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# EXISTENCE OF THE STARK-WANNIER QUANTUM RESONANCES

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**ABSTRACT.** In this paper we prove the existence of the Stark-Wannier quantum resonances for one-dimensional Schrödinger operators with smooth periodic potential and small external homogeneous electric field. Such a result extends the existence result previously obtained in the case of periodic potentials with a finite number of open gaps.

*Ams classification (MSC 2010):* 81Qxx, 81Q15.

*Keywords:* Quantum resonances, Stark-Wannier resonances, regular perturbation theory.

## 1. INTRODUCTION

Existence of quantum resonances for the one-dimensional Schrödinger operator with a periodic potential and an external homogeneous electric field has been a largely debated question among the solid state community since the seminal papers by Wannier [24, 25]. These resonances, named Stark-Wannier resonances, are strictly connected with the time behavior of the Bloch oscillators [10], which electron wave function  $\psi(x, t)$  is the solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H_F \psi$$

where  $H_F$  is the Stark-Wannier operator formally defined on  $L^2(\mathbb{R}, dx)$  as

$$H_F := H_B + Fx, \quad H_B := -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x),$$

$Fx$  represents the Stark potential associated to the external homogeneous electric field,  $H_B$  is the Bloch operator and  $V(x)$  represents the periodic potential of an one-dimensional crystal; here we assume to choose the units such that  $2m = 1$  and  $\hbar = 1$  is fixed.

Assuming that  $V(x)$  is regular enough then the spectrum of  $H_F$  covers the whole real axis. On the other side, if we neglect the interband coupling term then it turns out that the spectrum of such a decoupled band approximation consists of a sequence on infinite ladders of real eigenvalues. The crucial point is to understand what happen to these eigenvalues when we restore the interband coupling terms. This question has been largely debated [26, 27, 28] and at present it is expected that these ladders of real eigenvalues will turn into quantum resonances.

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This paper is dedicated to Professor Vincenzo Grecchi on the occasion of his 70th birthday.

Despite its apparently simplicity this problem is rather technically complicated to study because the external potential  $Fx$  is not a bounded perturbation and so the interband coupling term is quite hard to manage.

A first result in order to put on a rigorous ground this problem was due to Herbst and Howland [13], who gave a rigorous definition for quantum resonances for the Wannier-Stark operator by means of the complex translation method (in fact, this definition applies to the case of analytic periodic potential); that is a quantum resonance corresponds to a complex-valued eigenvalue of a non self-adjoint operator obtained by  $H_F$  by means of the analytic translation. More precisely, quantum resonances for the Stark-Wannier operator are defined by Herbst and Howland as the complex eigenvalues of the analytically translated operator

$$H_F^\theta = \mathcal{U}^\theta H_F (\mathcal{U}^\theta)^{-1} = -\frac{d^2}{dx^2} + V(x + \theta) + Fx + F\theta$$

where  $-R < \Im\theta < 0$ , the periodic potential was assumed to be analytic in a set containing the strip  $|\Im z| \leq R$ , and

$$\mathcal{U}^\theta : L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)$$

is the translation operator defined as

$$(\mathcal{U}^\theta \psi)(x) = \psi(x + \theta).$$

In particular, they proved that

$$\sigma_{ess}(H_F^\theta) = \{z : \Im z = F\Im\theta\}, \quad \sigma_d(H_F^\theta) \subseteq \{z : F\Im\theta < \Im z \leq 0\}$$

and, let

$$\mathcal{R} = \cup_{\theta : 0 < -\Im\theta < R} \sigma_d(H_F^\theta),$$

then the kernel of the resolvent operator  $[z - H_F]^{-1}$  has an analytic continuation from  $\Im z > 0$  to  $\{z : \Im z > -R\} \setminus \mathcal{R}$ , and if  $z_0 \in \mathcal{R}$  then the kernel of the resolvent operator  $[z_0 - H_F]^{-1}$  has a pole at this point. The points of  $\mathcal{R}$  with strictly negative imaginary part are called quantum resonances.

A second important result was obtained by proving that the spectrum of the Stark-Wannier operator is purely absolutely continuous [2]

$$\sigma(H_F) = \sigma_{ac}(H_F) = \mathbb{R};$$

hence, embedded eigenvalues are not admissible. By means of this result we can so exclude the possibility that the ladders of real eigenvalues for the decoupled band approximation will turn into real embedded eigenvalues when the coupling interband term is restored. In fact, this is not the case of singular periodic potentials; in particular, when the periodic potential is given by means of a periodic sequence of Dirac's  $\delta$  pointwise interactions then we expect to observe embedded eigenvalues [8], as well as in the case of a periodic sequence of Dirac's  $\delta$ -prime pointwise interactions [18].

Finally, the proof of existence of ladders of quantum resonances was given in the limit of small electric field and under a technical assumption: it had been assumed that the number of open gaps is finite [11]. We should recall that, in 3-dimensional crystals, the number of open gaps is usually finite; this is not the case of 1-dimensional crystals where we have a finite number of open gaps only for periodic potentials given by a class of elliptic functions.

In this paper we are able to remove such a technical assumption on the periodic potential, that is we prove the existence of quantum resonances for electric field small enough and for any periodic potential analytic in a finite strip containing the real axis. In our proof we develop some ideas already contained in [11], the main improvement is that we are able to manage here the interband coupling term (see Theorem 4 and Theorem 5), even in the case of infinitely many open gaps, by means of precise estimates of the asymptotic behavior of the Bloch functions around the Kohn branch points.

In our treatment we replace also the usual definition of quantum resonances, given by Herbst and Howland, by considering the complex eigenvalues of the non self-adjoint operator obtained by  $H_F$  in the *crystal momentum representation* (CMR) by means of the analytic distortion  $\hat{\psi}(p) \rightarrow e^{ip\theta}\hat{\psi}(p)$  for  $p$  large enough and  $\Im\theta < 0$ ; where  $\hat{\psi}$  represents the wave-function in the crystal momentum representation and  $p$  is the crystal momentum variable.

We close by mentioning the fact that the problem of the existence of the Stark-Wannier quantum resonances and the time behavior of Bloch oscillators have been the object of a large analysis, where many different techniques have been adopted; among all the papers we recall the works by Nenciu [19, 20] and Hövermann, Spohn and Teufel [14], where adiabatic methods has been applied, the works by Buslaev and Dmitrieva [5], where a WKB-type expansion has been adapted in order to match the asymptotic behaviour of the wavefunction in a neighbourhood of the turning points in the tilted band picture, the works by Bentosela and Grecchi [3] and by Combes and Hislop [6], where the problem of the existence of the Stark-Wannier resonances has been considered in the semiclassical limit  $\hbar \rightarrow 0$ , and finally we mention the paper by Avron [1], where he considered the spectral problem in the case of complex-valued  $F$ .

The paper is organized as follows.

In §2 we state our main result.

In §3 we collect some known results about band and Bloch functions.

In §4 we give the asymptotic behavior of the Bloch functions in the high energy limit.

In §5 we adapt the CMR to the analytic structure of the Bloch functions, the obtained representation is named *extended crystal momentum representation* (ECMR), and then we consider the Stark-Wannier operator  $H_F$  in the ECMR.

In §6 we define the analytic distortion in the ECMR.

In §7 we study the spectral properties of the decoupled band approximation of the analytically distorted Stark-Wannier operator in the ECMR.

In §8 we prove that the coupling term between the bands is a bounded operator and then we define quantum resonances as complex valued eigenvalues of the analytically distorted Stark-Wannier operator in the ECMR. We then restore the coupling term and we prove, by means of the regular perturbation theory, that the ladders of eigenvalues of the decoupled band approximation turns into complex eigenvalues of the distorted Stark-Wannier operator contained in the strip  $|\Im z| \leq \frac{1}{2}F|\Im\theta|$ . Furthermore, we prove that the essential spectrum of the distorted Stark-Wannier operator lies below the line  $\frac{1}{2}F\Im\theta$ , where  $\Im\theta < 0$ . In such a way the existence of Stark-Wannier quantum resonances will follow.

**Notation.** In the following:

- we denote by  $E_n(k)$  and  $\varphi_n(x, k)$  the band and Bloch functions where  $n \in \mathbb{N}$  and  $k$  is the quasi-momentum (or crystal momentum) variable belonging to the Brillouin zone  $\mathcal{B} = (-\frac{1}{2}b, +\frac{1}{2}b]$ ,  $b = \frac{2\pi}{a}$  where  $a$  is the period of the periodic potential;
- we denote by  $E(p)$  and

$$\varphi(x, p) = e^{ipx}u(x, p)$$

the multisheeted band and Bloch functions, where the quasimomentum  $p$  belongs to the complex plane  $\tilde{\mathbb{C}}$  with cuts connecting the Kohn branch points  $k_n = \pm\frac{1}{2}nb \pm i\kappa_n$ ,  $\kappa_n \geq 0$  (actually, if  $\kappa_n = 0$  then the gap is closed and there is no branch point);

- since we have to work with meromorphic functions  $f(p)$  defined on  $\tilde{\mathbb{C}}$  then, by construction,  $f(p)$  would have discontinuity points on  $\mathbb{R} \setminus \tilde{\mathbb{R}}$ , where  $\tilde{\mathbb{R}} = \mathbb{R} \cap \tilde{\mathbb{C}}$ ; we'll denote by  $f_A f(p)dp$ , where  $A$  in an interval in  $\mathbb{R}$ , the integral of the meromorphic function  $f(p)$  over the set  $A \cap \tilde{\mathbb{C}}$ ;
- we denote by  $\Gamma$  the complex path such that  $E(p)$  monotonically maps  $\Gamma$  onto  $\mathbb{R}$  (see Figure 1). We denote also by

$$\Gamma_1 = \left[-\frac{1}{2}Nb, +\frac{1}{2}Nb\right] \cap \tilde{\mathbb{R}} \quad \text{and} \quad \Gamma_c = \tilde{\mathbb{R}} \setminus \Gamma_1$$

where  $N$  is fixed and such that  $\kappa_N > 0$ ; by definition  $f_{\Gamma_1} f(p)dp$  coincides with  $\int_{\Gamma_1} f(p)dp$  and, similarly,  $f_{\Gamma_c} f(p)dp$  coincides with  $\int_{\Gamma_c} f(p)dp$ ;

- We define  $\mathcal{H}_E$  the Hilbert space of functions  $\hat{\psi}(p) \in L^2(\tilde{\mathbb{R}}, dp)$  such that

$$\frac{d^r \hat{\psi}(-\frac{1}{2}Nb \pm 0)}{dp^r} = \frac{d^r \hat{\psi}(\frac{1}{2}Nb \mp 0)}{dp^r}, \quad \forall r \in \mathbb{N}.$$

Hence, we may identify the vector  $\hat{\psi}$  with the vector  $(\hat{\psi}_1, \hat{\psi}_c) \in L^2(\Gamma_1) \otimes L^2(\Gamma_c)$  satisfying the periodic boundary conditions at the points  $\pm\frac{1}{2}Nb$ , that is

$$\psi_1\left(-\frac{1}{2}Nb + 0\right) = \psi_1\left(+\frac{1}{2}Nb - 0\right)$$

and

$$\psi_c\left(-\frac{1}{2}Nb - 0\right) = \psi_c\left(+\frac{1}{2}Nb + 0\right)$$

together with their derivatives. In particular, in the case  $N = 1$  then  $L^2(\Gamma_1)$  coincides with the space  $L^2(\mathcal{B})$ .

## 2. MAIN RESULT

Here we state the assumptions on the model and the main result.

**Theorem 1.** *Let  $V(x)$  be a non constant periodic function with period  $a$*

$$V(x) = V(x + a), \quad \forall x \in \mathbb{R},$$

*and mean value zero:*

$$\frac{1}{a} \int_0^a V(x)dx = 0. \quad (1)$$

We assume that  $V(x)$  is an analytic function in an open set containing the strip  $|\Im x| \leq R$  for some  $R > 0$ . Let  $N \geq 1$  any fixed integer number such that the  $N$ -th gap of the Bloch operator  $H_B$  is not empty. Then there exists  $F_N > 0$  such that for any  $F \in (0, F_N)$  the Stark-Wannier operator  $H_F$  admits  $N$  ladders of quantum resonances; that is the resolvent operator  $[H_F - z]^{-1}$  has an analytic continuation from the upper half-plane  $\Im z > 0$  to the lower half-plane  $\Im z < 0$ , with poles corresponding to the quantum resonances.

**Remark 1.** In fact, the assumption (1) on the mean value of  $V(x)$  is for sake of definiteness. Indeed, if the mean value of the potential is not zero then we can always recover condition (1) by means of the phase gauge  $\psi \rightarrow \psi e^{-it \int_0^a V(x) dx / a \hbar}$ .

**Remark 2.** The regular perturbation theory we apply in §7 gives that the imaginary part of the quantum resonances, which is strictly negative, is of order  $O(F^2)$  in the limit of small  $F$ . In fact, we expect that the imaginary part is exponentially small in the limit of small  $F$  as it has been proved in the case of periodic potential with a finite number of gaps [12]; however, we don't dwell here on such a problem.

### 3. PRELIMINARY RESULTS

Here we consider the spectral problem for Stark-Wannier operator

$$H_F := -\frac{d^2}{dx^2} + V(x) + Fx,$$

where  $F > 0$  is a given parameter and where  $V(x)$  is a (not constant) smooth periodic potential with period  $a$ :

$$V(x) = V(x + a)$$

and mean value zero.

We recall now some basic properties of the Bloch and band functions [22].

**3.1. Bloch Decomposition.** Let  $H_B$  be the Bloch operator formally defined on  $L^2(\mathbb{R}, dx)$  as

$$H_B := -\frac{d^2}{dx^2} + V(x).$$

Its spectrum is given by bands. Let  $k \in \mathcal{B}$  be the quasi-momentum (or crystal momentum) variable, the torus  $\mathcal{B} = \mathbb{R}/b\mathbb{Z} = (-\frac{1}{2}b, +\frac{1}{2}b]$ , where  $b = \frac{2\pi}{a}$ , is usually named Brillouin zone.

Let  $\varphi_n(x, k)$  denote the Bloch functions associated to the band functions  $E_n(k)$ ; where the Bloch functions are assumed to be normalized to 1 on the interval  $[0, a]$ :

$$\int_0^a \overline{\varphi_m(x, k)} \varphi_n(x, k) dx = \delta_m^n. \quad (2)$$

The band and Bloch functions satisfy to the following eigenvalues problem

$$H_B \varphi = E \varphi \quad (3)$$

with quasi-periodic boundary conditions (hereafter we denote  $' = \frac{\partial}{\partial x}$ , as usual)

$$\varphi(a, k) = e^{ika} \varphi(0, a) \quad \text{and} \quad \varphi'(a, k) = e^{ika} \varphi'(0, k).$$

The Bloch functions  $\varphi_n$  may be written as

$$\varphi_n(x, k) = e^{ikx} u_n(x, k)$$

where  $u_n(x, k)$  is a periodic function with respect to  $x$ :  $u_n(x + a, k) = u_n(x, k)$ . For any fixed  $k \in \mathcal{B}$  the spectral problem (3) has a sequence of real eigenvalues

$$E_1(k) \leq E_2(k) \leq \dots \leq E_n(k) \leq \dots ,$$

such that  $\lim_{n \rightarrow \infty} E_n(k) = +\infty$ .

As functions on  $k$ , both Bloch and band functions are periodic with respect to  $k$ :

$$E_n(k) = E_n(k + b) \quad \text{and} \quad \varphi_n(x, k) = \varphi_n(x, k + b) ,$$

and they satisfy to the following properties for any real-valued  $k$ :

$$\varphi_n(x, -k) = \overline{\varphi_n(x, k)} \quad \text{and} \quad E_n(-k) = E_n(k) .$$

Furthermore, if  $V(x)$  is an even potential then  $\varphi_n(-x, k) = \overline{\varphi_n(x, k)}$ ,  $\varphi_n(x, 0)$  are even functions while  $\varphi_n(x, b/2)$  are odd functions.

The band functions  $E_n(k)$  are monotone increasing (resp. decreasing) functions for any  $k \in [0, \frac{1}{2}b]$  if the index  $n$  is an odd (resp. even) natural number. The spectrum of  $H_B$  is purely absolutely continuous and it is given by bands:

$$\sigma(H_B) = \cup_{n=1}^{\infty} [E_n^b, E_n^t] \quad \text{where} \quad [E_n^b, E_n^t] = \{E_n(k), k \in \mathcal{B}\} .$$

In particular we have that

$$E_n^b = \begin{cases} E_n(0) & \text{for odd } n \\ E_n(b/2) & \text{for even } n \end{cases} \quad \text{and} \quad E_n^t = \begin{cases} E_n(b/2) & \text{for odd } n \\ E_n(0) & \text{for even } n \end{cases} .$$

The intervals  $(E_n^t, E_{n+1}^b)$  are named gaps; a gap  $(E_n^t, E_{n+1}^b)$  may be empty, that is  $E_{n+1}^b = E_n^t$ , or not. It is well known that, in the case of one-dimensional crystals, all the gaps are empty if, and only if, the periodic potential is a constant function. Because we assume that the periodic potential is not a constant function then one gap, at least, is not empty.

From the Bloch decomposition formula it follows that any vector  $\psi \in L^2$  can be written as [21]

$$\psi(x) = \sum_{n \in \mathbb{N}} \frac{1}{2\pi} \int_{\mathcal{B}} \varphi_n(x, k) \hat{\psi}_n(k) dk .$$

The family of functions  $\{\hat{\psi}_n(k)\}_n$  is called the crystal momentum representation (hereafter CMR) of the wave vector associated to  $\psi$  and it is defined as

$$\hat{\psi}_n(k) = \int_{\mathbb{R}} \overline{\varphi_n(x, k)} \psi(x) dx .$$

By construction any function  $\hat{\psi}_n(k)$  is a periodic function and the transformation

$$\psi \in L^2(\mathbb{R}, dx) \rightarrow U\psi = \hat{\psi} := \{\hat{\psi}_n\}_{n=1}^{\infty} \in \mathcal{H} := \otimes_{n=1}^{\infty} L^2(\mathcal{B}, dk) \quad (4)$$

is unitary:

$$\|\psi\|_{L^2(\mathbb{R}, dx)}^2 = \|\hat{\psi}\|_{\mathcal{H}}^2 := \sum_{n=1}^{\infty} \|\hat{\psi}_n\|_{L^2(\mathcal{B}, dk)}^2 .$$

**3.2. Analytic properties of the band and Bloch functions.** The band functions are analytic functions with respect to  $k \in \mathbb{C}$ ; more precisely, they are the branches of a single multisheeted function that has no singularities other than the square root branch points (hereafter named Kohn branch points)  $k_n = \frac{1}{2}nb \pm i\kappa_n$ ,  $n = \pm 1, \pm 2, \dots$ , for some  $\kappa_n = \kappa_{-n} \geq 0$  [16]. In fact, the Kohn branch points only occur in correspondence of open gaps. If the  $n$ -th gap is open (that is  $E_{n+1}^b > E_n^t$ ) then  $\kappa_n > 0$ . On the other side, if the  $n$ -th gap is empty (that is  $E_{n+1}^b = E_n^t$ ) then the Kohn branch points  $\pm k_n$  don't occur; actually, in such a case we have that  $\kappa_n = 0$  and the two branch points  $k_n$  and  $\bar{k}_n$  coincide and then the multisheeted band and Bloch functions are regular at this point.

Let  $\ell_n$  be the straight lines connecting the branch points  $\kappa_n$  and  $\bar{\kappa}_n$ , let  $\tilde{\mathbb{C}} = \mathbb{C} - \cup_{n=\pm 1, \pm 2, \dots} \ell_n$ . Then there exists an analytic multisheeted function  $E(p)$  (hereafter named *multisheeted band function*) such that

$$E_n(k) = \begin{cases} E\left[\frac{1}{2}(n-1)b + k\right] & \text{if } n \text{ is odd} \\ E\left[\frac{1}{2}nb - k\right] & \text{if } n \text{ is even} \end{cases}, k \in \left[0, \frac{1}{2}b\right]. \quad (5)$$

Furthermore, the multisheeted band function is such that  $E(\bar{p}) = \overline{E(p)}$ .

Let us denote by  $\Gamma$  the complex path defined as in Figure 1, then the multisheeted band function  $E(p)$  monotonically maps  $\Gamma$  onto  $\mathbb{R}$  [9]. In particular  $E(p)$  maps the real intervals  $[\frac{n-1}{2}b, \frac{n}{2}b]$  onto the interval band  $[E_n^b, E_n^t]$ , and it maps the clockwise path surrounding the cut  $\ell_n$  associated to the Kohn branch point  $k_n$  to the gap  $(E_n^t, E_{n+1}^b)$ .

Similarly, the Bloch functions  $\varphi_n(x, k)$  are the branches of a single multisheeted Bloch function  $\varphi(x, p)$  (hereafter named *multisheeted Bloch function*) that has no singularities other than the Kohn branch points. As before, we may write

$$\varphi(x, p) = e^{ipx} u(x, p)$$

where  $u(x, p) = u(x + a, p)$  is a periodic function.

Then the multisheeted band function and the multisheeted Bloch function are defined for any  $p \in \tilde{\mathbb{C}}$  and, in particular, on the set  $\tilde{\mathbb{R}} = \tilde{\mathbb{C}} \cap \mathbb{R}$ , are such that

$$E\left(-\frac{1}{2}nb \pm 0\right) = E\left(\frac{1}{2}nb \mp 0\right) \quad \text{and} \quad \varphi\left(x, -\frac{1}{2}nb \pm 0\right) = \varphi\left(x, \frac{1}{2}nb \mp 0\right) \quad (6)$$

together with their derivatives, for any  $n$  such that the  $n^{\text{th}}$  gap is open. In particular, we have that

$$E\left(-\frac{1}{2}nb + 0\right) = E\left(\frac{1}{2}nb - 0\right) = E_n^t$$

and

$$E\left(-\frac{1}{2}nb - 0\right) = E\left(\frac{1}{2}nb + 0\right) = E_{n+1}^b.$$

#### 4. ASYMPTOTIC BEHAVIOR OF THE MULTISHEETED BLOCH AND BAND FUNCTIONS

We consider the differential equation

$$-\varphi''(x) + V(x)\varphi(x) = E\varphi(x) \quad (7)$$

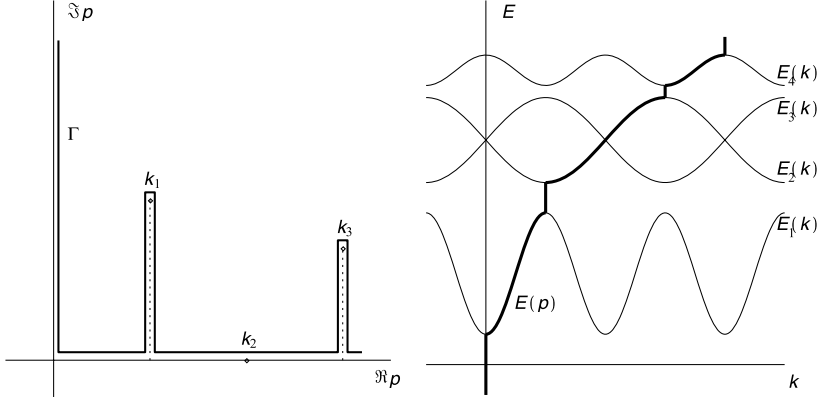


FIGURE 1. In the left panel we plot the path  $\Gamma$  in the quasimomentum complex plane where  $k_1$ ,  $k_2$  and  $k_3$  are the first 3 Kohn's branch points. In the right panel we plot the first 4 band functions and the multisheeted function  $E(p)$  which maps  $\Gamma$  onto the whole real axis (bold line). When  $p$  is real valued then  $E(p)$  takes values in the spectrum of  $H_B$ ; when  $p$  belongs to the imaginary axes then the energy  $E(p)$  lies below the spectrum of  $H_B$ ; and finally, when  $p$  belongs to the part of  $\Gamma$  surrounding the complex cuts  $\ell_n$  then the energy  $E(p)$  belongs to the gaps. In this picture we have the situation where the gap between the 2<sup>nd</sup> and 3<sup>rd</sup> band is empty, this means that the two square root Kohn's branch points  $k_2$  and  $\bar{k}_2$  coincides and then the multisheeted function  $E(p)$  is regular at this points, in fact. We should remark that the *periodic* band function  $E_2(k)$  and  $E_3(k)$ , and the associated Bloch functions, are *singular* at the point  $k = 0$ ; this "artificial" singularity is a consequence of the periodicity of the band functions and from the fact that the gap between the 2<sup>nd</sup> and the 3<sup>rd</sup> band is empty.

where  $V(x) = V(x+a)$  is a smooth periodic function. We denote by  $\phi_1(x, E)$  and  $\phi_2(x, E)$  two independent solutions of this equation with initial conditions

$$\phi_1(0, E) = 1, \phi_1'(0, E) = 0 \quad \text{and} \quad \phi_2(0, E) = 0, \phi_2'(0, E) = 1. \quad (8)$$

The general solution of equation (7) may be written as

$$\varphi := \varphi(x, E) = \alpha \phi_1(x, E) + \beta \phi_2(x, E) \quad (9)$$

where the parameters  $\alpha$  and  $\beta$  are chosen in order to normalize to one the solution  $\varphi$ :

$$\int_0^a |\varphi(x, E)|^2 dx = 1. \quad (10)$$

Let  $p \in \mathbb{R}$  and let

$$\lambda = e^{ipa}.$$

If we look for solutions satisfying the boundary conditions

$$\varphi(a, E) = \lambda\varphi(a, E) \quad \text{and} \quad \varphi'(a, E) = \lambda\varphi'(a, E) \quad (11)$$

then  $\lambda$  should satisfies the equation

$$\lambda^2 - 2\mu(E)\lambda + 1 = 0$$

where  $\mu(E) = \frac{1}{2} [\phi_1(a, E) + \phi_2'(a, E)]$ . In conclusion,  $p$  may be written as function of  $E$  (and viceversa) by means of the relation

$$\cos(pa) = \mu(E), \quad \text{that is} \quad \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right) = \mu(E).$$

The solution  $E = E(p)$  of the equation above is the multisheeted band function, and the associated wave-function  $\varphi = \varphi(x, p) = \varphi[x, E(p)]$  is the multisheeted Bloch function.

**Remark 3.** *In the case of symmetric potentials  $V(x) = V(-x)$  then it follows that  $\phi_1(a, E) = \phi_2'(a, E)$  (see, e.g., §1.3 [17]) and we have that*

$$\mu(E) = \phi_1(a, E).$$

We prove now that

**Lemma 1.** *Let  $\phi_1(x, E)$  and  $\phi_2(x, E)$  be the solutions of Eq. (7) satisfying the initial condition (8). Let  $\varphi(x, E)$  be the normalized solution of Eq. (7) satisfying the boundary conditions (11). Then*

$$\varphi(x, E) = C(E) \left[ \phi_1(x, E) + i\sqrt{E}\phi_2(x, E) \right], \quad C(E) = \left[ \sqrt{\frac{1}{a}} + O(E^{-3/2}) \right], \quad (12)$$

for large  $E$ . The above asymptotic behavior can be derived with respect to  $E$  term by term.

*Proof.* Hereafter, we drop out the dependence on  $E$  when this does not cause misunderstanding. As a first step we prove that the equation (7) has non-trivial solutions with  $\phi_2(a) = 0$  (or  $\phi_1'(a) = 0$ ) only for values of  $E$  such that  $\lambda = \pm 1$ . Indeed, the quasiperiodic boundary conditions implies that

$$\begin{cases} [\phi_1(a) - \lambda]\alpha + \phi_2(a)\beta = 0 \\ \phi_1'(a)\alpha + [\phi_2'(a) - \lambda]\beta = 0 \end{cases}$$

that is

$$\begin{cases} \frac{1}{2} \left[ \frac{1}{\lambda} - \lambda \right] \alpha + \phi_2(a)\beta = 0 \\ \phi_1'(a)\alpha + \frac{1}{2} \left[ \frac{1}{\lambda} - \lambda \right] \beta = 0 \end{cases}$$

If we assume that  $\lambda \neq \pm 1$  then  $\phi_2(a) \neq 0$  and  $\phi_1'(a) \neq 0$ . Indeed, if, for instance,  $\phi_2(a) = 0$  then  $\alpha = 0$  and finally even  $\beta$  should be equal to zero, that is we only have a trivial solution.

Thus, assuming that  $\lambda \neq \pm 1$  the wave function (9) may be written in the form [16] (adapted to the normalization condition (10))

$$\varphi(x) = \frac{\phi_2(a)\phi_1(x) + \frac{1}{2} \left( \frac{1}{\lambda} - \lambda \right) \phi_2(x)}{\sqrt{-2\phi_2(a) \frac{d\mu}{dE}}} = \frac{\phi_2(a)\phi_1(x) + i \sin(pa)\phi_2(x)}{\sqrt{-2\phi_2(a) \frac{d\mu}{dE}}} \quad (13)$$

where we observe that  $\frac{1}{2} \left( \frac{1}{\lambda} - \lambda \right) = i \sin(pa)$ . At  $\lambda = \pm 1$  the solution  $\varphi(x)$  follows from (13) by means of a continuity argument.

Recalling that the non trivial solutions  $\phi_{1,2}$  satisfy the following integral equation (see Lemma 1.7 [23]):

$$\phi_1(x, E) = \cos(\sqrt{E}x) + \frac{1}{\sqrt{E}} \int_0^x \sin[\sqrt{E}(x-y)] V(y) \phi_1(y, E) dy \quad (14)$$

$$\phi_2(x, E) = \frac{1}{\sqrt{E}} \sin(\sqrt{E}x) + \frac{1}{\sqrt{E}} \int_0^x \sin[\sqrt{E}(x-y)] V(y) \phi_2(y, E) dy \quad (15)$$

from which follows the asymptotic behavior for large  $E$ :

$$\begin{aligned} \phi_1(x, E) &= \cos(\sqrt{E}x) + O\left(\frac{1}{\sqrt{E}}\right), \\ \phi_2(x, E) &= \frac{\sin(\sqrt{E}x)}{\sqrt{E}} + O\left(\frac{1}{E}\right), \end{aligned}$$

where the asymptotic behavior is uniform for any  $x \in [0, a]$  and where the asymptotic behavior can be derived with respect to  $x$  and  $E$  term by term; e.g.:

$$\begin{aligned} \frac{\partial \phi_1(x, E)}{\partial x} &= -\sqrt{E} \sin(\sqrt{E}x) + O(1), \\ \frac{\partial \phi_1(x, E)}{\partial \sqrt{E}} &= -x \sin(\sqrt{E}x) + O\left(\frac{1}{\sqrt{E}}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \phi_2(x, E)}{\partial x} &= \cos(\sqrt{E}x) + O\left(\frac{1}{\sqrt{E}}\right), \\ \frac{\partial \phi_2(x, E)}{\partial \sqrt{E}} &= \frac{x \cos(\sqrt{E}x)}{\sqrt{E}} + O\left(\frac{1}{E}\right). \end{aligned}$$

In order to have an asymptotic expression with more terms we iterate equations (14) and (15) obtaining the following asymptotic behavior for large  $E$ , which are uniform for any  $x \in [0, a]$

$$\begin{aligned} \phi_1(x) &= \cos(\sqrt{E}x) + \frac{1}{2\sqrt{E}} \sin(\sqrt{E}x) Q(x) + \frac{1}{4E} [V(x) - V(0)] \cos(\sqrt{E}x) + g_1(x, E) \\ \phi_2(x) &= \frac{\sin(\sqrt{E}x)}{\sqrt{E}} - \frac{1}{2E} \cos(\sqrt{E}x) Q(x) + \frac{1}{4E^{3/2}} [V(x) - V(0)] \sin(\sqrt{E}x) + g_2(x, E) \end{aligned}$$

where  $Q(x) = \int_0^x V(y) dy$  and where

$$\begin{aligned} g_1(x, E) &= -\frac{1}{4E} \int_0^x \cos[\sqrt{E}(x-2y)] V'(y) dy + \frac{1}{2E} \int_0^x V(z) \phi_1(z, E) dz \times \\ &\quad \times \left[ \int_z^x \cos[\sqrt{E}(x+z-2y)] V(y) dy - \cos[\sqrt{E}(x-z)] [Q(z) - Q(x)] \right] \\ g_2(x, E) &= -\frac{1}{4E^{3/2}} \int_0^x \sin[\sqrt{E}(x-2y)] V'(y) dy + \frac{1}{2E^{3/2}} \int_0^x V(z) \phi_2(z, E) dz \times \\ &\quad \times \left[ \int_z^x \cos[\sqrt{E}(x+z-2y)] V(y) dy - \cos[\sqrt{E}(x-z)] [Q(z) - Q(x)] \right] \end{aligned}$$

are such that

$$\begin{aligned} g_1(x, E) &\sim E^{-3/2}, \quad \frac{\partial g_1(x, E)}{\partial x} \sim E^{-1} \quad \text{and} \quad \frac{\partial g_1(x, E)}{\partial E} \sim E^{-3/2} \\ g_2(x, E) &\sim E^{-2}, \quad \frac{\partial g_2(x, E)}{\partial x} \sim E^{-3/2} \quad \text{and} \quad \frac{\partial g_2(x, E)}{\partial E} \sim E^{-2} \end{aligned}$$

by integrating by part. In particular, recalling that  $Q(a) = \int_0^a V(x)dx = 0$ , we have that

$$\begin{aligned} \mu(E) &= \cos(\sqrt{E}a) + \frac{\sin(\sqrt{E}a)}{2\sqrt{E}} \int_0^a V(y)dy + O(E^{-1}) \\ &= \cos(\sqrt{E}a) + O(E^{-1}) \\ \phi_2(a) &= \frac{1}{\sqrt{E}} \sin(\sqrt{E}a) + O(E^{-3/2}) \\ \frac{d\mu}{dE} &= -\frac{a}{2\sqrt{E}} \sin(\sqrt{E}a) + O(E^{-2}) \end{aligned}$$

and thus  $p = \sqrt{E} + O(E^{-1})$ ,  $\sin(pa) = \sin(\sqrt{E}a) + O(E^{-1})$  and

$$\begin{aligned} \varphi(x, E) &= \sqrt{-\frac{\phi_2(a, E)}{2\frac{d\mu}{dE}}} \phi_1(x, E) + \frac{\sin(pa)}{\sqrt{-2\phi_2(a, E)\frac{d\mu}{dE}}} \phi_2(x, E) \\ &= \left[ \sqrt{\frac{1}{a}} + O(E^{-3/2}) \right] \left[ \phi_1(x, E) + i\sqrt{E}\phi_2(x, E) \right] \end{aligned}$$

proving the asymptotic behavior (12).

We have only to discuss the case of non symmetric periodic potentials. In the case of non symmetric potential the expression of the wavefunction  $\varphi$  given by (13) must be replaced by (see Appendix A)

$$\varphi(x) = \frac{\phi_2(a)\phi_1(x) - (\phi_1(a) - \lambda)\phi_2(x)}{\sqrt{-2\phi_2(a)\frac{d\mu}{dE}}} \quad (16)$$

and even in such a case the asymptotic arguments as above may be applied. The Lemma 1 is thus completely proved.  $\square$

Now, we are ready to prove the estimate of  $u(x, E)$  for any complex value  $x$  belonging to the box

$$L = \{x \in \mathbb{C} : \Re x \in [0, a], \Im x \in [-R, 0]\}.$$

In fact, the solution of equation (7) may be extended to complex values  $x$  in the strip  $|\Im x| \leq R$  because the potential  $V$  is an analytic function on such a strip.

**Theorem 2.** *The multsheeted Bloch function  $\varphi(x, E) = e^{ipx}u(x, E)$  is such that*

$$u(x, E) = \frac{1}{\sqrt{a}} - i\frac{Q(x)}{2\sqrt{aE}} + O(E^{-1}), \quad \text{where } Q(x) = \int_{\gamma_{0,x}} V(y)dy, \quad (17)$$

as  $E \rightarrow \infty$  uniformly with respect to  $x \in L$ , and  $\gamma_{0,x}$  is any complex path contained in the box  $L$  and connecting the complex points 0 and  $x$ . The asymptotic behavior

can be derived with respect to  $x$  and  $E$  term by term; in particular,

$$\frac{\partial u(x, E)}{\partial E} = i \frac{Q(x)}{4E^{3/2}\sqrt{a}} + O(E^{-2})$$

and

$$\frac{\partial^2 u(x, E)}{\partial E^2} = -i \frac{3}{8} \frac{Q(x)}{E^{5/2}\sqrt{a}} + O(E^{-3}).$$

*Proof.* From (12) it follows that

$$\varphi(x, E) = C(E)\phi(x, E), \quad \phi(x, E) := \phi_1(x, E) + i\sqrt{E}\phi_2(x, E)$$

where, recalling (14) and (15), it follows that

$$\phi(x, E) = e^{i\sqrt{E}x} + \frac{1}{\sqrt{E}} \int_{\gamma_{0,x}} \sin[\sqrt{E}(x-y)] V(y)\phi(y, E) dy.$$

If we choose the path  $\gamma_{0,x}$  such that  $\Im x \leq \Im y$  for any  $y \in \gamma_{0,x}$ , and if we set

$$\phi(x, E) = e^{i\sqrt{E}x} v(x, E)$$

then  $v(x, E)$  satisfies to the following integral equation

$$\begin{aligned} v(x, E) &= 1 + \frac{1}{\sqrt{E}} \int_{\gamma_{0,x}} \sin[\sqrt{E}(x-y)] e^{-i\sqrt{E}(x-y)} V(y) v(y, E) dy \\ &= 1 + \frac{1}{\sqrt{E}} \int_{\gamma_{0,x}} \sin[\sqrt{E}(x-y)] e^{-i\sqrt{E}(x-y)} V(y) dy + r(x, E) \end{aligned}$$

where

$$\begin{aligned} r(x, E) &= \frac{1}{E} \int_{\gamma_{0,x}} \sin[\sqrt{E}(x-y)] e^{-i\sqrt{E}(x-y)} V(y) \times \\ &\quad \times \int_{\gamma_{0,y}} \sin[\sqrt{E}(y-z)] e^{-i\sqrt{E}(y-z)} V(z) v(z, E) dz dy \end{aligned}$$

and

$$\sin[\sqrt{E}(x-y)] e^{-i\sqrt{E}(x-y)} = \frac{1 - e^{-2i\sqrt{E}(x-y)}}{2i}$$

is such that

$$\left| e^{-2i\sqrt{E}(x-y)} \right| = e^{2\sqrt{E}(\Im x - \Im y)} \leq 1, \quad \text{for any } y \in \gamma_{0,x}.$$

Hence

$$v(x, E) = 1 - i \frac{Q(x)}{2\sqrt{E}} + \frac{i}{2\sqrt{E}} \int_{\gamma_{0,x}} e^{-2i\sqrt{E}(x-y)} V(y) dy + r(x, E)$$

If we set

$$\nu(E) = \sup_{x \in L} |v(x, E)|$$

then from the equation above it turns out that

$$\nu \leq 1 + \frac{C_1}{\sqrt{E}} + \frac{C_2}{E} \nu$$

for some positive constants  $C_1$  and  $C_2$ . From this fact, it follows that  $\nu \leq C$  for some positive constant  $C$  uniformly with respect to  $E$ . Hence we can conclude that

$$v(x, E) = 1 - i \frac{Q(x)}{2\sqrt{E}} + \mathcal{O}(E^{-1})$$

in the limit of large  $E$ . Finally

$$u(x, E) = e^{-ipx} \varphi(x, E) = e^{-ipx} C(E) \phi(x, E) = C(E) e^{-ipx} e^{i\sqrt{E}x} v(x, E)$$

from which (17) follows for any  $x \in L$ . We close by underlining that, by construction, the asymptotic behavior can be derived term by term.  $\square$

### 5. STARK-WANNIER OPERATOR IN THE EXTENDED CRYSTAL MOMENTUM REPRESENTATION

The CMR of the Stark-Wannier operator is usually defined on  $\mathcal{H} = \otimes_{n=1}^{+\infty} L^2(\mathcal{B}, dk)$  as (see [4])

$$\left[ U H_F U^{-1} \hat{\psi} \right]_m(k) = iF \frac{\partial \hat{\psi}_m(k)}{\partial k} + E_m(k) \hat{\psi}_m(k) + F \sum_{n \in \mathbb{N}} X_{m,n}(k) \hat{\psi}_n(k),$$

where  $U$  is defined by (4) and the interband coupling terms are defined as

$$X_{m,n}(k) = \frac{i}{a} \int_0^a \frac{\overline{u_m(x, k)}}{u_m(x, k)} \frac{\partial u_n(x, k)}{\partial k} dx. \quad (18)$$

This usual approach is not suitable in order to apply analytic distortion techniques; hence, as a preliminary step, by making use of the analytic properties of the multisheeted band and Bloch functions  $E(p)$  and  $\varphi(x, p)$ , we define the Bloch decomposition formula and the CMR in an equivalent form which will be useful in the sequel (it will be called *extended* crystal momentum representation, or simply ECMR).

The transformation  $U$  can be rewritten as  $\hat{\psi} = U_E \psi$  where

$$\hat{\psi}(p) = \int_{\mathbb{R}} \overline{\varphi(x, p)} \psi(x) dx, p \in \tilde{\mathbb{R}}.$$

The inverse transformation is defined as

$$\left( U_E^{-1} \hat{\psi} \right)(x) = \frac{1}{2\pi} \int_{\tilde{\mathbb{R}}} \varphi(x, p) \hat{\psi}(p) dp.$$

**Lemma 2.** *Let  $\psi(x)$  be a rapidly decreasing function: that is for any  $\alpha, \beta \in \mathbb{N}$  there exists a positive constant  $C = C_{\alpha, \beta}$  such that*

$$\left| x^\alpha \frac{d^\beta \psi(x)}{dx^\beta} \right| \leq C, \quad \forall x \in \mathbb{R}.$$

*Then  $\hat{\psi} = U_E \psi$  is a rapidly decreasing function, too. And vice versa.*

*Proof.* Let

$$\hat{\psi}(p) = \int_{\mathbb{R}} e^{-ipx} \overline{u(x, p)} \psi(x) dx.$$

Then, by integrating by parts

$$p^\alpha \hat{\psi}(p) = \int_{\mathbb{R}} p^\alpha e^{-ipx} \overline{u(x, p)} \psi(x) dx = (-i)^\alpha \int_{\mathbb{R}} e^{-ipx} \frac{\partial^\alpha}{\partial x^\alpha} \left[ \overline{u(x, p)} \psi(x) \right] dx$$

is bounded because  $\overline{u(x, p)}$  is a smooth periodic function with respect to  $x$  and  $\psi(x)$  is a rapidly decreasing function. For the same reason it follows that

$$\frac{d^\beta \hat{\psi}(p)}{dp^\beta} = \int_{\mathbb{R}} \frac{\partial^\beta \left( e^{-ipx} \overline{u(x, p)} \right)}{\partial p^\beta} \psi(x) dx$$

converges. Finally, the estimates are uniformly with respect to  $p$  because of Theorem 2. Similarly, the vice versa follows.  $\square$

**Theorem 3.** *If we define*

$$\tilde{H}_F = U_E H_F U_E^{-1}$$

*then it acts as*

$$\left[ \tilde{H}_F \hat{\psi} \right] (p) = E(p) \hat{\psi}(p) + iF \frac{\partial \hat{\psi}(p)}{\partial p} + F \left( X \hat{\psi} \right) (p), \quad p \in \tilde{\mathbb{R}}, \quad (19)$$

*where  $X$  represents the interband coupling term:*

$$\left( X \hat{\psi} \right) (p) = \sum_{j \in \mathbb{Z}} C_j(p) \hat{\psi}(p - jb).$$

*where*

$$C_j(p) = \frac{i}{a} \int_0^a e^{-ibjx} \overline{u(x, p)} \frac{\partial u(x, p - bj)}{\partial p} dx.$$

*Proof.* Indeed, let  $\hat{\psi}$  be a test function (e.g.  $\hat{\psi}$  is a rapidly decreasing function)

$$\left[ H_F U_E^{-1} \hat{\psi} \right] (x) = \frac{1}{2\pi} \int_{\tilde{\mathbb{R}}} H_F \varphi(x, p) \hat{\psi}(p) dp$$

where

$$H_F \varphi(x, p) = H_B \varphi(x, p) + Fx \varphi(x, p) = E(p) \varphi(x, p) + Fx \varphi(x, p)$$

and where

$$\begin{aligned} \int_{\tilde{\mathbb{R}}} x \varphi(x, p) \hat{\psi}(p) dp &= \int_{\tilde{\mathbb{R}}} x e^{ipx} u(x, p) \hat{\psi}(p) dp = -i \int_{\tilde{\mathbb{R}}} \frac{\partial e^{ipx}}{\partial p} u(x, p) \hat{\psi}(p) dp \\ &= -i \left[ \varphi(x, p) \hat{\psi}(p) \right]_{\partial \tilde{\mathbb{R}}} + i \int_{\tilde{\mathbb{R}}} e^{ipx} \frac{\partial u(x, p)}{\partial p} \hat{\psi}(p) dp + i \int_{\tilde{\mathbb{R}}} \varphi(x, p) \frac{\partial \hat{\psi}(p)}{\partial p} dp \end{aligned}$$

because  $\varphi(x, p)$  and  $\hat{\psi}(p)$  have the same values, together with their derivatives, at the boundary points of  $\tilde{\mathbb{R}}$  (see Eq. (6)). Hence (19) follows where

$$\left( X \hat{\psi} \right) (p) = \frac{i}{2\pi} \int_{\tilde{\mathbb{R}}} \overline{\varphi(x, p)} \int_{\tilde{\mathbb{R}}} e^{ip'x} \frac{\partial u(x, p')}{\partial p} \hat{\psi}(p') dp' dx. \quad (20)$$

In particular, a formal calculation lead us to the following result

$$\begin{aligned}
(X\hat{\psi})(p) &= \frac{i}{2\pi} \int_{\tilde{\mathbb{R}}} \left[ \int_{\mathbb{R}} \overline{\varphi(x,p)} e^{ip'x} \frac{\partial u(x,p')}{\partial p} dx \right] \hat{\psi}(p') dp' & (21) \\
&= \frac{i}{2\pi} \int_{\tilde{\mathbb{R}}} \left[ \int_{\mathbb{R}} \overline{u(x,p)} e^{i(p'-p)x} \frac{\partial u(x,p')}{\partial p} dx \right] \hat{\psi}(p') dp' \\
&= \frac{i}{2\pi} \int_{\tilde{\mathbb{R}}} \left[ \int_{\mathbb{R}} \sum_{j \in \mathbb{Z}} \alpha_j(p,p') e^{i(p'-p+bj)x} dx \right] \hat{\psi}(p') dp' \\
&= i \sum_{j \in \mathbb{Z}} \int_{\tilde{\mathbb{R}}} \alpha_j(p,p') \hat{\psi}(p') \delta(p' - p + bj) dp' & (22) \\
&= \sum_{j \in \mathbb{Z}} C_j(p) \hat{\psi}(p - jb)
\end{aligned}$$

because  $\overline{u(x,p)} \frac{\partial u(x,p')}{\partial p}$  is a periodic function with respect to  $x$  with period  $a$  and where we set

$$\alpha_j(p,p') = \frac{1}{a} \int_0^a e^{-ibjx} \overline{u(x,p)} \frac{\partial u(x,p')}{\partial p} dx, \quad b = \frac{2\pi}{a},$$

and

$$C_j(p) := \alpha_j(p, p - jb) = \frac{i}{a} \int_0^a e^{-ibjx} \overline{u(x,p)} \frac{\partial u(x, p - bj)}{\partial p} dx.$$

In fact, we only have to justify the exchange of integration from (20) to (21) and the exchange of the operation of sum, with respect to  $j$ , and integration from (21) to (22). To this end we make use of the fact that  $\hat{\psi}$  is a rapidly decreasing function, from a standard mollifier argument and from the following technical result concerning the decreasing behavior of  $C_j(p)$  with respect of  $j$  and  $p$ .

**Lemma 3.** *Let the periodic potential regular enough, that is  $V \in C^r$  for some  $r > 0$ , then there exists a positive constant  $C := C_r$  independent of  $j$  and  $p$  such that*

$$|C_j(p)| \leq C[|p| + 1]^{-2}[|j| + 1]^{-r}, \quad \forall p \in \tilde{\mathbb{R}} \quad \text{and} \quad j \in \mathbb{Z}.$$

*Proof.* First of all we remark that from the results obtained in Theorem 2 it follows that (let us denote  $u(x, E) = u[x, E(p)]$  instead of  $u(x, p)$  where this does not cause misunderstanding)

$$\begin{aligned}
C_j(p) &= \frac{i}{a} \int_0^a e^{-ibjx} \overline{u(x,p)} \frac{\partial u(x, p - bj)}{\partial p} dx \\
&= \frac{i}{a} \int_0^a e^{-ibjx} \overline{u(x, E)} \Big|_{E=E(p)} \frac{\partial u(x, E)}{\partial E} \Big|_{E=E(p-bj)} \frac{dE(p - bj)}{dp} dx
\end{aligned}$$

is uniformly bounded for any  $j$  and any  $p$ :

$$|C_j(p)| \leq C, \quad \forall j \in \mathbb{Z}, \quad \forall p \in \tilde{\mathbb{R}},$$

for some positive constant  $C > 0$ . Furthermore, equation (2) can be rewritten as

$$\delta_j^0 = \frac{1}{a} \int_0^a \overline{\varphi(x,p)} \varphi(x, p - bj) dx = \frac{1}{a} \int_0^a \overline{u(x,p)} u(x, p - bj) e^{-ibjx} dx$$

then it follows that

$$C_j(p) = -\frac{i}{a} \int_0^a e^{-ibjx} u(x, p - bj) \frac{\overline{\partial u(x, p)}}{\partial p} dx = \overline{C_{-j}(p - bj)}. \quad (23)$$

Then, we estimate the terms  $C_j(p + bj)$  by making use of the asymptotic estimate for  $j$  and  $p$  large. To this end let  $E = E(p)$  and  $E_j = E(p + bj)$ ; then, from Theorem 2, recalling that  $E(p) \sim p^2$  and integrating by parts  $r$  times it follows that

$$\begin{aligned} C_j(p + bj) &= \frac{i}{a} \int_0^a e^{-ibjx} \overline{u(x, E_j)} \frac{\partial u(x, E)}{\partial E} \frac{dE}{dp} dx \\ &\sim \frac{2ip}{a^2} \int_0^a e^{-ijbx} \left[ 1 - i \frac{Q(x)}{2\sqrt{E_j}} \right] [1 + O(E_j^{-1})] i \frac{Q(x)}{4E^{3/2}} [1 + O(E^{-1/2})] 2\sqrt{E} dx \\ &\sim O(p^{-2}j^{-r}) [1 + O(E_j^{-1})] [1 + O(E^{-1/2})] \end{aligned}$$

provided that  $V \in C^r$ .  $\square$

Thus Theorem 3 is completely proved since the set of rapidly decreasing functions is a dense set in  $L^2(\tilde{R}, dp)$ .  $\square$

**Remark 4.** From (23) it turns out

$$C_0(p) = \overline{C_0(p)}$$

is real valued. In particular, if the periodic potential is a symmetric function, then  $\overline{u(x, p)} = u(-x, p)$  and we have that

$$\begin{aligned} C_0(p) &= \frac{i}{a} \int_{-a/2}^{+a/2} u(-x, p) \frac{\partial u(x, p)}{\partial p} dx = \frac{i}{a} \int_{-a/2}^{+a/2} u(x, p) \frac{\partial u(-x, p)}{\partial p} dx \\ &= \frac{i}{a} \int_{-a/2}^{+a/2} u(x, p) \frac{\overline{\partial u(x, p)}}{\partial p} dx = -\frac{i}{a} \int_{-a/2}^{+a/2} \overline{u(x, p)} \frac{\partial u(x, p)}{\partial p} dx = -C_0(p) \end{aligned}$$

Hence  $C_0(p) \equiv 0$  is the case of symmetric periodic potentials.

Now, we have the following property

**Theorem 4.** Let  $V \in C^2$ . Then the linear operator  $X$  from  $L^2(\tilde{\mathbb{R}}, dp)$  to  $L^2(\tilde{\mathbb{R}}, dp)$  is bounded.

*Proof.* Let us fix a real and positive number  $M > 0$  large enough and let us compute the norm of  $X\hat{\psi}$ :

$$\begin{aligned} \|X\hat{\psi}\|_{L^2(\tilde{R}, dp)}^2 &= \int_{\tilde{\mathbb{R}}} \left| \sum_{j \in \mathbb{Z}} C_j(p) \hat{\psi}(p - bj) \right|^2 dp = \int_{\tilde{\mathbb{R}}} \left| \sum_{j \in \mathbb{Z}} C_j(p + bj) \hat{\psi}(p) \right|^2 dp \\ &\leq \int_{\tilde{\mathbb{R}}} \left[ \sum_{j \in \mathbb{Z}} |C_j(p + bj)| \right]^2 |\hat{\psi}(p)|^2 dp \\ &\leq 2 \int_{\tilde{\mathbb{R}}} \left[ \sum_{j^*} |C_j(p + bj)| \right]^2 |\hat{\psi}(p)|^2 dp + 2 \int_{\tilde{\mathbb{R}}} \left[ \sum_{j^{**}} |C_j(p + bj)| \right]^2 |\hat{\psi}(p)|^2 dp \end{aligned}$$

where  $\sum_{j^*}$  is the sum for all  $j \in \mathbb{Z}$  such that  $|p + bj| > M$  and  $|j| > M$ ;  $\sum_{j^{**}}$  is the sum for all  $j \in \mathbb{Z}$  such that  $|p + bj| \leq M$  or  $|j| \leq M$ . The second integral is simply estimated as follow

$$\int_{\tilde{\mathbb{R}}} \left[ \sum_{j^{**}} |C_j(p + bj)| \right]^2 |\hat{\psi}(p)|^2 dp \leq C^2 M^2 \|\hat{\psi}\|_{L^2(\tilde{\mathbb{R}}, dp)}^2.$$

since from Lemma 3 it follows that  $|C_j(p + bj)| \leq C$  for some positive constant  $C$ . Concerning the first integral we estimate the terms  $C_j(p + bj)$  by making use again of Lemma 3 obtaining that

$$\begin{aligned} \int_{\tilde{\mathbb{R}}} \left[ \sum_{j^*} |C_j(p + bj)| \right]^2 |\hat{\psi}(p)|^2 dp &\leq \int_{\tilde{\mathbb{R}}} \left[ \sum_{j^*} C[|p + j|^2 + 1]^{-2} |j|^{-r} \right]^2 |\hat{\psi}(p)|^2 dp \\ &\leq C \|\hat{\psi}\|_{L^2(\tilde{\mathbb{R}}, dp)}^2 \end{aligned}$$

provided that  $r \geq 2$ . Thus, the boundedness of the linear operator  $X$  follows.  $\square$

**Remark 5.** *In fact, the boundedness of the interband operator has been previously proved for potentials such that the associated Bloch operator has spectrum with only a finite number of open gaps [11] or, in the opposite situation, for potentials such that all the gaps are open and with width bigger than a positive constant  $C$ , for some  $C > 0$  fixed (see [18] in the case of periodic potential given by a periodic sequence of  $\delta'$  distributions). With Theorem 4 we eventually fill that gap between these two opposite situations.*

## 6. ANALYTIC DISTORTION OF THE STARK-WANNIER OPERATOR IN THE ECMR

Since translation in the  $x$ -space corresponds, by means of the Fourier transformation, to multiply by an exponential function in the momentum representation then we adopt the following strategy: we consider the Stark-Wannier operator  $\tilde{H}_F$  in the ECMR and its analytic deformation consists by multiplying the vector  $\hat{\psi}$  by  $e^{i\theta p}$ , for  $p$  outside a given interval.

Let  $N \geq 1$  be fixed and such that the gap between the  $N$ -th band and the  $(N + 1)$ -th band is not empty:  $E_{N+1}^b - E_N^b > 0$ . Let  $\Gamma_1 = (-\frac{1}{2}Nb, +\frac{1}{2}Nb) \cap \tilde{\mathbb{R}}$  and  $\Gamma_c = \tilde{\mathbb{R}} \setminus \Gamma_1$ . Then we define

**Definition 1.** *Let  $\mathcal{H}_E$  be the space of vectors  $\hat{\psi} \in L^2(\tilde{\mathbb{R}})$  satisfying the periodic boundary conditions at  $\pm\frac{1}{2}Nb$ ; that is*

$$\frac{d^r \hat{\psi}(-\frac{1}{2}Nb \pm 0)}{dp^r} = \frac{d^r \hat{\psi}(+\frac{1}{2}Nb \mp 0)}{dp^r}, \quad \forall r \in \mathbb{N}. \quad (24)$$

Hence,  $\mathcal{H}_E$  can be identified with  $L^2(\Gamma_1) \otimes L^2(\Gamma_c)$  and  $\hat{\psi}$  with a couple of vectors  $(\hat{\psi}_1, \hat{\psi}_c)$  such that  $\hat{\psi}_1(-\frac{1}{2}Nb + 0) = \hat{\psi}_1(+\frac{1}{2}Nb - 0)$  and  $\hat{\psi}_c(-\frac{1}{2}Nb - 0) = \hat{\psi}_c(+\frac{1}{2}Nb + 0)$  together with their derivatives. The vector  $(\hat{\psi}_1, \hat{\psi}_c) = U_E \psi$  will be denoted the EMCR of  $\psi$  and the unitary transformation from  $L^2(\mathbb{R}, dx)$  to  $\mathcal{H}_E$  acts as

$$\hat{\psi}_1(p) = \int_{\mathbb{R}} \overline{\varphi(x, p)} \psi(x) dx, \quad p \in \Gamma_1, \quad \hat{\psi}_c(p) = \int_{\mathbb{R}} \overline{\varphi(x, p)} \psi(x) dx, \quad p \in \Gamma_c$$

with inverse

$$\psi(x) = \frac{1}{2\pi} \int_{\Gamma_1} \varphi(x, p) \hat{\psi}_1(p) dp + \frac{1}{2\pi} \int_{\Gamma_c} \varphi(x, p) \hat{\psi}_c(p) dp.$$

We define now the analytic distortion.

**Definition 2.** Let  $\theta \in \mathbb{C}$  be fixed and such that  $-R \leq \Im\theta \leq 0$ , we define the analytic deformation

$$\hat{\psi}^\theta = (\hat{\psi}_1^\theta, \hat{\psi}_c^\theta) = \tilde{\mathcal{U}}^\theta \hat{\psi} = U_E^\theta \psi, \quad U_E^\theta := \mathcal{U}^\theta U_E$$

as follows

$$\begin{cases} \hat{\psi}_1^\theta(p) = \hat{\psi}_1(p) \\ \hat{\psi}_c^\theta(p) = e^{ip\theta} \hat{\psi}_c(p) \end{cases},$$

where  $\hat{\psi}_1$  and  $\hat{\psi}_c$  are defined in Definition 1, with inverse

$$\psi(x) = \frac{1}{2\pi} \int_{\Gamma_1} \varphi(x, p) \hat{\psi}_1^\theta(p) dp + \frac{1}{2\pi} \int_{\Gamma_c} e^{-ip\theta} \varphi(x, p) \hat{\psi}_c^\theta(p) dp,$$

$U_E^\theta$  is an unitary transformation for real  $\theta$ .

We define the analytic distortion of the Stark-Wannier operator in the ECMR as follows

$$\tilde{H}_F^\theta = U_E^\theta H_F (U_E^\theta)^{-1}. \quad (25)$$

Now, we are going to give an explicit expression of  $\tilde{H}_F^\theta$ . Let  $\hat{\psi}^\theta$  be a test function such that  $e^{-ip\theta} \hat{\psi}_c^\theta(p)$  is a rapidly decreasing function (where the set of such functions is dense in  $\mathcal{H}_E$ ), then

$$\begin{aligned} [H_F (U_E^\theta)^{-1} \hat{\psi}^\theta](x) &= \frac{1}{2\pi} \int_{\Gamma_1} [E(p)\varphi(x, p) + Fx\varphi(x, p)] \hat{\psi}_1^\theta(p) dp + \\ &+ \frac{1}{2\pi} \int_{\Gamma_c} [E(p)\varphi(x, p) + Fx\varphi(x, p)] e^{-ip\theta} \hat{\psi}_c^\theta(p) dp. \end{aligned}$$

In particular, we have that

$$\begin{aligned} \int_{\Gamma_c} x\varphi(x, p) e^{-ip\theta} \hat{\psi}_c^\theta(p) dp &= \int_{\Gamma_c} x e^{ipx} u(x, p) e^{-ip\theta} \hat{\psi}_c^\theta(p) dp = \\ &= \int_{\Gamma_c} \frac{\partial(-ie^{ipx})}{\partial p} u(x, p) e^{-ip\theta} \hat{\psi}_c^\theta(p) dp \\ &= \left[ -i\varphi(x, p) e^{-ip\theta} \hat{\psi}_c^\theta(p) \right]_{\partial\Gamma_c} + i \int_{\Gamma_c} e^{ipx} \left[ u(x, p) \frac{\partial e^{-ip\theta} \hat{\psi}_c^\theta(p)}{\partial p} + \frac{\partial u(x, p)}{\partial p} e^{-ip\theta} \hat{\psi}_c^\theta(p) \right] dp \\ &= i \int_{\Gamma_c} \varphi(x, p) \frac{\partial e^{-ip\theta} \hat{\psi}_c^\theta(p)}{\partial p} dp + i \int_{\Gamma_c} e^{ipx} \frac{\partial u(x, p)}{\partial p} e^{-ip\theta} \hat{\psi}_c^\theta(p) dp \\ &= i \int_{\Gamma_c} \left[ -i\theta \hat{\psi}_c^\theta(p) + \frac{\partial \hat{\psi}_c^\theta(p)}{\partial p} \right] e^{-i\theta p} \varphi(x, p) dp + i \int_{\Gamma_c} e^{ipx} \frac{\partial u(x, p)}{\partial p} e^{-ip\theta} \hat{\psi}_c^\theta(p) dp \end{aligned}$$

because both functions  $e^{-ip\theta} \hat{\psi}_c^\theta(p)$  and  $\varphi(x, p)$  have the same value at the boundary points of  $\Gamma_c$  (and  $e^{-ip\theta} \hat{\psi}_c^\theta(p)$  goes to zero as  $p$  goes to infinity). Similarly, we treat

the integral over the set  $\Gamma_1$ . Hence,

$$\left[ \tilde{H}_F^\theta \hat{\psi}^\theta \right] (p) = \begin{cases} E(p) \hat{\psi}_1^\theta(p) + iF \frac{\partial \hat{\psi}_1^\theta(p)}{\partial p} + F h_1^\theta(p) & , \text{ if } p \in \Gamma_1 \\ E(p) \hat{\psi}_c^\theta(p) + iF \frac{\partial \hat{\psi}_c^\theta(p)}{\partial p} + F h_c^\theta(p) + F \theta \hat{\psi}_c^\theta & , \text{ if } p \in \Gamma_c \end{cases}$$

where

$$h_1^\theta(p) = \frac{i}{2\pi} \int_{\mathbb{R}} \overline{\varphi(x, p)} \left\{ \int_{\Gamma_1} e^{iqx} \frac{\partial u(x, q)}{\partial q} \hat{\psi}_1^\theta(q) dq + \int_{\Gamma_c} e^{iqx} \frac{\partial u(x, q)}{\partial q} e^{-iq\theta} \hat{\psi}_c^\theta(q) dq \right\} dx,$$

and

$$h_c^\theta(p) = \frac{ie^{ip\theta}}{2\pi} \int_{\mathbb{R}} \overline{\varphi(x, p)} \left\{ \int_{\Gamma_1} e^{iqx} \frac{\partial u(x, q)}{\partial q} \hat{\psi}_1^\theta(q) dq + \int_{\Gamma_c} e^{iqx} \frac{\partial u(x, q)}{\partial q} e^{-iq\theta} \hat{\psi}_c^\theta(q) dq \right\} dx.$$

By making use of the same arguments as in Theorem 3 then we can exchange the integration sets obtaining that

$$\begin{aligned} h_1^\theta(p) &= \int_{\Gamma_1} Y(p, q) \hat{\psi}_1^\theta(q) dq + \int_{\Gamma_c} Y(p, q) e^{-iq\theta} \hat{\psi}_c^\theta(q) dq, \quad p \in \Gamma_1 \\ h_c^\theta(p) &= e^{ip\theta} \int_{\Gamma_1} Y(p, q) \hat{\psi}_1^\theta(q) dq + \int_{\Gamma_c} Y(p, q) e^{i(p-q)\theta} \hat{\psi}_c^\theta(q) dq, \quad p \in \Gamma_c \end{aligned}$$

where

$$Y(p, q) = \frac{i}{2\pi} \int_{\mathbb{R}} e^{-i(p-q)x} \overline{u(x, p)} \frac{\partial u(x, q)}{\partial q} dx.$$

**Remark 6.** We should point out that  $Y(p, q)$  is well defined because the  $N$ -th gap is open; if not the periodic boundary conditions of the band and Bloch functions at the points  $\pm \frac{1}{2}Nb$  would imply an artificial singularity for the Bloch function at  $q = \mp \frac{1}{2}Nb \pm 0$  and thus  $Y(p, q)$  and  $\frac{\partial u(x, q)}{\partial q}$  are not well defined in these points.

**Lemma 4.** If the periodic potential  $V(x)$  is an analytic function in a set containing the strip  $|\Im x| \leq R$  then the Fourier series of the periodic function  $\overline{u(x, p)} \frac{\partial u(x, q)}{\partial q}$

$$\overline{u(x, p)} \frac{\partial u(x, q)}{\partial q} = \sum_{j \in \mathbb{Z}} \alpha_j(p, q) e^{ijbx} \quad (26)$$

has coefficients

$$\alpha_j(p, q) = \frac{1}{a} \int_0^a \overline{u(x, p)} \frac{\partial u(x, q)}{\partial q} e^{-ijbx} dx$$

satisfying the following estimates

$$|\alpha_j(p, q)|, \left| \frac{\partial \alpha_j(p, q)}{\partial p} \right|, \left| \frac{\partial \alpha_j(p, q)}{\partial q} \right| \leq C e^{-|j|bR}. \quad (27)$$

for some positive constant independent of  $j$ ,  $p$  and  $q$ .

*Proof.* Let us assume for a moment  $j > 0$ . Then, from the Cauchy Theorem it follows that

$$\begin{aligned} \alpha_j(p, q) &= \frac{1}{a} \int_0^a \overline{u(x, -p)} \frac{\partial u(x, q)}{\partial q} e^{-ijbx} dx \\ &= e^{-jbR} \frac{1}{a} \int_0^a \overline{u(x - iR, -p)} \frac{\partial u(x - iR, q)}{\partial q} e^{-ijbx} dx \end{aligned}$$

where  $u(x - iR, -p)$  and  $\frac{\partial u(x - iR, q)}{\partial q} = \frac{\partial u(x - iR, E)}{\partial E} \frac{dE}{dq}$  are uniformly bounded since Theorem 2. That is it follows that

$$|\alpha_j(p, q)| \leq C(p, q)e^{-|j|bR}$$

where

$$C(p, q) = \max_{-R \leq \Im x \leq 0} \left| u(x, p) \frac{\partial u(x, q)}{\partial q} \right|. \quad (28)$$

In fact, from the asymptotic behavior in Theorem 2 then it follows that  $\frac{\partial u(x, q)}{\partial q} = \frac{\partial u(x, E)}{\partial E} \frac{dE}{dq}$ , where  $\left| \frac{dE}{dq} \right| \leq C|q|$  and  $\left| \frac{\partial u(x, E)}{\partial E} \right| = O(q^{-3})$ ; hence, there exists a positive constant  $C$  such that

$$C(p, q) \leq C$$

for any  $p$  and  $q$ . In the case  $j = -|j| < 0$  then we estimate its complex conjugate

$$\overline{\alpha_j(p, q)} = \frac{1}{a} \int_0^a u(x, p) \frac{\partial u(x, -q)}{\partial q} e^{-i|j|bx} dx$$

by means of the same arguments.

In a similar way we get the estimates of the derivatives of  $\alpha_j$ . With more details, the estimate (27) of the derivative

$$\frac{\partial \alpha_j(p, q)}{\partial p} = \frac{1}{a} \int_0^a \frac{\partial u(x, p)}{\partial p} \frac{\partial u(x, q)}{\partial q} e^{-ijbx} dx$$

immediately follows by means of the same arguments as above. In order to get the estimate (27) of the derivative

$$\frac{\partial \alpha_j(p, q)}{\partial q} = \frac{1}{a} \int_0^a \frac{\partial^2 u(x, q)}{\partial q^2} e^{-ijbx} dx$$

we have to control the second derivative, with respect to  $q$ , of  $u(x, q)$ ; it comes by deriving both sides of the asymptotic behavior given in Theorem 2 obtaining

$$\frac{\partial^2 u(x, q)}{\partial q^2} = \left( \frac{dE}{dq} \right)^2 \frac{\partial^2 u(x, E)}{\partial E^2} \sim q^{-3}.$$

The Lemma thus follows.  $\square$

From this fact and recalling that the Fourier transform of  $e^{ibjx}$  is the Dirac's  $\delta$  distribution  $\delta(x - jb)$  the it follows that

$$Y(p, q) = \sum_{j \in \mathbb{Z}} \alpha_j(p, q) \delta(p - q - jb)$$

where the coefficients  $\alpha_j(p, q)$  satisfy the above estimates. Collecting all these facts and remembering that  $C_0(p)$  is a real valued function as discussed in Remark 4, we can conclude that

**Lemma 5.** *We have that the analytically distorted ECMR of the Stark-Wannier operator (25) is given by  $\tilde{H}_{F, \eta}^\theta$  for  $\eta = F$ , where  $\tilde{H}_{F, \eta}^\theta$  is formally defined on  $\hat{\psi} \in \mathcal{H}_E$*

as

$$\left(\tilde{H}_{F,\eta}^\theta \hat{\psi}\right)(p) = \begin{cases} \left[ iF \frac{d}{dp} + E(p) \right] \hat{\psi}_1(p) + F \left( K_{11} \hat{\psi}_1 \right)(p) + \eta \left( K_{1c}^\theta \hat{\psi}_c \right)(p) & , p \in \Gamma_1 \\ \left[ iF \frac{d}{dp} + E(p) + FC_0(p) \right] \hat{\psi}_c(p) + F\theta \hat{\psi}_c(p) + \\ + \eta \left( K_{c1}^\theta \hat{\psi}_1 \right)(p) + \eta \left( K_{cc}^\theta \hat{\psi}_c \right)(p) & , p \in \Gamma_c \end{cases}$$

where  $K_{11}$  is the intraband interaction among the first  $N$  bands defined as

$$\left( K_{11} \hat{\psi}_1 \right)(p) = \sum_{j \in \mathbb{Z} : p-bj \in \Gamma_1} \alpha_j(p, p-bj) \hat{\psi}_1(p-bj), \quad p \in \Gamma_1, \quad (29)$$

$K_{1c}$  and  $K_{c1}$  are the interband term between the first  $N$  bands and the remainder bands defined as

$$\begin{aligned} \left( K_{1c}^\theta \hat{\psi}_c \right)(p) &= \sum_{j \in \mathbb{Z} : p-bj \in \Gamma_c} \alpha_j(p, p-bj) e^{-i(p-bj)\theta} \hat{\psi}_c(p-bj), \quad p \in \Gamma_1 \\ \left( K_{c1}^\theta \hat{\psi}_1 \right)(p) &= \sum_{j \in \mathbb{Z} : p-bj \in \Gamma_1} \alpha_j(p, p-bj) e^{ip\theta} \hat{\psi}_1(p-bj), \quad p \in \Gamma_c \end{aligned}$$

and finally where  $K_{cc}^\theta$  is defined as

$$\left( K_{cc}^\theta \hat{\psi}_c \right)(p) = \sum_{j \in \mathbb{Z} \setminus \{0\} : p-bj \in \Gamma_c} \alpha_j(p, p-bj) e^{-ijb\theta} \hat{\psi}_c(p-bj), \quad p \in \Gamma_c.$$

In conclusion, we are able to obtain the explicit expression of the analytically distorted ECMR of the Stark-Wannier operator preparing the ground in order to apply the regular perturbation theory. In fact, by construction it turns out that  $\tilde{H}_{F,0}^\theta$  is the decoupled band approximation where the coupling term between the first  $N$  bands and the other bands is neglected.

## 7. SPECTRAL ANALYSIS OF THE UNPERTURBED OPERATOR $\tilde{H}_{F,0}^\theta$

Hereafter, let us fix  $N = 1$  for the sake of simplicity. We introduce a change of notation: in the case of  $N = 1$  then  $\Gamma_1$  simply coincides with the Brillouin zone  $\mathcal{B}$ . The vectors  $\hat{\psi} \in \mathcal{H}_E$  can be written as  $\hat{\psi} = (\hat{\psi}_1(k), \hat{\psi}_c(p))$  where  $\hat{\psi}_1(k)$  satisfies the periodic boundary conditions on the Brillouin zone  $\mathcal{B}$  and where  $\hat{\psi}_c(p)$  is such that  $\hat{\psi}_c(-\frac{1}{2}b - 0) = \hat{\psi}_c(\frac{1}{2}b + 0)$  together with its derivatives. The integral operator  $K_{11}$  defined in (29) simply reduces to the multiplication operator  $X_{1,1}$  defined in (18).

The unperturbed operator  $\tilde{H}_{F,0}^\theta$  can be written on  $\mathcal{H}_E$  as  $\tilde{H}_1 \otimes \tilde{H}_c^\theta$ , where  $\tilde{H}_1$  and  $\tilde{H}_c^\theta$  are formally defined on  $L^2(\mathcal{B}, dk)$  and  $L^2(\Gamma_c, dp)$  as

$$\begin{aligned} \tilde{H}_1 &= iF \frac{\partial}{\partial k} + E_1(k) + FX_{1,1}(k) \\ \tilde{H}_c^\theta &= iF \frac{\partial}{\partial p} + E(p) + FC_0(p) + F\theta \end{aligned}$$

and its spectrum is given by the union of the spectrum of the operators  $\tilde{H}_1$  and  $\tilde{H}_c^\theta$ .

**Lemma 6.** *The resolvent of  $\tilde{H}_1$  on a vector  $\phi_1 \in L^2(\mathcal{B}, dk)$  is given by*

$$\left( \left[ \tilde{H}_1 - z \right]^{-1} \phi_1 \right)(k) = -\frac{i}{F} \left\{ \int_{-b/2}^k e^{-\frac{i}{F} \int_k^q Q_1(s) ds} \phi_1(q) dq - \frac{\int_{\mathcal{B}} e^{-\frac{i}{F} \int_k^q Q_1(s) ds} \phi_1(q) dq}{1 - e^{\frac{i}{F} b(z - \mathcal{E}_{1,0})}} \right\} \quad (30)$$

where we set  $Q_1(k) = E_1(k) + FX_{1,1}(k) - z$ . The spectrum of the operator  $\tilde{H}_1$  on  $L^2(\mathcal{B}, dk)$  is purely discrete and it consists of a ladder of simple eigenvalues

$$\mathcal{E}_{1,j} = \frac{1}{b} \int_{\mathcal{B}} [E_1(k) + FX_{1,1}(k)] dk + jFa, \quad j \in \mathbb{Z}. \quad (31)$$

*Proof.* Indeed, formula (30) for the resolvent directly comes by means of a simple calculation, and the eigenvalue equation has normalized eigenvector

$$\hat{\psi}_{1,j}(k) = \frac{1}{\sqrt{b}} e^{-\frac{i}{F} \int_{-b/2}^k [E_1(q) + FX_{1,1}(q) - E] dq} \quad (32)$$

where  $E = \mathcal{E}_{1,j}$  in order to have  $\hat{\psi}_{1,j}(-b/2) = \hat{\psi}_{1,j}(+b/2)$ .  $\square$

**Lemma 7.** *The spectrum of the operator  $\tilde{H}_c^\theta$  is purely essential and it coincides with the line  $\Im z = F\Im\theta$ .*

*Proof.* Now, we look for the resolvent of  $\tilde{H}_c^\theta$ , let  $\phi \in L^2(\Gamma_c, dp)$  and we consider the solution of the equation (see, e.g., [16])

$$\left[ iF \frac{\partial}{\partial p} + E(p) + FC_0(p) + F\theta - z \right] \hat{\psi}_c = \phi$$

for  $p \in \Gamma_c$  and with the condition

$$\hat{\psi}_c \left( -\frac{1}{2}b - 0 \right) = \hat{\psi}_c \left( +\frac{1}{2}b + 0 \right).$$

If we denote by

$$\Gamma_+ = \tilde{\mathbb{R}} \cap \left[ \frac{1}{2}b, +\infty \right) \quad \text{and} \quad \Gamma_- = \left( -\infty, -\frac{1}{2}b \right] \cap \tilde{\mathbb{R}}$$

then, by means of a direct calculation it turns out that (hereafter  $f_A$  denotes the integral of a meromorphic function on the set  $A\tilde{\mathbb{C}}$ )

$$\hat{\psi}_c(p) = \begin{cases} -\frac{i}{F} f_p^{+\infty} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq & \text{if } \Im(z - F\theta) > 0 \\ \frac{i}{F} f_{b/2}^p e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq + C_+ e^{\frac{i}{F} \int_{b/2}^p Q_c(q) dq} & \text{if } \Im(z - F\theta) < 0 \end{cases}, \quad p \in \Gamma_+$$

and

$$\hat{\psi}_c(p) = \begin{cases} \frac{i}{F} f_p^{-b/2} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq + C_- e^{\frac{i}{F} \int_{-b/2}^p Q_c(q) dq} & \text{if } \Im(z - F\theta) > 0 \\ -\frac{i}{F} f_{-\infty}^p e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq & \text{if } \Im(z - F\theta) < 0 \end{cases}, \quad p \in \Gamma_-$$

where we set here

$$Q_c(p) = E(p) + F\theta + FC_0(p) - z.$$

Now, we have to require that  $\hat{\psi}_c(-b/2 - 0) = \hat{\psi}_c(+b/2 + 0)$ , which implies that

$$\begin{aligned} C_+ &= -\frac{i}{F} \int_{-\infty}^{-b/2} e^{-\frac{i}{F} \int_{-b/2}^q Q_c(s) ds} \phi(q) dq \\ C_- &= -\frac{i}{F} \int_{b/2}^{+\infty} e^{-\frac{i}{F} \int_{b/2}^q Q_c(s) ds} \phi(q) dq \end{aligned}$$

Then it follows that the spectrum of  $\tilde{H}_c^\theta$  coincides with the line  $\mathbb{R} + iF\Im\theta$  and thus it is purely essential.  $\square$

In conclusion

$$\sigma(\tilde{H}_{F,0}^\theta) = \sigma_d(\tilde{H}_{F,0}^\theta) \cup \sigma_{ess}(\tilde{H}_{F,0}^\theta)$$

where

$$\begin{aligned} \sigma_d(\tilde{H}_{F,0}^\theta) &= \sigma(\tilde{H}_1) = \{\mathcal{E}_{1,j}, j \in \mathbb{Z}\} \\ \sigma_{ess}(\tilde{H}_{F,0}^\theta) &= \sigma(\tilde{H}_c^\theta) = \mathbb{R} + iF\Im\theta \end{aligned}$$

Now, let us fix  $\Im\theta < 0$  and let  $\gamma_j = \partial\Omega_j$  where

$$\Omega_j = \left[ \mathcal{E}_{1,j} - \frac{1}{2}Fa, \mathcal{E}_{1,j} + \frac{1}{2}Fa \right] \times i \left[ \frac{1}{2}F\Im\theta, -\frac{1}{2}F\Im\theta \right]$$

The following estimates for the resolvent operators of  $\tilde{H}_1$  and  $\tilde{H}_c^\theta$  for  $z \in \gamma_j$  hold true

**Lemma 8.** *There exists a positive constant  $C > 0$  independent of  $F$  and  $j$  such that*

$$\|[\tilde{H}_1 - z]^{-1}\| \leq \frac{C}{F}, \quad \forall z \in \gamma_j \quad (33)$$

and

$$\|[\tilde{H}_c^\theta - z]^{-1}\| \leq \frac{C}{F}, \quad \forall z \in \gamma_j. \quad (34)$$

*Proof.* First of all we remark that the Stark-Wannier spectral problem is invariant with respect the translation  $E \rightarrow E + Fa$ , then the above estimates, if hold true, are independent of  $j$  and thus we can assume, for argument's sake,  $j = 0$ . The proof of the estimate (33) directly comes from formula (30) and from the following estimate

$$\min_{z \in \gamma_0} |1 - e^{\frac{i}{F}b(z - \mathcal{E}_{1,0})}| \geq C > 0$$

for some positive constant  $C$  independent of  $F$ . In order to prove estimate (34) we observe that  $\Im(z - F\theta) > 0$  for any  $z \in \gamma_0$ ; then from the proof of Lemma 7 it follows that the resolvent  $\hat{\psi} = [\tilde{H}_c^\theta - z]^{-1}\phi$ , where  $\phi \in L^2(\Gamma_c, dp)$  is a rapidly decreasing test function, is given by

$$\hat{\psi}(p) = \begin{cases} -\frac{i}{F} \int_p^{+\infty} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq & \text{if } p \in \Gamma_+ \\ \frac{i}{F} \int_p^{-b/2} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \phi(q) dq + \\ -\frac{i}{F} e^{\frac{i}{F} \int_{-b/2}^p Q_c(q) dq} \int_{b/2}^{+\infty} e^{-\frac{i}{F} \int_{b/2}^q Q_c(s) ds} \phi(q) dq & \text{if } p \in \Gamma_- \end{cases}. \quad (35)$$

For  $p \in \Gamma_+$  we have that

$$|\hat{\psi}(p)| \leq \frac{1}{F} \int_p^{+\infty} e^{c(p-q)} |\phi(q)| dq, \quad c = \frac{1}{F} \Im(z - F\theta) \geq a \quad (36)$$

for some  $a > 0$  independent of  $F$  since  $z \in \gamma_0$ . Then, it turns out that

$$|\hat{\psi}(p)| \leq \frac{1}{F} \|\phi\|_{L^\infty(\Gamma_+)} \int_p^{+\infty} e^{c(p-q)} dq \leq \frac{C}{F} \|\phi\|_{L^\infty(\Gamma_+)}$$

for some  $C > 0$ . Furthermore

$$\begin{aligned}
\int_{b/2}^{+\infty} |\hat{\psi}(p)| dp &\leq \frac{1}{F} \int_{b/2}^{+\infty} dp \int_p^{+\infty} e^{c(p-q)} |\phi(q)| dq \\
&\leq \frac{1}{F} \int_{b/2}^{+\infty} |\phi(q)| dq \int_{b/2}^p e^{c(p-q)} dp \\
&= \frac{1}{\Im(z - F\theta)} \int_{b/2}^{+\infty} [1 - e^{-c(q-b/2)}] |\phi(q)| dq \\
&\leq \frac{C}{F} \|\phi\|_{L^1(\Gamma_+)}
\end{aligned}$$

for some  $C > 0$ . Then, we have that

$$\|\hat{\psi}\|_{L^\infty(\Gamma_+)} \leq \frac{C}{F} \|\phi\|_{L^\infty(\Gamma_+)} \quad \text{and} \quad \|\hat{\psi}\|_{L^1(\Gamma_+)} \leq \frac{C}{F} \|\phi\|_{L^1(\Gamma_+)}$$

for some  $C > 0$ . From this fact and from the Riesz-Thorin interpolation theorem it follows that

$$\|\hat{\psi}\|_{L^2(\Gamma_+)} \leq \frac{C}{F} \|\phi\|_{L^2(\Gamma_+)}.$$

Similarly, we have the same result for  $p \in \Gamma_-$ ; that is:

$$|\hat{\psi}(p)| \leq \frac{1}{F} \left\{ \int_p^{-b/2} e^{c(p-q)} |\phi(q)| dq + e^{c(p+b/2)} \int_{b/2}^{+\infty} e^{-c(q-b/2)} |\phi(q)| dq \right\}, p \in \Gamma_-$$

where the first integral can be estimated as in the case  $p \in \Gamma_+$ , while the second integral is simply estimated as follows

$$\begin{aligned}
\left\| e^{c(p+b/2)} \int_{b/2}^{+\infty} e^{-c(q-b/2)} |\phi(q)| dq \right\|_{L^2(\Gamma_-)} &\leq C \left| \int_{b/2}^{+\infty} e^{-c(q-b/2)} |\phi(q)| dq \right| \\
&\leq C \left\| e^{-c(q-b/2)} \right\|_{L^2(\Gamma_+, dq)} \|\phi(q)\|_{L^2(\Gamma_+, dq)} \\
&\leq C \|\phi(q)\|_{L^2(\Gamma_+, dq)}
\end{aligned}$$

for some  $C > 0$ . Hence, we can conclude that

$$\begin{aligned}
\|\hat{\psi}\|_{L^2(\Gamma_c)}^2 &= \|\hat{\psi}\|_{L^2(\Gamma_+)}^2 + \|\hat{\psi}\|_{L^2(\Gamma_-)}^2 \\
&\leq \frac{C}{F^2} \|\phi\|_{L^2(\Gamma_+)}^2 + \frac{C}{F^2} [\|\phi\|_{L^2(\Gamma_-)} + \|\phi\|_{L^2(\Gamma_+)}]^2 \\
&\leq \frac{C}{F^2} [\|\phi\|_{L^2(\Gamma_+)}^2 + \|\phi\|_{L^2(\Gamma_-)}^2] \\
&= \frac{C}{F^2} \|\phi\|_{L^2(\Gamma_c)}^2
\end{aligned}$$

Then, the Lemma follows because the set of rapidly decreasing functions belonging to  $L^2(\Gamma_c, dp)$  is dense in  $L^2(\Gamma_c, dp)$ .  $\square$

## 8. PERTURBATION RESULTS AND PROOF OF THEOREM 1.

**8.1. Norm estimate of the perturbation.** We then denote by  $K^\theta$  the perturbation defined on  $\mathcal{H}_E$  as

$$K^\theta \hat{\psi} = \left( K_{1c}^\theta \hat{\psi}_c, K_{c1}^\theta \hat{\psi}_1 + K_{cc}^\theta \hat{\psi}_c \right)$$

where  $K_{1c}^\theta$ ,  $K_{c1}^\theta$  and  $K_{cc}^\theta$  are defined in Lemma 5, and thus

$$\tilde{H}_{F,\eta}^\theta = \tilde{H}_{F,0}^\theta + \eta K^\theta.$$

**Lemma 9.** *Let  $V(x)$  be an analytic function in an open set containing the strip  $|\Im\theta| \leq R$ . Then, the linear operator  $K^\theta$  is bounded for any  $\theta \in \mathbb{C}$  such that  $0 < |\Im\theta| < R$ :*

$$\|K^\theta\| \leq C_1$$

for some  $C_1$  independent of  $F$ .

*Proof.* The proof basically mimic the proof of Theorem 4 where, in the case of analytic potential  $V(x)$ , the exponential term  $e^{-ijb\theta}$  in  $K_{cc}^\theta$  (and similarly the other exponential terms in  $K_{1c}^\theta$  and  $K_{c1}^\theta$ ) are supported by the exponential estimate (27) since we choose  $|\Im\theta| < R$ .  $\square$

From this fact and since

$$\left\| \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1} \right\| \leq \frac{C}{F}, \quad \forall z \in \gamma_0$$

then it follows that the perturbative series

$$\left[ \tilde{H}_{F,\eta}^\theta - z \right]^{-1} = \sum_{j=0}^{\infty} \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1} \left( -\eta K^\theta \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1} \right)^j$$

converges for any  $\eta$  such that  $|\eta| < CF$  for some  $C > 0$ . In fact, we don't have an estimate of such a constant  $C$  and thus, in order to get results about the spectrum of  $\tilde{H}_{F,F}^\theta$  we need to improve the result of Lemma 9 and obtain the convergence of the perturbation series for  $\eta = F$ . To this end we set

$$\mathcal{K}^\theta(z) = \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1} \eta K^\theta \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1}.$$

and the perturbation series takes the form

$$\left[ \tilde{H}_{F,\eta}^\theta - z \right]^{-1} = \left( \left[ \tilde{H}_{F,0}^\theta - z \right]^{-1} - \mathcal{K}^\theta(z) \right) \sum_{j=0}^{\infty} (\eta K^\theta \mathcal{K}^\theta(z))^j \quad (37)$$

We prove that

**Theorem 5.** *Let  $\theta$  be in a given box  $[-\Lambda, +\Lambda] \times i[-R, 0]$  for some  $\Lambda > 0$  fixed. Then, for  $F$  small enough*

$$\|\mathcal{K}^\theta(z)\| \leq C_2 \eta F^{-3/2}, \quad (38)$$

for any  $z \in \gamma_0$  and for some  $C_2$  independent of  $F$  and  $\theta$ .

*Proof.* The vector  $\mathcal{K}^\theta \hat{\psi}$  may be written as  $(\mathcal{K}_{1c}^\theta \hat{\psi}_c, \mathcal{K}_{c1}^\theta \hat{\psi}_1 + \mathcal{K}_{cc}^\theta \hat{\psi}_c)$  where

$$\begin{aligned} \mathcal{K}_{1c}^\theta &= \eta [\tilde{H}_1 - z]^{-1} K_{1,c}^\theta [\tilde{H}_c^\theta - z]^{-1} \\ \mathcal{K}_{c1}^\theta &= \eta [\tilde{H}_c^\theta - z]^{-1} K_{c,1}^\theta [\tilde{H}_1 - z]^{-1} \\ \mathcal{K}_{cc}^\theta &= \eta [\tilde{H}_c^\theta - z]^{-1} K_{c,c}^\theta [\tilde{H}_c^\theta - z]^{-1} \end{aligned}$$

In order to estimate these terms we consider at first  $\mathcal{K}_{cc}^\theta \hat{\psi}_c$ . Let assume for a moment that  $p \in \Gamma_+$ , then by (35) it follows that:

$$\left[ \mathcal{K}_{cc}^\theta \hat{\psi}_c \right] (p) = -\frac{i\eta}{F} \int_p^{+\infty} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \left( K_{cc}^\theta [\tilde{H}_c^\theta - z]^{-1} \hat{\psi}_c \right) (q) dq = g_+(p) + g_-(p)$$

where by (28)

$$g_{\pm}(p) = -i \frac{\eta}{F} \int_p^{+\infty} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \int_{\Gamma_{\pm}} Y(q, s) e^{i(q-s)\theta} \left( [\tilde{H}_c^\theta - z]^{-1} \hat{\psi}_c \right) (s) ds dq$$

We consider, at first, the integral over  $\Gamma_+$  obtaining that (see Lemma 5)

$$g_+(p) = -\frac{\eta}{F^2} \int_p^{+\infty} e^{\frac{i}{F} \int_q^p Q_c(s) ds} \sum_{j \in \mathbb{Z} \setminus \{0\} : q-jb \in \Gamma_+} \alpha_j(q, q-jb) e^{ijb\theta} \int_{q-jb}^{+\infty} e^{\frac{i}{F} \int_\zeta^{q-jb} Q_c(\xi) d\xi} \hat{\psi}_c(\zeta) d\zeta dq$$

As a first step we should remark that we can exchange integration and sum operations because

$$\begin{aligned} \left| e^{\frac{i}{F} \int_q^p Q_c(s) ds} \right| &= \left| e^{-\frac{1}{F} \int_q^p \Im Q_c(s) ds} \right| = e^{c(p-q)} \\ \left| e^{\frac{i}{F} \int_\zeta^{q-jb} Q_c(s) ds} \right| &= \left| e^{-\frac{1}{F} \int_\zeta^{q-jb} \Im Q_c(s) ds} \right| = e^{c(q-jb-\zeta)} \end{aligned}$$

for some positive constant  $c > 0$  defined in (36) and where  $p < q$  and  $q - jb < \zeta$  in the domain of integration, and (see (27) and (28))

$$|\alpha_j(q, q-jb)| \leq C e^{-|j|b(R-\Im\theta)}, \quad \Im\theta < R,$$

uniformly with respect to  $q$ .

Hence  $g_+(p)$  may be written as

$$\begin{aligned} g_+(p) &= -\frac{\eta}{F^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} e^{ijb\theta} \int_{p-jb}^{+\infty} \hat{\psi}_c(\zeta) d\zeta \times \\ &\quad \times \left[ \int_p^{\zeta+jb} \chi(q-jb-\frac{1}{2}b) \alpha_j(q, q-jb) e^{\frac{i}{F} \int_q^p Q_c(s) ds + \frac{i}{F} \int_\zeta^{q-jb} Q_c(s) ds} dq \right] \\ &= -\frac{\eta}{F^2} \sum_{j \in \mathbb{Z} \setminus \{0\}} e^{ijb\theta} \int_{p-jb}^{+\infty} e^{\frac{i}{F} \int_\zeta^p Q_c(s) ds} V_j(p, \zeta) \hat{\psi}_c(\zeta) d\zeta \end{aligned} \quad (39)$$

where  $\chi(x)$  is the Heaviside function (i.e.  $\chi(x) = 1$  for any  $x \geq 0$  and  $\chi(x) = 0$  elsewhere), and

$$V_j(p, \zeta) = \int_p^{\zeta+jb} \chi(q-jb-\frac{1}{2}b) \alpha_j(q, q-jb) e^{-\frac{i}{F} \int_q^{q-jb} Q_c(s) ds} dq.$$

By means of a stationary phase argument we obtain the following estimate.

**Lemma 10.** *We have that*

$$|V_j(p, \zeta)| \leq CF^{1/2} |\zeta + jb - p| e^{-cjb - |j|bR}$$

for some positive constants  $c, C > 0$  independent of  $j$  and  $F$  ( $c$  is defined in (36)).

*Proof.* The function  $V_j(p, \zeta)$  can be written in the following form

$$V_j(p, \zeta) = \int_{q_1}^{q_2} v_j(q) dq, \quad v_j(q) := \alpha_j(q, q-jb) e^{-\frac{i}{F} \int_q^{q-jb} E(s) ds} e^{-i \int_q^{q-jb} [C_0(s) + \theta - z/F] ds} \quad (40)$$

where we set  $q_2 = \zeta + jb$  and  $q_1 = \max[p, (j + \frac{1}{2})b]$ , and where  $z \sim F$  since  $z \in \gamma_0$ .

Hence, the stationary points are the solutions  $q \in \mathbb{C}$  of the equation

$$E(q-jb) = E(q).$$

Recalling that  $E(q)$  is a monotone increasing function on  $\Gamma$  and that

$$E(-p) = E(p) \quad \text{and} \quad E(\bar{p}) = \overline{E(p)}$$

then the solution on  $\tilde{\mathbb{C}}$  of the previous equation are only the Kohn branch points  $k_j$ . Hence, we could apply the stationary phase method obtaining the estimate

$$|V_j(p, \zeta)| \leq C_j F |\zeta + jb - p| e^{-cjb - |j|bR} \quad (41)$$

where  $C_j \leq C/g_j$  and  $g_j$  is the gap between the two bands; indeed, by integration by parts we have that (40) has a denominator of the form  $E(q - jb) - E(q)$  where

$$\min_{q \in \mathbb{R}} |E(q - jb) - E(q)| = \left| E\left(\frac{1}{2}jb \pm 0\right) - E\left(-\frac{1}{2}jb \mp 0\right) \right| = E_{j+1}^t - E_j^b := g_j$$

In order to have an estimate uniform with respect to  $j$  let us fix  $\mu \in (0, 1)$  and let

$$\begin{aligned} S_1 &= \{q : q \in [q_1, q_2] \cap \tilde{\mathbb{R}}, |E(q - jb) - E(q)| \geq F^\mu\} \\ S_2 &= \{q : q \in [q_1, q_2] \cap \tilde{\mathbb{R}}, |E(q - jb) - E(q)| < F^\mu\} \end{aligned}$$

and let, with obvious meaning of notation,  $V_j(p, \zeta) = \int_{S_1} v_j(q) dq + \int_{S_2} v_j(q) dq$ . The integral over the set  $S_1$  may be estimated by integration by parts obtaining that

$$\begin{aligned} \left| \int_{S_1} v_j(q) dq \right| &\leq F(q_2 - q_1) e^{-cjb} \max_q \left| \frac{d}{dq} \frac{\alpha_j(q, q - jb)}{E(q - jb) - E(q)} \right| F^{-\mu} \\ &\leq C |\zeta + jb - p| F^{1-\mu} e^{-cjb - |j|bR} \end{aligned}$$

since Lemma 4. In order to estimate the second integral we simply observe that  $E(p)$  has a square branch point at the Kohn's branch points and that  $E(p) \sim p^2$  for large  $p$ , thus we have that the measure of the set  $S_2$  is of order  $F^\mu/j$ . From this fact and from Lemma 4 again then it follows that

$$\left| \int_{S_2} v_j(q) dq \right| \leq C F^\mu e^{-cjb - |j|bR}.$$

The result follows by choosing  $\mu = \frac{1}{2}$ .  $\square$

As a consequence of such a result it follows that

$$\begin{aligned} |g_+(p)| &\leq \frac{\eta}{F^2} \sum_j e^{-jb\Im\theta} \int_{p-jb}^{+\infty} e^{c(p-\zeta)} |V_j(p, \zeta)| |\hat{\psi}_c(\zeta)| d\zeta \\ &\leq C \frac{\eta F^{1/2}}{F^2} \sum_j e^{-jb\Im\theta} \int_{p-jb}^{+\infty} e^{c(p-\zeta)} |\zeta + jb - p| e^{-cjb - |j|bR} |\hat{\psi}_c(\zeta)| d\zeta \\ &\leq C \eta F^{-3/2} \|\hat{\psi}_c\|_{L^\infty} \end{aligned}$$

and similarly

$$\begin{aligned}
\|g_+(p)\|_{L^1(\Gamma_+)} &\leq \frac{\eta}{F^2} \sum_{j \neq 0} e^{-jb\Im\theta} \int_{\frac{1}{2}b}^{+\infty} dp \left| \int_{p-jb}^{+\infty} e^{c(p-\zeta)} |V_j(p, \zeta)| |\hat{\psi}_c(\zeta)| d\zeta \right| \\
&\leq \frac{\eta}{F^2} \sum_{j \neq 0} e^{-jb\Im\theta} \int_{\frac{1}{2}b-jb}^{+\infty} |\hat{\psi}_c(\zeta)| d\zeta \int_{\frac{1}{2}b}^{\zeta-jb} e^{c(p-\zeta)} |V_j(p, \zeta)| dp \\
&\leq \frac{\eta}{F^2} \sum_{j \neq 0} e^{-jb\Im\theta} \int_{\frac{1}{2}b-jb}^{+\infty} |\hat{\psi}_c(\zeta)| d\zeta \int_{\frac{1}{2}b}^{\zeta-jb} e^{c(p-\zeta)} C F^{1/2} |\zeta + jb - p| e^{-cjb - |j|bR} dp \\
&\leq C \eta F^{-3/2} \|\hat{\psi}_c\|_{L^1(\Gamma_c)}
\end{aligned}$$

Since a similar estimate still holds true for the vector  $g_-(p)$  and when  $p \in \Gamma_-$ ; then, by means of the Riesz-Thorin interpolation argument already applied in the proof of Lemma 8, the estimate (38) follows. Similarly we have the estimates of the other two terms  $\mathcal{K}_{1c}^\theta \hat{\psi}_c$  and  $\mathcal{K}_{c1}^\theta \hat{\psi}_1$ .

The proof of Theorem 5 is so completed.  $\square$

**8.2. Proof of Theorem 1.** As a result of Theorem 5 it follows that the series (37) converges for any  $\eta$  such that  $C_1 C_2 \eta^2 < F^{3/2}$ . In particular, if  $F$  is small enough then the series (37) converges for  $\eta = F$ . We have proved that

**Theorem 6.** *Let  $\theta$  be fixed and such that  $-R < \Im\theta < 0$ , let  $F$  small enough, then the spectrum of  $\tilde{H}_F^\theta := \tilde{H}_{F,F}^\theta$  inside  $\gamma_0$  consists of one, and only, non degenerate eigenvalue  $\mathcal{E}_{1,0}(F)$*

*Proof.* This result holds true because, from the convergence of the perturbation series (37), it turns out that  $\gamma_0 \subset \rho(\tilde{H}_{F,\eta}^\theta)$  for any  $\eta \in [0, F]$ . On the other side,  $\tilde{H}_{F,\eta}^\theta$  is an analytic family of operator of type (A). Hence the dimension of the range of the projection

$$P^\theta(\eta) = -\frac{1}{2\pi i} \oint_{\gamma_0} [\tilde{H}_{F,\eta}^\theta - z]^{-1} dz$$

is constant for any  $\eta \in [0, F]$ , where  $\dim(\text{Ran}(\tilde{H}_{F,0}^\theta))=1$ . Hence,  $\dim(\text{Ran}(\tilde{H}_{F,F}^\theta))=1$   $\square$

**Remark 7.** *Similarly, the perturbation argument gives that  $\sigma_{ess}(\tilde{H}_{F,F}^\theta)$  is contained in the strip  $-CF \leq \Im z \leq -\frac{1}{2}|\Im\theta|F$  for some  $C > 0$ . In fact, we would expect that the essential spectrum coincides with the line  $\Im z = -|\Im\theta|F$  as for the unperturbed problem.*

**Remark 8.** *By construction it turns out that the spectrum of  $\tilde{H}_{F,F}^\theta$  in the strip  $|\Im z| < \frac{1}{2}|\Im\theta|F$  is given by a ladder of simple eigenvalues*

$$\mathcal{E}_{1,j}(F) = \mathcal{E}_{1,0}(F) + Fja.$$

*Furthermore, a direct computation gives that the first perturbative term is of order  $\mathcal{O}(F^2)$ . Indeed, let  $\hat{\psi}_0 = (\hat{\psi}_{1,0}, \hat{\psi}_{c,0})$  be the unperturbed eigenvector associated to the eigenvalue  $\mathcal{E}_{1,0}$  given in (31), where  $\psi_{c,0} \equiv 0$  and  $\psi_{1,0}$  is given in (32). Then, the simple eigenvalue  $\mathcal{E}_{1,0}(F)$  is given by*

$$\mathcal{E}_{1,0}(F) = \frac{\langle \phi, \tilde{H}_{F,F}^\theta P^\theta(F) \hat{\psi}_{1,0} \rangle}{\langle \phi, P^\theta(F) \hat{\psi}_{1,0} \rangle}$$

where  $\phi$  is any vector such that  $\langle \phi, P^\theta(F)\hat{\psi}_{1,0} \rangle \neq 0$ , e.g.  $\phi = \hat{\psi}_{1,0}$ . In particular, an straightforward calculation gives that the first perturbative term is exactly zero:  $\mathcal{E}_{1,0}(F) = \mathcal{E}_{1,0} + O(F^2)$ .

**Remark 9.** If  $N$  is bigger than 1 then the same arguments apply again obtaining that

$$\dim(\text{Ran}(\tilde{H}_{F,F}^\theta)) = N$$

for  $F$  small enough, i.e.  $0 < F < F_N$  for some  $F_N > 0$ . In such a case the spectrum of  $\tilde{H}_{F,F}^\theta$  in the strip  $|\Im z| < \frac{1}{2}|\Im \theta|F$  is given by  $N$  ladders of simple eigenvalues. In fact,  $F_N$  will depend on the norm of the perturbation and on the distance of the unperturbed eigenvalues from the boundary  $\gamma_0$ , then we expect that  $F_N \rightarrow 0$  as  $N \rightarrow +\infty$ .

We are ready now to conclude the proof of Theorem 1. Let

$$\mathcal{R} := \{\mathcal{E}_{1,0}(F) + Faj, j \in \mathbb{Z}\} = \sigma\left(\tilde{H}_{F,F}^\theta\right) \cap \{z \in \mathbb{C} : |\Im z| \leq \frac{1}{2}|\Im \theta|F\}$$

Furthermore, we remark that  $K^\theta$  is a bounded operator-valued analytic function on  $|\Im \theta| < R$ . Hence  $\tilde{H}_{F,F}^\theta$  is an analytic family of type (A). By means of standard arguments (see [7]) then it follows that the eigenvalues of  $\tilde{H}_{F,F}^\theta$  are  $\theta$ -independent (as long as the essential spectrum does not intersect these eigenvalues). Hence  $\mathcal{R}$  does not depend on  $\theta$  for  $F$  small enough and

$$\mathcal{R} \subset \{z \in \mathbb{C} : -\frac{1}{2}|\Im \theta|F < \Im z < 0\}.$$

Indeed, it follows that  $\Im \mathcal{E}_{1,0}(F) < 0$ ; If not then  $\mathcal{E}_{1,0}(F)$  is an eigenvalue of  $\tilde{H}_{F,F}^\theta$  for any  $\theta$ , even for  $\theta = 0$ , and then it is an eigenvalue of  $H_F$ , in contradiction with the fact that the spectrum of  $H_F$  is purely absolutely continuous. The set  $\mathcal{R}$  is then the set of quantum resonances; indeed, by making use of the standard arguments discussed in [7], it follows that the function

$$\langle \hat{\psi}, [\tilde{H}_F - z]^{-1}\hat{\psi} \rangle \tag{42}$$

has an analytic continuation from  $\Im z > 0$  to the set  $\{\Im z > -\frac{1}{2}F|\Im \theta|\} \setminus \mathcal{R}$ . If  $z_0 \in \mathcal{R}$  then there exists an  $\hat{\psi}$  such that (42) has an analytic continuation to  $\Im z > -\frac{1}{2}F|\Im \theta|$  with poles in the points of  $\mathcal{R}$ .

#### APPENDIX A. DERIVATION OF (16)

In order to derive formula (16) we make use of the same ideas as in [16]. If the periodic potential is not a symmetric function then we may write (see §4 in [16])

$$\varphi(x, E) = \frac{\chi(x, E)}{N^{1/2}(E)}$$

where  $N$  is a normalization factor and where

$$\chi(x, E) = \phi_1(x, E)\phi_2(a, E) - [\phi_1(a, E) - \lambda(E)]\phi_2(x, E).$$

Following the same argument by [16] we only have to estimate the integral

$$I := \int_0^a \chi(x, E)\tilde{\chi}(x, E + \delta E)dx$$

where

$$\tilde{\chi}(x, E) = \bar{\chi}(x, E) = \phi_1(x, E)\phi_2(a, E) - \left[ \phi_1(a, E) - \frac{1}{\lambda(E)} \right] \phi_2(x, E).$$

By making use of the quasiperiodic boundary conditions

$$\left\{ \begin{array}{l} \chi(a, E) = \lambda(E)\chi(0, E) \\ \frac{\partial\chi(a, E)}{\partial x} = \lambda(E)\frac{\partial\chi(0, E)}{\partial x} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \tilde{\chi}(a, E) = \frac{1}{\lambda(E)}\tilde{\chi}(0, E) \\ \frac{\partial\tilde{\chi}(a, E)}{\partial x} = \frac{1}{\lambda(E)}\frac{\partial\tilde{\chi}(0, E)}{\partial x} \end{array} \right.$$

a straightforward calculation yields to

$$I = -\lambda'(E) \left( 1 - \frac{1}{\lambda^2(E)} \right) \phi_2(a, E)\delta E + O((\delta E)^2)$$

as in the case of symmetric potentials. Hence  $N(E) = -2\phi_2(a, E)\frac{d\mu}{dE}$  holds true even in the case of not symmetric potential too, proving (16).

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