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REGULARITY OF MEAN CURVATURE FLOW OF GRAPHS ON LIE GROUPS FREE UP TO STEP 2

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ABSTRACT. We consider (smooth) solutions of the mean curvature flow of graphs over bounded domains in a Lie group free up to step two (and not necessarily nilpotent), endowed with a one parameter family of Riemannian metrics σ_ϵ collapsing to a subRiemannian metric σ_0 as $\epsilon \rightarrow 0$. We establish $C^{k,\alpha}$ estimates for this flow, that are uniform as $\epsilon \rightarrow 0$ and as a consequence prove long time existence for the subRiemannian mean curvature flow of the graph. Our proof extend to the setting of every step two Carnot group (not necessarily free) and can be adapted following our previous work in [10] to the total variation flow.

1. INTRODUCTION

The mean curvature flow is the motion of a surface where each point is moving in the direction of the normal with speed equal to the mean curvature. In the case where the evolution of graphs $S_t = \{(x, u(x, t))\} \subset \mathbb{R}^n \times \mathbb{R}$ is considered, then, provided enough regularity is assumed, the function u satisfies the equation

$$\partial_t u = \sqrt{1 + |\nabla u|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Given appropriate boundary/initial conditions, global in time solutions asymptotically converge to minimal graphs.

In this paper we study long time existence of graph solutions of the mean curvature flow in a special class of degenerate Riemannian ambient spaces: The setting of sub-Riemannian manifolds [22], [40]. In particular we will focus on a class of Lie groups endowed with a metric structure (G, σ_0) that arises as limit of collapsing left-invariant Riemannian structures (G, σ_ϵ) .

Our approach to the existence of global (in time) smooth solutions is based on a Riemannian approximation scheme. We study graph solutions of the mean curvature flow in the Riemannian spaces (G, σ_ϵ) where G is a group and σ_ϵ is a family of Riemannian metrics that 'collapse' as $\epsilon \rightarrow 0$ to a sub-Riemannian metric σ_0 in G .

Key words and phrases. mean curvature flow, sub-Riemannian geometry, Carnot groups
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Our results are analogue to those we proved for the total variation flow proved in [10]. The main difference is that we remove here the assumption that the group G is a Carnot group, i.e. we also consider non-nilpotent groups such as the group of rigid Euclidean motions \mathcal{RT} . In fact the results in the present paper yield, with minor modifications of the proof, the regularity and long time existence of the total variation flow in the same extended class of groups. The main new technical challenge that distinguishes the study of the total variation flow from the mean curvature flow is that the equation studied here is not in divergence form, which makes it necessary to completely change the proof of the $C^{1,\alpha}$ regularity.

1.1. Lie group structure. Let G be an analytic and simply connected Lie group with topological dimension n .

A subRiemannian manifold on G is a triplet $(G; \Delta; \sigma_0)$ where Δ denotes a left invariant bracket-generating subbundle of TM , and σ_0 is a positive definite smooth, bilinear form on Δ , (see for instance Montgomery [32]). We fix a orthonormal horizontal basis X_1, \dots, X_m of Δ . We will say that the group has step 2 if $\{X_i\}_{i=1, \dots, m} \cup \{[X_i, X_j]\}_{i, j=1, \dots, m}$ span the whole tangent space at every point. If in addition the vector fields

$$\{X_i\}_{i=1, \dots, m} \text{ and } \{[X_i, X_j]\}_{i, j=1, \dots, m}$$

are linearly independent we say that the group is free up to step 2. We can complete X_1, \dots, X_m to a basis (X_1, \dots, X_n) of the Lie algebra \mathcal{G} , choosing a basis of the second layer of the tangent space. We denote by (X_1, \dots, X_n) (resp. (X_1^r, \dots, X_n^r)) the left invariant (resp. right invariant) translations of the frames (X_1, \dots, X_n) . We will say that the vector fields X_1, \dots, X_m have degree 1 and denote $d(X_i) = 1$ while their commutators of the form $[X_i, X_j]$ have degree 2. Throughout the paper we will assume for simplicity that the horizontal frame above is self-adjoint, the results continue to hold with minimal modifications of the proofs if this hypothesis is not satisfied.

As prototypes for this class of spaces we highlight the following:

- The standard example for such families is the Heisenberg group \mathbb{H}^1 . This is a Lie group whose underlying manifold is \mathbb{R}^3 and is endowed with a group law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - (x_2 y_1 - x_1 y_2)).$$

With respect to such law one has that the vector fields

$$X_1 = \partial_{x_1} - x_2 \partial_{x_3} \text{ and } X_2 = \partial_{x_2} + x_1 \partial_{x_3}$$

are left-invariant. Together with their commutator $[X_1, X_2] = 2\partial_{x_3}$ they yield a basis of \mathbb{R}^3 .

- A second example is given by the classical group of rigid motions of the plane, also known as the *roto-translation* group \mathcal{RT} . This is a Lie group with underlying manifold $\mathbb{R}^2 \times S^1$ and a group law $(x_1, x_2, \theta_1)(y_1, y_2, \theta_2) = (x_1 + y_1 \cos \theta - y_2 \sin \theta, x_2 + y_1 \sin \theta + y_2 \cos \theta, \theta_1 + \theta_2)$. The horizontal distribution is given by

$$\Delta = \text{span}\{X_1, X_2\}, \text{ with } X_1 = \cos \theta \partial_{x_1} + \sin \theta \partial_{x_2}, \text{ and } X_2 = \partial_{\theta}.$$

The subRiemannian metric σ_0 is defined so that X_1 and X_2 form an orthonormal basis. Note that X_1, X_2 and $[X_1, X_2] = X_3 = -\sin\theta\partial_{x_1} + \cos\theta\partial_{x_2}$ span the tangent space at every point.

The assumption that Δ is bracket generating, allows to use the results from [35], and define a control distance $d_0(x, y)$ associated to the distribution X_1, \dots, X_m , which is called *the Carnot-Carathéodory metric* (denote by $d_{r,0}$ the corresponding right invariant distance). We let $\nabla_0 = (X_1, \dots, X_m)$ denote the *horizontal gradient* operator. If $\phi \in C^\infty(G)$ we set $\nabla_0\phi = \sum_{i=1}^m X_i\phi X_i$ and $|\nabla_0\phi|^2 = \sum_{i=1}^m (X_i\phi)^2$.

We define a family of left invariant Riemannian metrics $\sigma_\epsilon, \epsilon > 0$ in \mathcal{G} by requesting that

$$\{X_1^\epsilon, \dots, X_n^\epsilon\} := \{X_1, \dots, X_m, \epsilon X_{m+1}, \dots, \epsilon X_n\}$$

is an orthonormal frame. We will denote by d_ϵ the corresponding distance functions. Correspondingly we use ∇_ϵ , (resp. ∇_ϵ^r) to denote the left (resp. right) invariant gradients. In particular, if $\phi \in C^\infty(G)$ we set $\nabla_\epsilon\phi = \sum_{i=1}^n X_i^\epsilon\phi X_i^\epsilon$ and $|\nabla_\epsilon\phi|^2 = \sum_{i=1}^n (X_i^\epsilon\phi)^2$.

Our results rest in a crucial way on the use of the celebrated Rothschild and Stein approximation theorem [38]. Let X_1, \dots, X_m be a bracket generating family of smooth vector fields, free up to step 2, denote by X_1, \dots, X_n its completion to a basis of the tangent bundle TG and set for every $u \in \mathbb{R}^n$,

$$(1.1) \quad \Phi_x(u) = \exp\left(\sum_{i=1}^n u_i X_i\right)(x).$$

We will use the following special case of the Rothschild-Stein osculating theorem [38, Theorem 5], to approximate a neighborhood of any point in G with a neighborhood of the identity in the free group $G_{m,2}$ with m generators Y_1, \dots, Y_m and step 2.

Theorem 1.1. *Let X_1, \dots, X_m be a family of smooth vector fields in G that are free up to rank 2 at every point, as defined above. Let $G_{m,2}$ be the free Lie group of step two, with m generators Y_1, \dots, Y_m and set Y_1, \dots, Y_n to be the basis obtained by the original generators and their commutators. For every $x \in G$ there exists a neighborhood V of x and a neighborhood U of the identity in $G_{m,2}$ such that:*

- (a) *the map $\Phi_x : U \rightarrow V$ is a diffeomorphism onto its image. We will denote by Θ_x its inverse map. Here we have denoted points in U by their coordinates (u_1, \dots, u_n) in the basis Y_1, \dots, Y_n .*
- (b) *we have*

$$(1.2) \quad d\Theta_x(X_i) = Y_i + R_i, \quad i = 1, \dots, m$$

where R_i is a vector field of local degree less or equal than zero, depending smoothly on x . This means that the operator R_i is represented as

$$R_i = \sum_{h=1}^n \sigma_{ih}(u) X_h,$$

where each σ_{ih} has a Taylor expansion of homogeneous functions of degree larger or equal than $d(X_h)$.

1.2. The mean curvature flow. If a surface M is represented as a 0-level set of a function f , the points where the horizontal gradient of the defining function does not vanish are called non characteristic. If we consider graphs, all points are non characteristics. At these points several equivalent definitions of the horizontal mean curvature h_0 have been proposed. To quote a few: h_0 can be defined in terms of the first variation of the area functional [16, 15, 19, 26, 37, 13], as horizontal divergence of the horizontal unit normal or as limit of the mean curvatures h_ϵ of suitable Riemannian approximating metrics σ_ϵ [13]. If the surface is not regular, the notion of curvature can be expressed in the viscosity sense (we refer to [3], [4], [42], [41], [29], [2], [31], [6] for viscosity solutions of PDE in the sub-Riemannian setting).

The mean curvature flow of a graph in $G \times \mathbb{R}$ is characterized by the fact that each point of the evolving graph moves in the direction of the *upward* unit normal with speed equal to the mean curvature. In the setting of the approximating Riemannian metrics (G, σ_ϵ) this flow is smooth and in terms of the functions $t \rightarrow u_\epsilon(\cdot, t) : \Omega \subset G \rightarrow \mathbb{R}$ describing the evolving graphs, the relevant equation reads:

$$(1.3) \quad \frac{\partial u_\epsilon}{\partial t} = W_\epsilon h_\epsilon = W_\epsilon \sum_{i=1}^n X_i^\epsilon \left(\frac{X_i^\epsilon u_\epsilon}{W_\epsilon} \right) = \sum_{i,j=1}^n a_{ij}^\epsilon (\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon u_\epsilon \quad \text{for } x \in \Omega, t > 0,$$

where, h_ϵ is the mean curvature of the graph of $u_\epsilon(\cdot, t)$ and

$$(1.4) \quad W_\epsilon^2 = 1 + |\nabla_\epsilon u_\epsilon|^2 = 1 + \sum_{i=1}^n (X_i^\epsilon u_\epsilon)^2 \quad \text{and} \quad a_{ij}^\epsilon(p) = \delta_{ij} - \frac{p_i p_j}{1 + |p|^2},$$

for all $p \in \mathbb{R}^n$. In the sub-Riemannian limit $\epsilon = 0$ the equation reads

$$(1.5) \quad \frac{\partial u}{\partial t} = \sqrt{1 + |\nabla_0 u|^2} \sum_{i=1}^m X_i \left(\frac{X_i u}{\sqrt{1 + |\nabla_0 u|^2}} \right),$$

with $u := \lim_{\epsilon \rightarrow 0} u_\epsilon$, for $x \in \Omega$ and $t > 0$. Mean curvature flow in the setting of Carnot group has been studied by [6] and [12]. See also the recent [20] as well as [21] for a probabilistic interpretation of the flow.

The aim of this paper is to establish uniform (in the parameter ϵ as $\epsilon \rightarrow 0$) estimates and determine the asymptotic behavior of solutions to the initial value problem for the mean curvature motion of graphs over bounded domains of a group G ,

$$(1.6) \quad \begin{cases} \partial_t u_\epsilon = h_\epsilon W_\epsilon & \text{in } Q = \Omega \times (0, T) \\ u_\epsilon = \varphi & \text{on } \partial_p Q. \end{cases}$$

Here $\partial_p Q = (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0, T))$ denotes the parabolic boundary of Q .

The classical parabolic theory yields local existence and uniqueness for smooth solutions u_ϵ of (1.6) of the problem, under suitable assumptions on the boundary data. Our main goal

consists in proving that estimates are stable as ϵ tends to 0, thus providing estimates also for the solution of the limit problem.

Our first result consists in showing that if the initial/boundary data is sufficiently smooth then the solutions of (1.6) are Lipschitz up to the boundary uniformly in $\epsilon > 0$.

Theorem 1.2. (*Global gradient bounds*) *Let G be a Lie group of step two, $\Omega \subset G$ a bounded, open, convex set (in the sense of definition 3.3 below) and $\varphi \in C^2(\bar{\Omega})$. For $1 \geq \epsilon > 0$ denote by $u_\epsilon \in C^2(\Omega \times (0, T)) \cap L^\infty((0, \infty), C^1(\bar{\Omega}))$ the non-negative unique solution of the initial value problem (1.6). There exists $C = C(G, \|\varphi\|_{C^2(\bar{\Omega})}) > 0$ such that*

$$(1.7) \quad \sup_{\bar{\Omega} \times (0, T)} |\nabla_1 u_\epsilon| \leq C.$$

In particular, since $\sup_{\bar{\Omega} \times (0, T)} |\nabla_\epsilon u_\epsilon| \leq \sup_{\bar{\Omega} \times (0, T)} |\nabla_1 u_\epsilon|$ one has uniform Lipschitz bounds for u_ϵ .

Having established Lipschitz bounds, the next step is to recognize that right derivatives $X_i^r u_\epsilon$ of the solutions of (1.6) are solutions of a divergence form, degenerate parabolic PDE. We prove that weak solutions of such PDE satisfy a Harnack inequality and consequently obtain $C^{1,\alpha}$ interior estimates for the original solution u_ϵ , which are uniform in $\epsilon > 0$.

At this point one rewrites the PDE in (1.6) in non-divergence form and invokes the *stable* Schauder estimates for subriemannian equations (see [10], [7]) to prove local higher regularity and long time existence.

Theorem 1.3. *In the hypothesis of Theorem 1.2 there exists a unique solution $u_\epsilon \in C^\infty(\Omega \times (0, \infty)) \cap L^\infty((0, \infty), C^1(\bar{\Omega}))$ of the initial value problem*

$$(1.8) \quad \begin{cases} \partial_t u_\epsilon = W_\epsilon h_\epsilon & \text{in } Q = \Omega \times (0, \infty) \\ u_\epsilon = \varphi & \text{on } \partial_p Q \end{cases}$$

and that for each $K \subset\subset \Omega$ and for each $h \in \mathbb{N}$ there exists $C_h = C_h(G, \varphi, h, K, \Omega) > 0$ not depending on ϵ such that

$$(1.9) \quad \|u_\epsilon\|_{C^h(K \times [0, +\infty])} \leq C_h.$$

Since the estimates are uniform in ϵ and in time, and with respect to ϵ , we will deduce the following corollary:

Corollary 1.4. *Under the assumptions of the Theorem 1.2, as $\epsilon \rightarrow 0$ the solutions u_ϵ converge uniformly (with all its derivatives) on compact subsets of Q to the unique, smooth solution $u_0 \in C^\infty(\Omega \times (0, \infty)) \cap L^\infty((0, \infty), C^1(\bar{\Omega}))$ of the sub-Riemannian mean curvature flow (1.5) in $\Omega \times (0, \infty)$ with initial data φ .*

Corollary 1.5. *Under the assumptions of Theorem 1.2, as $T \rightarrow \infty$ the solutions $u_\epsilon(\cdot, t)$ converge uniformly on compact subsets of Ω to the unique solution of the minimal surface equation*

$$h_\epsilon = 0 \quad \text{in } \Omega$$

with boundary value φ , while $u_0 = \lim_{\epsilon \rightarrow 0} u_\epsilon \in C^\infty(\Omega) \cap Lip(\bar{\Omega})$ is the unique solution of the sub-Riemannian minimal surfaces equation $h_0 = 0$ in Ω , with boundary data φ .

Regularity of minimal surfaces in the special case of Heisenberg group has been investigated in [23, 36, 16, 15, 8, 9, 18, 33, 37, 39].

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2. STRUCTURE STABILITY IN THE RIEMANNIAN LIMIT

If $x \in G$ and $r > 0$, we will denote by

$$B(x, r) = \{y \in G \mid d_0(x, y) < r\}$$

the balls in the Carnot-Carathéodory control distance corresponding to the subRiemannian metric σ_0 . For each $\epsilon > 0$ we also define the distance function d_ϵ corresponding to the Riemannian metric σ_ϵ ,

$$d_\epsilon(x, y) = \inf \left\{ \int_0^1 |\gamma'|_{\sigma_\epsilon}(s) ds \text{ with } \gamma : [0, 1] \rightarrow G \right.$$

a Lipschitz curve s. t. $\gamma(0) = x, \gamma(1) = y$ $\left. \right\}$.

Set

$$B_\epsilon(x, r) = \{y \in G \mid d_\epsilon(x, y) < r\}.$$

Note that in the definition of d_ϵ , if the curve for which the infimum is achieved happens to be horizontal then $d_\epsilon(x, y) = d_0(x, y)$. In general we have $\sup_{\epsilon > 0} d_\epsilon(x, y) = d_0(x, y)$ and it is well known that (G, d_ϵ) converges in the Gromov-Hausdorff sense as $\epsilon \rightarrow 0$ to the sub-Riemannian space (G, d_0) . (See for instance [25] and references therein).

2.1. Stability of the homogenous structure as $\epsilon \rightarrow 0$. If we denote by dx the Haar measure in G , and by $|\Omega|$ the corresponding measure of a subset $\Omega \subset G$, then Rea and two of the authors have proved in [7, 11] that

Proposition 2.1. *There exist constants $C, R > 0$ independent of ϵ such that for every $x \in G$ and $R > r > 0$,*

$$|B_\epsilon(x, 2r)| \leq C |B_\epsilon(x, r)|.$$

Having this property the spaces (G, d_ϵ, dx) are called *homogenous* with constant $C > 0$ independent of ϵ (see [17]).

Let $\tau > 0$ and consider the space $\tilde{G} = G \times (0, \tau)$ with its product Lebesgue measure $dxdt$. In \tilde{G} define the pseudo-distance function

$$(2.1) \quad \tilde{d}_\epsilon((x, t), (y, s)) = \max(d_\epsilon(x, y), \sqrt{|t - s|}).$$

Proposition 2.1 tells us that $(\tilde{G}, \tilde{d}_\epsilon, dxdt)$ is a homogeneous space with constant independent of $\epsilon \geq 0$. Likewise, the Poincaré inequality holds for all ϵ near zero, with constant independent of ϵ .

2.2. Stability of Schauder estimates. Let us recall uniform estimates in spaces of Hölder continuous functions for solutions of second order sub-elliptic differential equations in non divergence form

$$L_{\epsilon,A}u \equiv \partial_t u - \sum_{i,j=1}^n a_{ij}^\epsilon(x,t) X_i^\epsilon X_j^\epsilon u = 0,$$

in a cylinder $Q = \Omega \times (0, T)$ that are stable as $\epsilon \rightarrow 0$. We will assume ellipticity

$$\Lambda^{-1}|\eta|^2 \leq a_{ij}^\epsilon(x,t)\eta^i\eta^j \leq \Lambda|\eta|^2,$$

with $\Lambda > 0$, and for a.e. $(x, t) \in Q$, all $\eta \in \mathbb{R}^n$ and $\epsilon \in [0, 1]$.

Let us start with the definition of classes of Hölder continuous functions in this setting

Definition 2.2. Let $0 < \alpha < 1$, $Q \subset \mathbb{R}^{n+1}$ and u be defined on Q . We say that $u \in C_{\epsilon,X}^\alpha(Q)$ if there exists a positive constant M such that for every $(x, t), (x_0, t_0) \in Q$

$$(2.2) \quad |u(x, t) - u(x_0, t_0)| \leq M \tilde{d}_\epsilon^\alpha((x, t), (x_0, t_0)).$$

We put

$$\|u\|_{C_{\epsilon,X}^\alpha(Q)} = \sup_{(x,t) \neq (x_0,t_0)} \frac{|u(x, t) - u(x_0, t_0)|}{\tilde{d}_\epsilon^\alpha((x, t), (x_0, t_0))} + \sup_Q |u|.$$

Iterating this definition, if $k \geq 1$ we say that $u \in C_{\epsilon,X}^{k,\alpha}(Q)$ if for all $i = 1, \dots, m$ $X_i u \in C_{\epsilon,X}^{k-1,\alpha}(Q)$. Where we have set $C_{\epsilon,X}^{0,\alpha}(Q) = C_{\epsilon,X}^\alpha(Q)$.

Internal Schauder estimates for these type of operators are well known. We recall the results of Capogna and Han [14] for uniformly subelliptic operators, of Bramanti and Brandolini [5] for heat-type operators, and the results of Lunardi [30] and Gutiérrez and Lanconelli [24], which apply to a large class of squares of vector fields plus a drift term. Schauder estimates uniform in ϵ have been proved by the authors in [10] in the setting of Carnot Groups and by two of us in [7] in the setting of general Hörmander type vector fields.

These result can be stated as

Proposition 2.3. *Let w be a smooth solution of $L_{\epsilon,A}w = f$ on Q . Let K be a compact sets such that $K \subset\subset Q$, set $2\delta = d_0(K, \partial_p Q)$ and denote by K_δ the δ -tubular neighborhood of K . Assume that there exists a constant $C > 0$ such that*

$$\|a_{ij}^\epsilon\|_{C_{\epsilon,X}^{k,\alpha}(K_\delta)} \leq C,$$

for any $\epsilon \in (0, 1)$. There exists a constant $C_1 > 0$ depending on α, C, δ but independent of ϵ , such that

$$\|w\|_{C_{\epsilon,X}^{k+2,\alpha}(K)} \leq C_1 \left(\|f\|_{C_{\epsilon,X}^{k,\alpha}(K_\delta)} + \|w\|_{C_{\epsilon,X}^{k+1,\alpha}(K_\delta)} \right).$$

3. GRADIENT ESTIMATES

In this section we prove Theorem 1.2. The proof is carried out in two steps: First we use the maximum principle to establish interior L^∞ bounds for the full gradient of the solution $\nabla_1 u$ of (1.6) with respect to the Lipschitz norm of u on the parabolic boundary. Next, we construct appropriate barriers and invoke the comparison principle established in [6] to prove boundary gradient estimates. The combination of the two will yield the uniform global Lipschitz bounds.

3.1. Interior gradient estimates. Recalling that the right invariant vector fields X_j^r commute with the left invariant frame X_i , $i = 1, \dots, n$ it is easy to show through a direct computation the following result.

Lemma 3.1. *Let $u_\epsilon \in C^3(Q)$ be a solution to (1.3) and denote $v_0 = \partial_t u_\epsilon$, $v_i = X_i^r u_\epsilon$ for $i = 1, \dots, n$. Then for every $h = 0, \dots, n$ the function v_h is a solution of*

$$(3.1) \quad -\partial_t v_h + \sum_{i,j,k=1}^n \frac{\partial a_{ij}^\epsilon}{\partial p_k} (\nabla_\epsilon u) X_i^\epsilon X_j^\epsilon u_\epsilon X_k^\epsilon v_h + \sum_{i,j=1}^n a_{ij}^\epsilon (\nabla_\epsilon u) X_i^\epsilon X_j^\epsilon v_h = 0.$$

Note that in order to prove L^∞ bounds on the horizontal gradient of solutions of (1.6) one *cannot* invoke Lemma 3.1 with differentiation along the horizontal left invariant frame, because such vector fields do not commute.

Let us explicitly note that the right derivatives $X_k^{1,r}$ are a basis of the tangent space, as well as X_k^1 , so that it is possible to represent each family of vector fields as linear combination of the other. In particular, in the Carnot setting, it has been proved in [38] that there exist homogenous polynomials c_{kj} such that

$$(3.2) \quad X_k^1 = \sum_j c_{kj} X_j^{1,r}.$$

In the general Lie groups setting this assertion is true only locally and the functions c_{kj} are polynomials in the local exponential variables independent of ϵ .

In view of this observation and from the weak maximum principle one may easily deduce that:

Proposition 3.2. *Let $u_\epsilon \in C^3(Q)$ be a solution to (1.6) with Ω bounded. There exists $C = C(G, \|\varphi\|_{C^2(\Omega)}) > 0$ such that*

$$\sup_Q |\nabla_1 u_\epsilon| \leq \sup_{\partial_p Q} (|\nabla_1 u_\epsilon| + |\partial_t u_\epsilon|),$$

where ∇_1 is the full σ_1 -Riemannian gradient.

3.2. Barrier functions and boundary gradient estimate. In [6, Section 4.2] it is shown that in a step two Carnot group coordinate planes (i.e. images under the exponential of level sets of the form $x_k = 0$) solve the minimal surface equation $h_0 = 0$. In the same paper it is also shown that this may fail for step three or higher. In order to adapt the construction

of the barrier to the present non-nilpotent setting we will need a refinement of this result, based on the following definition of convex set:

Definition 3.3. For every point $x_0 \in \partial\Omega$ consider the canonical coordinates $\Phi_{x_0}(u)$ defined in (1.1) and centered at x_0 (so that x_0 is represented by the origin in these coordinates). Assume that Ω has a tangent plane Π at the point $u = 0$ and assume that $\Phi_{x_0}^{-1}(\Omega)$ is lying on one side of the plane. If this happens for every $x_0 \in \partial\Omega$ we say that Ω is convex.

In a step two Carnot group this definition is equivalent to the one in [10], and every set that is Euclidean convex when expressed in exponential coordinates satisfies the condition. Using Darboux coordinates one can see that the same holds for the root-translation group \mathcal{RT} . This ad-hoc definition of convexity is not implied, and does not imply, the notion of horizontal convexity as defined for instance in [29].

Lemma 3.4. *Let G be a step two Carnot group. If $f : G \rightarrow \mathbb{R}$ is linear (in exponential coordinates) then for every $\epsilon \geq 0$, the matrix with entries $X_i^\epsilon X_j^\epsilon f$ is anti-symmetric, in particular every level set of f satisfies $h_\epsilon = 0$.*

The previous lemma and the Rothschild and Stein local osculation result will lead to the construction of a barrier in the present setting, and to establish a priori Lipschitz estimates at the boundary for solutions. We begin by recalling an immediate consequence of the proof of [6, Theorem 3.3].

Lemma 3.5. *For each $\epsilon \geq 0$, if v_ϵ is a bounded subsolution and w_ϵ is a bounded supersolution of (1.6) then $v_\epsilon(x, t) \leq w_\epsilon(x, t)$ for all $(x, t) \in Q$.*

Let $u_\epsilon \in C^2(Q)$ be a solution of (1.6), and express the evolution PDE in non-divergence form

$$(3.3) \quad \partial_t u_\epsilon = h_\epsilon W_\epsilon = a_{ij}^\epsilon (\nabla_\epsilon u) X_i^\epsilon X_j^\epsilon u_\epsilon.$$

Set $v_\epsilon = u_\epsilon - \varphi$ so that v_ϵ solves the homogenous 'boundary' value problem

$$(3.4) \quad \begin{cases} \partial_t v_\epsilon = a_{ij}^\epsilon (\nabla_\epsilon v_\epsilon + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon v_\epsilon + b^\epsilon & \text{in } Q = \Omega \times (0, T) \\ v_\epsilon = 0 & \text{on } \partial_p Q, \end{cases}$$

with $b^\epsilon(x) = a_{ij}^\epsilon (\nabla_\epsilon v_\epsilon(x) + \nabla_\epsilon \varphi(x)) X_i^\epsilon X_j^\epsilon \varphi(x)$. We define our (weakly) parabolic operator for which the function v_ϵ is a solution

$$(3.5) \quad P(v) = a_{ij}^\epsilon (\nabla_\epsilon v + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon v + b^\epsilon - \partial_t v.$$

In the following we construct for each point $p_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ a *barrier function* for Q, v_ϵ : i.e.,

Lemma 3.6. *Let G be a Lie group free up to step two and $\Omega \subset G$ convex in the sense of Definition 3.3. For each point $p_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ and for every $\epsilon > 0$ there exist a parabolic neighborhood V_ϵ of p_0 and a positive function $w_\epsilon \in C^2(Q)$ such that*

$$(3.6) \quad P(w_\epsilon) \leq 0 \text{ in } V_\epsilon \cap Q \text{ with } w_\epsilon \geq v_\epsilon \text{ in } \partial_p V_\epsilon \cap Q.$$

Proof. For every $x_0 \in \partial\Omega$ we can select exponential coordinates locally around the point x_0 . The point x_0 has coordinates 0 in the variables u .

In these coordinates there exists an hyperplane tangent to the open set Ω defined by an equation of the form $\Pi(u) = \sum_{i=1}^n a_i u_i = 0$ with $\Pi > 0$ in Ω , $\Pi(0) = 0$, and normalized as $\sum_{d(i)=1,2} a_i^2 = 1$. Following the standard argument (see for instance [28, Chapter 10]) we select the barrier at $(x_0, t_0) \in \partial\Omega \times (0, T)$ independent of time with

$$(3.7) \quad \tilde{w}_\epsilon = \psi(\Pi)$$

with ψ solution of

$$(3.8) \quad \psi'' + \nu(\psi')^2 = 0,$$

in particular

$$(3.9) \quad \psi(s) = \frac{1}{\nu} \log(1 + ks),$$

with k and ν chosen appropriately so that conditions (3.6) will hold. We choose a neighborhood $V = O \times (0, T)$ such that $\Pi \cap O \cap \partial\Omega = \{x_0\}$. By an appropriate choice of k sufficiently large we can easily obtain $\tilde{w}_\epsilon(0) = 0$ and $\tilde{w}_\epsilon \circ \Theta_{x_0} \geq \nu_\epsilon$ in $\partial_p V \cap Q$.

We denote by P_Y the operator which has the same expression of P , but with respect to the left invariant osculating frame $\{Y_i^\epsilon\}_{i=1,\dots,n}$ in the nilpotent osculating free group $G_{m,2}$, i.e. $P_Y(v) = a_{ij}^\epsilon (\nabla_{Y_i^\epsilon} v_\epsilon + \nabla_{Y_j^\epsilon} \varphi) Y_i^\epsilon Y_j^\epsilon v_\epsilon + b^\epsilon - \partial_t v$.

To estimate $P_Y(\tilde{w}_\epsilon) \leq 0$ we begin by observing that \tilde{w}_ϵ satisfies

$$(3.10) \quad P_Y(\tilde{w}_\epsilon) = \psi' a_{ij}^\epsilon (\nabla_{Y_i^\epsilon} \tilde{w}_\epsilon + \nabla_{Y_j^\epsilon} \varphi) Y_i^\epsilon Y_j^\epsilon \Pi + \frac{\psi''}{(\psi')^2} \mathcal{F} + b_\epsilon,$$

with $\mathcal{F} = a_{ij}^\epsilon (\nabla_{Y_i^\epsilon} \tilde{w}_\epsilon + \nabla_{Y_j^\epsilon} (\varphi \circ \Phi_{x_0})) Y_i^\epsilon Y_j^\epsilon \tilde{w}_\epsilon$.

We first note that

$$(3.11) \quad a_{ij}^\epsilon \left(\nabla_{Y_i^\epsilon} \tilde{w}_\epsilon + \nabla_{Y_j^\epsilon} (\varphi \circ \Phi_{x_0}) \right) (Y_i^\epsilon Y_j^\epsilon) \Pi = 0$$

as a_{ij}^ϵ is symmetric and $X_i^\epsilon X_j^\epsilon \Pi$ is anti-symmetric in view of Lemma 3.4. We can now estimate the remaining terms of (3.10)

$$(3.12) \quad \frac{\psi''}{(\psi')^2} \mathcal{F} + b_\epsilon$$

in a parabolic neighborhood of $u = 0$. We first note that Lemma 3.4 implies

$$\begin{aligned} \frac{\epsilon^2}{2} &\leq \max\left(\sum_{d(i)=1} a_i^2, \epsilon^2 \sum_{d(k)=1} a_k^2 \right) \leq |\nabla_{Y_i^\epsilon} \Pi| = \\ &= \sum_{d(i)=1} \left(a_i + \sum_{d(k)=2, d(j)=1} c_{ij}^k a_k x_j \right)^2 + \epsilon^2 \sum_{d(k)=2} a_k^2 \leq C(G)(1 + \epsilon^2), \end{aligned}$$

for some constant $C(G) > 0$. Consequently, for $\psi' \gg 1$ sufficiently large one finds

$$(3.13) \quad \mathcal{F} \geq \frac{|\nabla_{Y,\epsilon}\tilde{w}_\epsilon|^2}{1 + |\nabla_{Y,\epsilon}\tilde{w}_\epsilon + \nabla_{Y,\epsilon}\varphi|^2} \geq C(G) \frac{|\nabla_{Y,\epsilon}\tilde{w}_\epsilon|^2}{1 + |\psi'|^2 + |\nabla_{Y,\epsilon}\varphi|^2} \geq C(G)\epsilon^2 > 0,$$

with $C(G) > 0$ a constant depending only on G (not always the same along the chain of inequalities). In view of the definition of b_ϵ and (3.8) with an appropriate choice of $\nu = \nu(G, \epsilon, \phi) > 0$ and $k = k(G, \phi) \gg 1$ in (3.9), we conclude

$$(3.14) \quad \frac{\Phi''}{(\Phi')^2} \mathcal{F} + b_\epsilon \leq \left(\frac{\psi''}{(\psi')^2} + \nu - 1 \right) \mathcal{F} \leq -C(G)\epsilon^2.$$

It follows that

$$P_Y(\tilde{w}_\epsilon) \leq -C(G)\epsilon^2.$$

To conclude, we set $w_\epsilon = \tilde{w}_\epsilon \circ \Theta_{x_0}$. In view of the relation (1.2) between the vector fields X and Y , it is now immediate to see that there exists a neighborhood V_ϵ of p_0 , depending on ϵ , such that

$$P(w_\epsilon) \leq 0 \text{ in } V_\epsilon \cap Q.$$

□

Proposition 3.7. *Let G be a Lie group free up to step two, $\Omega \subset G$ convex in the sense of Definition 3.3 and $\varphi \in C^2(\bar{\Omega})$. For $\epsilon > 0$ denote by $u_\epsilon \in C^2(\Omega \times (0, T)) \cap C^1(\bar{\Omega} \times (0, T))$ the non-negative unique solution of the initial value problem (1.6). There exists $C = C(G, \|\varphi\|_{C^2(\bar{\Omega})}) > 0$ such that*

$$(3.15) \quad \sup_{\partial\Omega \times (0, T)} |\nabla_\epsilon u_\epsilon| \leq \sup_{\partial\Omega \times (0, T)} |\nabla_1 u_\epsilon| \leq C.$$

Proof. In view of Lemma 3.5, a comparison with the barrier constructed above yields that

$$(3.16) \quad 0 \leq \frac{v_\epsilon(x, t)}{\text{dist}_{\sigma_1}(x, x_0)} \leq \frac{w_\epsilon(x, t)}{\text{dist}_{\sigma_1}(x, x_0)} \leq C(k, \nu),$$

in $V_\epsilon \cap Q$, with $\text{dist}_{\sigma_1}(x, x_0)$ being the distance between x and x_0 in the Riemannian metric σ_1 , concluding the proof of the boundary gradient estimates at the point p_0 . □

4. REGULARITY PROPERTIES IN THE $C^{k,\alpha}$ SPACES

In this section we will prove uniform estimates for solution of (1.3) in the $C_{\epsilon, X}^{k,\alpha}$ Hölder spaces. This is accomplished in two steps and follows a strategy originally introduced by Trudinger (see notes in [28, Chapter 7]). First we establish $C^{1,\alpha}$ regularity,

4.1. Regularity properties in the $C^{1,\alpha}$ spaces.

Remark 4.1. Let u be a smooth solution of (1.3). Note that the function $v_k = X_k^r u$ is then a solution of the equation

$$(4.1) \quad -\partial_t v_k + \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon v_k \right) + \sum_{i,j,h=1}^n a^{i,j,h} X_i^\epsilon X_j^\epsilon u_\epsilon X_h^\epsilon v_k = 0,$$

where

$$a^{i,j,h} = \frac{\partial a_{ij}^\epsilon}{\partial p_h} - \frac{\partial a_{ih}^\epsilon}{\partial p_j}.$$

Indeed the equation provided in (3.1) in non divergence form can be re-written in divergence form as

$$-\partial_t v_k + \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon v_k \right) + \sum_{i,j,h=1}^n \frac{\partial a_{ij}^\epsilon}{\partial p_h} X_i^\epsilon X_j^\epsilon u_\epsilon X_h^\epsilon v_k - \sum_{i,j,h=1}^n \frac{\partial a_{ij}^\epsilon}{\partial p_h} X_j^\epsilon X_k^r u_\epsilon X_i^\epsilon X_h^\epsilon u_\epsilon = 0$$

and the latter is equivalent to (4.1).

Remark 4.2. Starting from (4.1) one can immediately see that the function $z = |\nabla_\epsilon^r u_\epsilon|^2$ is solution of

$$-\partial_t z + \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon z \right) + \sum_{i,j,h=1}^n a^{i,j,h} X_i^\epsilon X_j^\epsilon u_\epsilon X_h^\epsilon z - 2 \sum_{i,j,h,k=1}^n a_{ij}^\epsilon X_i^\epsilon v_k X_j^\epsilon v_k = 0.$$

Lemma 4.3. *for every $k = 1, \dots, n$ and for every $\delta > 0$ the functions*

$$w_k^\pm = \pm v_k + \delta z$$

satisfy the inequality

$$-\partial_t w_k^\pm + \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon w_k^\pm \right) \geq -C_0 |\nabla_\epsilon w_k^\pm|^2 - C_1,$$

for suitable constants C_0 and C_1 .

Proof. For simplicity we temporarily drop the \pm superscript. Adding the equations satisfied by v_k and z we see that for every $k = 1, \dots, n$ and for every $\delta > 0$ the functions $w_k^\pm = \pm v_k + \delta z$ satisfy

$$(4.2) \quad \begin{aligned} -\partial_t w_k + \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon w_k \right) &= - \sum_{i,j,h=1}^n a^{i,j,h} X_i^\epsilon X_j^\epsilon u_\epsilon X_h^\epsilon w_k + 2\delta \sum_{i,j,s=1}^n a_{ij}^\epsilon X_i^\epsilon v_s X_j^\epsilon v_s \\ &\geq 2\delta \lambda \sum_{s=1}^n |\nabla_\epsilon X_s^{1,r} u_\epsilon|^2 - \sup_{i,j,h} |a^{i,j,h}| |X_i^\epsilon X_j^\epsilon u_\epsilon| |\nabla_\epsilon w_k|, \end{aligned}$$

where $\lambda > 0$ is the smallest eigenvalue of a_{ij}^ϵ and is independent of $\epsilon > 0$. Using the notation introduced in (3.2) we deduce that

$$|X_i^\epsilon X_j^\epsilon u_\epsilon| = \left| \sum_{j,s=1}^n X_i^\epsilon (c_{sj} X_s^{1,r} u_\epsilon) \right| \leq C_2 + C_3 \sum_{s=1}^n |\nabla_\epsilon X_s^{1,r} u_\epsilon|.$$

Consequently, using Schwarz's inequality,

$$\begin{aligned} -\partial_t w_k + \sum_{ij} X_i \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon w_k \right) &\geq \\ &\geq 2\delta\lambda \sum_{s=1}^n |\nabla_\epsilon X_s^{1,r} u_\epsilon|^2 - \sum_{s=1}^n \delta\lambda |\nabla_\epsilon X_s^1 u_\epsilon|^2 - C_0 |\nabla_\epsilon w_k|^2 - C_1, \end{aligned}$$

completing the proof. \square

Next we set $(x_0, t_0) \in Q = \Omega \times (0, T)$ and for $r > 0$, let $Q_\epsilon(r) = \{(x, t) \in Q \mid d_\epsilon(x, x_0) < r \text{ and } |t - t_0| \leq r^2\}$. Define

$$W_k^\pm = \sup_{Q_\epsilon(4r)} w_k^\pm$$

and observe that

$$\partial_t (W_k^\pm - w_k^\pm) - \sum_{i,j=1}^n X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon (W_k^\pm - w_k^\pm) \right) \geq -C_0 |\nabla_\epsilon (W_k^\pm - w_k^\pm)|^2 - C_1.$$

In order to invoke the weak Harnack inequality and derive the C^α estimates, we need to eliminate the quadratic term on the right hand side. Following [28, Chapter 12, Sec. 3] we define

$$\bar{w}_k = \frac{\lambda}{2C_0} \left(1 - \exp\left(\frac{2C_0}{\lambda} (w_k^\pm - W_k^\pm)\right) \right)$$

and observe that this new functions satisfies

$$-\partial_t \bar{w}_k + \sum_{ij} X_i^\epsilon \left(a_{ij}^\epsilon (\nabla_\epsilon u) X_j^\epsilon \bar{w}_k \right) + g \leq 0,$$

where $g = C_1 \left(\frac{2C_0}{\lambda} \bar{w}_k + 1 \right)$, for the constants λ, C_0, C_1 from Lemma 4.3. In view of the weak Harnack inequality [7, Proposition 7.6], one has that for some constant $C_4 > 0$ independent of ϵ and for $Q_\epsilon^-(r) = \{(x, t) \in Q \mid d_\epsilon(x, x_0) < r \text{ and } t_0 - 3r^2 < t < t_0 + 2t^2\}$,

$$\int_{Q_\epsilon^-(r)} \bar{w}_k \, dxdt \leq C_4 \left(\inf_{Q_\epsilon(r)} \bar{w}_k + \sup_{Q_\epsilon(r)} |g| r^2 \right).$$

Following the argument in [28, Chapter 12, Sec. 3] we obtain

Proposition 4.4. *Let u_ϵ be a solution of the mean curvature flow PDE (1.3) in $Q = \Omega \times (0, T) \subset G \times \mathbb{R}$. Let K be a compact sets such that $K \subset\subset Q$, set $2\delta = d_0(K, \partial_p Q)$ and denote by K_δ the δ -tubular neighborhood of K in d_0 . There exists constants $C > 0$ and $\alpha \in (0, 1)$ depending on δ and on the Lipschitz norm of u in K_δ , but independent of ϵ , such that*

$$\|u_\epsilon\|_{C_{\epsilon, X}^{1, \alpha}(K_\delta)} \leq C.$$

4.2. Regularity properties in the $C^{k,\alpha}$ spaces. Once obtained the interior $C^{1,\alpha}$ estimate of the solution uniform in ϵ , we write the mean curvature flow equation in non divergence form:

$$\partial_t u - \sum_{i,j=1}^n a_{ij}^\epsilon(x,t) X_i^\epsilon X_j^\epsilon u = 0.$$

Applying Schauder estimates in Proposition 2.3 we immediately deduce the proof of Theorem 1.3.

Proof of Theorem 1.3. Since the solution is of class $C^{1,\alpha}$, and the norm is bounded uniformly in ϵ then u_ϵ it is a solution of a divergence form equation

$$\partial_t u_\epsilon - \sum_{i,j=1}^n a_{ij}^\epsilon(x,t) X_i^\epsilon X_j^\epsilon u_\epsilon = 0,$$

with a_{ij}^ϵ of class C^α such that for every K be a compact sets such that $K \subset\subset Q$ and $2\delta = d_0(K, \partial_p Q)$ there exists a positive constant C_0 such that

$$\|a_{ij}^\epsilon\|_{C_{\epsilon,X}^\alpha(K_\delta)} \leq C_0,$$

for every $\epsilon \in (0, 1)$. Consequently, by Proposition 2.3 there exists a constant C_2 such that

$$\|u_\epsilon\|_{C^2(K)} \leq C_2.$$

The conclusion immediately follows by induction. □

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