5]. We highlight our findings are independent of the marginal distribution function $F_{X_i}$, $i = 1, 2$, provided that they are continuous. Moreover, we consider a sequence $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots \in \mathbb{R}^2$ of *standard max-stable* rv (*i.e.* with Negative-Exp(1) margins and such that $\boldsymbol{M}_n \overset{d}{=} \boldsymbol{\eta}_1/n$) and FGM copula. This is also the framework of [6], which deals with single complete records.

## 2 Complete Records Occurrences

**Theorem 1.** *Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots$ be a sequence of bivariate standard max-stable rv with FGM copula. Then, the probability of appearance of a CR is*

$$\mathsf{P}\left(I_j^R = 1, I_k^R = 1\right) = \sum_{i=1}^{16} p_i \tag{4}$$

*for $j, k \in \mathbb{N}, j < k$. The $p_i$'s are given in the proof for readability.*

*Proof.* For what concerns the probability of the joint events we have

$$\begin{aligned}
&\mathsf{P}\big(I_j^{\mathrm{CR}}=1, I_k^{\mathrm{CR}}=1\big) = \mathsf{P}(\boldsymbol{\eta}_j > \boldsymbol{M}_{j-1}, \boldsymbol{\eta}_k > \boldsymbol{M}_{k-1}) \\
&= \mathsf{P}\left(\boldsymbol{\eta}_j > \frac{\boldsymbol{\eta}_1}{j-1}, \boldsymbol{\eta}_k > \max\left(\boldsymbol{\eta}_j, \frac{\boldsymbol{\eta}_2}{k-j-1}\right)\right) \\
&= \int_{(-\infty,0]^2} \int_{(-\infty,\boldsymbol{y}]} \mathsf{P}(\boldsymbol{\eta}_1 < (j-1)\boldsymbol{x}, \boldsymbol{\eta}_2 < (k-j-1)\boldsymbol{y})\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{x})\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{y}) \\
&= \int_{(-\infty,0]^2} \mathsf{P}(\boldsymbol{\eta}_2 < (k-j-1)\boldsymbol{y}) \int_{(-\infty,\boldsymbol{y}]} \mathsf{P}(\boldsymbol{\eta}_1 < (j-1)\boldsymbol{x})\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{x})\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{y}) \,.
\end{aligned}$$

Let us compute the internal integral, which will yield a function of $\boldsymbol{y}$.

$$\begin{aligned}
&\int_{(-\infty,\boldsymbol{y}]} \mathsf{P}(\boldsymbol{\eta}_1 < (j-1)\boldsymbol{x})\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{x}) = \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathsf{P}\Big(\eta_1^{(1)} < (j-1)x_1, \eta_1^{(2)} < (j-1)x_2\Big)\, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{x}) = \\
&= \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathrm{e}^{(j-1)(x_1+x_2)} \left(1+\lambda\left(1-\mathrm{e}^{(j-1)x_1}\right)\left(1-\mathrm{e}^{(j-1)x_2}\right)\right)\cdot \\
&\qquad\qquad \cdot\, \mathrm{e}^{x_1+x_2}\left(1+\lambda\left(2\mathrm{e}^{x_1}-1\right)\left(2\mathrm{e}^{x_2}-1\right)\right)\, \mathrm{d}x_1\, \mathrm{d}x_2 = \\
&= \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathrm{e}^{j(x_1+x_2)}\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&\quad + \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathrm{e}^{j(x_1+x_2)}\lambda\left(1-\mathrm{e}^{(j-1)x_1}\right)\left(1-\mathrm{e}^{(j-1)x_2}\right)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&\quad + \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathrm{e}^{j(x_1+x_2)}\lambda\left(2\mathrm{e}^{x_1}-1\right)\left(2\mathrm{e}^{x_2}-1\right)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&\quad + \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} \mathrm{e}^{j(x_1+x_2)}\lambda^2\left(1-\mathrm{e}^{(j-1)x_1}\right)\left(1-\mathrm{e}^{(j-1)x_2}\right)\left(2\mathrm{e}^{x_1}-1\right)\left(2\mathrm{e}^{x_2}-1\right)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
I_1 &= \frac{1}{j^2}\mathrm{e}^{j(y_1+y_2)} \\
I_2 &= \lambda\prod_{i=1}^{2}\left(\frac{\mathrm{e}^{jy_i}}{j} - \frac{\mathrm{e}^{(2j-1)y_i}}{2j-1}\right) \\
I_3 &= \lambda\prod_{i=1}^{2}\left(2\frac{\mathrm{e}^{(j+1)y_i}}{j+1} - \frac{\mathrm{e}^{jy_i}}{j}\right) \\
I_4 &= \lambda^2\prod_{i=1}^{2}\left(\frac{2}{j+1}\mathrm{e}^{(j+1)y_i} - \frac{\mathrm{e}^{jy_i}}{j} - \frac{\mathrm{e}^{2jy_i}}{j} + \frac{\mathrm{e}^{(2j-1)y_i}}{2j-1}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{(-\infty,0]^2} \mathsf{P}(\boldsymbol{\eta}_2 < (k-j-1)\boldsymbol{y}) \int_{(-\infty,\boldsymbol{y}]} \mathsf{P}(\boldsymbol{\eta}_1 < (j-1)\boldsymbol{x}) \, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{x}) \, \mathrm{d}\mathsf{P}_{\boldsymbol{\eta}}(\boldsymbol{y}) \\
&= \int_{-\infty}^{0} \int_{-\infty}^{0} \left[ \sum_{i=1}^{4} I_i \right] \cdot F\left((k-j-1)y_1, (k-j-1)y_2\right) f\left(y_1, y_2\right) \, \mathrm{d}y_1 \, \mathrm{d}y_2 \\
&== \sum_{i=1}^{4} \int_{-\infty}^{0} \int_{-\infty}^{0} I_i \cdot F\left((k-j-1)y_1, (k-j-1)y_2\right) f\left(y_1, y_2\right) \, \mathrm{d}y_1 \, \mathrm{d}y_2,
\end{aligned}$$

where $F$ and $f$ are as in Equation (2) and Equation (3), respectively. Solving the bivariate integral above, requires the computation of the following 16 integrals, each one relative to a different $p_i$ in Equation (4). This is because $F\left((k-j-1)y_1, (k-j-1)y_2\right) f\left(y_1, y_2\right)$ is a sum of four different contributes, each of them being multiplied with each one of the 4 integrals.

**For $F \cdot f \cdot I_1$:** The corresponding double integral is the sum of the following

$$\begin{aligned}
p_1 &= \frac{1}{j^2 k^2} \\
p_2 &= \frac{\lambda}{j^2} \left( \frac{1}{k} - \frac{1}{2k-j-1} \right)^2 \\
p_3 &= \frac{\lambda}{j^2} \left( \frac{2}{k+1} - \frac{1}{k} \right)^2 \\
p_4 &= \frac{\lambda^2}{j^2} \left( \frac{2}{k+1} - \frac{1}{k} - \frac{2}{2k-j} + \frac{1}{2k-j-1} \right)^2
\end{aligned}$$

**For $F \cdot f \cdot I_2$:** The corresponding double integral is the sum of the following

$$\begin{aligned}
p_5 &= \lambda \left( \frac{1}{jk} - \frac{1}{(2j-1)(k+j-1)} \right)^2 \\
p_6 &= \lambda \left( \frac{1}{jk} - \frac{1}{(2j-1)(k+j-1)} - \frac{1}{j(2k-j-1)} - \frac{1}{(2j-1)(2k-3)} \right)^2 \\
p_7 &= \lambda \left( \frac{2}{j(k+1)} - \frac{2}{(2j-1)(k+j)} - \frac{1}{jk} - \frac{1}{(2j-1)(k+j-1)} \right)^2 \\
p_8 &= \lambda^2 \left( \frac{2}{j(k+1)} - \frac{2}{(2j-1)(k+j)} - \frac{1}{jk} + \frac{1}{(2j-1)(k+j-1)} - \frac{2}{j(2k-j)} + \frac{2}{(2j-1)(2k-1)} \right. \\
&\quad \left. + \frac{1}{j(2k-j-1)} - \frac{2}{(2j-1)(2k-3)} \right)^2
\end{aligned}$$

**For** $F \cdot f \cdot I_3$**:** The corresponding double integral is the sum of the following

$$p_9 = \lambda\Big(\frac{2}{(j+1)(k+1)} - \frac{1}{jk}\Big)^2$$

$$p_{10} = \lambda\Big(\frac{2}{(j+1)(k+1)} - \frac{1}{jk} - \frac{2}{(j+1)(2k-j)} + \frac{1}{j(2k-j-1)}\Big)^2$$

$$p_{11} = \lambda\Big(\frac{4}{(j+1)(k+2)} - \frac{2}{j(k+1)} - \frac{2}{(j+1)(k+1)} + \frac{1}{jk}\Big)^2$$

$$p_{12} = \lambda^2\Big(\frac{4}{(j+1)(k+2)} - \frac{2}{j(k+1)} - \frac{2}{(j+1)(k+1)} + \frac{1}{jk} - \frac{k}{(j+1)(2k-j+1)} + \frac{2}{j(2k-j)} + \frac{2}{(j+1)(2k-j)} - \frac{1}{j(2k-j-1)}\Big)^2$$

**For** $F \cdot f \cdot I_4$**:** The corresponding double integral is the sum of the following

$$p_{13} = \lambda^2\Big(\frac{2}{(j+1)(k+1)} - \frac{1}{jk} - \frac{1}{j(k+j)} + \frac{1}{(2j-1)(k+j-1)}\Big)^2$$

$$p_{14} = \lambda\Big(\frac{2}{(j+1)(k+1)} - \frac{1}{jk} - \frac{1}{j(k+j)} + \frac{1}{(2j-1)(k+j-1)} - \frac{2}{(j+1)(2k-j)} + \frac{1}{j(2k-j-1)} + \frac{1}{j(2k-1)} - \frac{1}{(2j-1)(2k-2)}\Big)^2$$

$$p_{15} = \lambda^2\Big(\frac{4}{(j+1)(k+1)} - \frac{2}{j(k+1)} - \frac{2}{j(k+j+1)} + \frac{2}{(2j-1)(k+j)} - \frac{2}{(j+1)(k+1)} + \frac{1}{jk} + \frac{1}{j(k+j)} - \frac{1}{(2j-1)(k+j-1)}\Big)^2$$

$$p_{16} = \lambda^2\Big(\frac{4}{(j+1)(k+2)} - \frac{2}{j(k+1)} - \frac{2}{j(k+j+1)} + \frac{2}{(2j-1)(k+j)} - \frac{2}{(j+1)(k+1)} + \frac{1}{jk} + \frac{1}{j(k+j)} - \frac{1}{(2j-1)(k+j-1)} + - \frac{4}{(j+1)(k+1)} + \frac{2}{j(2k-j)} + \frac{1}{jk} - \frac{2}{(2j-1)(2k-1)} + + \frac{2}{(j+1)(2k-j)} - \frac{1}{j(2k-j-1)} - \frac{1}{j(2k-1)} + \frac{1}{(2j-1)(2k-2)}\Big)^2.$$

This concludes the proof.

## 3 Conclusion

In conclusion, this work addresses the challenge of studying the probability of occurrence of two complete records over independent and identically distributed bivariate sequences of standard max-stable random variables with the FGM copula. The authors suggest te use of mathematical softwares to simply the expressions given in the proof of Theorem 1. Ongoing projects regard the extension the case of complete records to sequences characterized by a general copula, and the study of alterntive definitions of multivariate records.

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